

A Weighted Turán-Type Monotonicity Lemma for Modified Bessel Functions via the Calibrated Amos Bound, with an Application to the Ordering of a Surface Expansion in Two-Dimensional Lattice Gauge Theory

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July 2026

Abstract

For $x > 0$ and real $m \geq 1$ define, with I_μ the modified Bessel function of the first kind,

$$\varphi_m(x) = \frac{(m-1) I_{m-1}(x)^2 + (m+1) I_{m+1}(x)^2}{m I_m(x)^2}.$$

We prove that $\varphi_m(x) < \varphi_{m+1}(x)$ for every $x > 0$ and every real $m \geq 1$ — a weighted Turán-type monotonicity statement we have not found in the literature, although every ingredient of the proof is classical. The proof is fully elementary and uses exactly the arsenal of the companion note on unit-step order-monotonicity: the three-term recurrence and the Amos-type bound $\rho_m < x/(a + \sqrt{a^2 + x^2})$, $a = m + \frac{1}{2}$, for $\rho_m = I_{m+1}/I_m$. The engine is a new exact factorization: with $u = \rho_m(x)$, $c = 1/x$, $S = u + 1/u$, $P = 1/u - u$,

$$\frac{1}{2}(\varphi_{m+1} - \varphi_m) = (S - 3c)(P - (2m+1)c) + (2m+1)c^2,$$

an algebraic identity once ρ_{m-1} and ρ_{m+1} are eliminated by the recurrence. The second factor is positive by the calibrated Amos bound — it is precisely the unit-step inequality of the companion note — and the first because the same bound forces $u < x/(2m+1) \leq x/3$. As an application we obtain the strict ordering $c_{mn} < 0$ ($m < n$) of the 2×2 determinants of the coefficient arrays of a π -local surface expansion arising in an audit-first programme on two-dimensional $SU(2)$ lattice gauge theory; this is the structural half of a positivity problem whose remaining half — an alternating-parity dominance with near-perfect mirror cancellation — we state precisely and leave open. The full chain of lemmas is machine-checked in Lean 4/Mathlib (no `sorry`; standard axiom oracle), with the analytic inputs carried as named hypotheses, and independently audited numerically at high precision.

1 Statement

Throughout, $x > 0$, I_μ is the modified Bessel function of the first kind, all values $I_\mu(x)$ for $\mu \geq 0$ are positive, and $\rho_\mu(x) = I_{\mu+1}(x)/I_\mu(x) \in (0, 1)$.

Theorem 1 (weighted Turán-type unit-step monotonicity). *For every $x > 0$ and every real $m \geq 1$,*

$$\varphi_m(x) = \frac{(m-1) I_{m-1}(x)^2 + (m+1) I_{m+1}(x)^2}{m I_m(x)^2} < \varphi_{m+1}(x), \quad (1)$$

strictly. Equivalently, in terms of the consecutive ratios $\rho_\mu = I_{\mu+1}/I_\mu$:

$$\frac{1}{m} \left(\frac{m-1}{\rho_{m-1}^2} + (m+1)\rho_m^2 \right) < \frac{1}{m+1} \left(\frac{m}{\rho_m^2} + (m+2)\rho_{m+1}^2 \right).$$

The weights $(m-1, m, m+1)$ are not decorative: they arise from a specific surface expansion (Section 3), and — as the proof shows — they are precisely the weights for which the difference factorizes exactly over the classical Amos bound [1, 2, 3]. We have not found (1) stated in the Turán-type literature for modified Bessel functions [4, 5, 6], though it sits squarely in its tradition: for fixed x , log-concavity of $\mu \mapsto I_\mu(x)$ (the classical Turán inequality [4, 5]) controls the *first-order* behaviour of the ratio sequence, while (1) is a statement about a *weighted second-order* combination.

2 The proof

We use two classical facts, valid for all real orders ([7, 10.29.1]): the three-term recurrence, in the two forms

$$\frac{1}{\rho_{m-1}} = \rho_m + \frac{2m}{x}, \quad \rho_{m+1} = \frac{1}{\rho_m} - \frac{2(m+1)}{x}, \quad (2)$$

and the Amos-type upper bound (Amos [1]; proved best of its form and valid for all real orders ≥ 0 by Segura [2, 3]):

$$\rho_m(x) < U_m(x) := \frac{x}{a+s}, \quad a = m + \frac{1}{2}, \quad s = \sqrt{a^2 + x^2}. \quad (3)$$

Step 1: elimination

Write $u = \rho_m(x) \in (0, 1)$ and $c = 1/x$. Substituting (2) into the ρ -form of (1) and clearing the positive denominators $m, m+1$, the claim $\varphi_{m+1} - \varphi_m > 0$ becomes $\Delta_m > 0$ where

$$\Delta_m = \frac{m^2}{u^2} - (m^2 - 1) \left(u + 2mc \right)^2 + m(m+2) \left(\frac{1}{u} - 2(m+1)c \right)^2 - (m+1)^2 u^2.$$

Step 2: the exact factorization

Lemma 2. *With $S = u + 1/u$ and $P = 1/u - u$,*

$$\frac{1}{2}(\varphi_{m+1} - \varphi_m) = \frac{\Delta_m}{2m(m+1)} = (S - 3c)(P - (2m+1)c) + (2m+1)c^2.$$

Proof. Expanding Δ_m and collecting, the coefficients of $1/u^2$ and u^2 are $\pm 2m(m+1)$, the linear-in- c terms are $-4m(m+1)c[(m-1)u + (m+2)/u]$, and the c^2 terms sum to $8m(m+1)(2m+1)c^2$; dividing by $2m(m+1)$,

$$\frac{\Delta_m}{2m(m+1)} = \left(\frac{1}{u^2} - u^2 \right) - \frac{2}{x} \left((m-1)u + \frac{m+2}{u} \right) + \frac{4(2m+1)}{x^2}.$$

Now $(m-1)u + (m+2)/u = \frac{1}{2}[(2m+1)S + 3P]$ and $1/u^2 - u^2 = PS$, so the right side equals $PS - c(2m+1)S - 3cP + 4(2m+1)c^2 = (S - 3c)(P - (2m+1)c) + (2m+1)c^2$. $\square$

Step 3: positivity of both terms

The second factor is strictly positive: $P - (2m + 1)c > 0$ is *exactly* the unit-step order-monotonicity inequality $1/\rho_m - \rho_m > (2m + 1)/x$ of the companion note [8], which follows from (3) by the exact calibration $1/U_m - U_m = (2a)/x = (2m + 1)/x$ together with the strict decrease of $t \mapsto 1/t - t$ on $(0, \infty)$.

The first factor is strictly positive: since $s > a$ in (3), $U_m < x/(2a) = x/(2m + 1)$, so $u < x/(2m + 1) \leq x/3$ for $m \geq 1$, hence $S > 1/u > (2m + 1)/x \geq 3/x = 3c$.

The third term $(2m + 1)c^2$ is positive. This proves Theorem 1. $\square$

Remark 3 (the calibration, for the third time). The bound U_m of (3) is the positive root of $1/t - t = (2m + 1)/x$: it is the extremal function of the threshold that appears in the factorization. In the companion note [8] this exact calibration was the engine of the unit-step monotonicity of $(\log I_\nu)'$ and was shown to be *equivalent* to the Amos bound; here the same calibrated threshold reappears, multiplied against a second factor whose positivity is again a consequence of the same bound. Within the audit programme from which these notes emerge, this is the third lock opened by the same key, which suggests the calibration identity, rather than any particular inequality, is the structural invariant of this family of problems.

3 Application: ordering of the π -local surface expansion

The following objects arise in an audit-first research programme on the two-dimensional $SU(2)$ lattice gauge theory with Wilson action (Griffiths-II positivity on 2-complexes; see the programme records [9] for context). For $\beta > 0$ let $r_k = I_{k+1}(\beta)/I_1(\beta)$, $k \geq 0$, and define the coefficient arrays ($k \geq 1$)

$$A_k = r_{k-1}^2 \sum_{j \in \{k-2, k\}, j \geq 0} (j+1) r_j^2, \quad B_k = k r_{k-1}^4.$$

A direct computation shows $A_k/B_k = \varphi_k(\beta)$ for every $k \geq 1$ — including $k = 1$, where the boundary truncation of the sum (the $j = -1$ term is absent) is annihilated by the weight $(k-1)$ in φ_k , so no separate edge case is needed.

Corollary 4 (determinant ordering). *For all $\beta > 0$ and all integers $1 \leq m < n$,*

$$c_{mn} = A_m B_n - A_n B_m = B_m B_n (\varphi_m(\beta) - \varphi_n(\beta)) < 0.$$

Proof. $B_k > 0$, and $\varphi_m < \varphi_n$ by Theorem 1 iterated over unit steps from m to n . $\square$

Open problem 5 (what remains open, stated honestly). The positivity lemma that Corollary 4 serves is: for $t = \pi - \varepsilon$, $\varepsilon \in (0, 0.4]$, and all $\beta > 0$ (the application requires only a finite window of couplings; the numerics below support the full range),

$$W(t) = \sum_{1 \leq m < n} c_{mn} [(m-n) \sin((m+n)t) + (m+n) \sin((n-m)t)] < 0.$$

Writing $p = n - m$, $q = n + m$, the substitution $t = \pi - \varepsilon$ gives the *exact identity*, valid for every pair,

$$W = \sum_{m < n} |c_{mn}| (-1)^{m+n} [q \sin(p\varepsilon) - p \sin(q\varepsilon)],$$

where Corollary 4 supplied $c_{mn} = -|c_{mn}|$. For the pairs with $q\varepsilon \leq \pi$ the bracket is strictly positive (decrease of $\sin(y)/y$ on $(0, \pi]$); for $q\varepsilon > \pi$ it is not sign-definite, so the sum is a signed-weight competition in which — numerically — the odd-parity classes prevail at every (β, ε) tested. The claim $W < 0$ is thus an *alternating-parity dominance* over the bracket weights, exact in the rewriting and open in the dominance. High-precision numerics (mpmath, up to 100 significant digits) show the two parity classes cancel to relative accuracy $\approx e^{-c\beta}$ with $c \approx 2.03$ – 2.17 — e.g. true totals $W \approx -2.9 \cdot 10^{-8}$ at $\beta = 8$, $-5.0 \cdot 10^{-29}$ at $\beta = 32$, $-5.1 \cdot 10^{-95}$ at $\beta = 100$ (always negative) — so any proof must be structural rather than estimate-based, and any numerical exploration below working precision $\gtrsim 2.2\beta + 20$ digits sees pure rounding noise. We record the phenomenology and the reduction here and make *no claim*: the problem is open.

4 Machine verification

The full chain of Section 2 is machine-checked in Lean 4/Mathlib (file `PhiMonotone.lean` accompanying this note; checked with `lake env lean` against Lean 4.30.0-rc2, Mathlib commit `cd3b69b`): `amos_calibration` (the exact calibration), `amos_small` ($u < x/(2m+1)$ from (3)), `phi_unit_step` (Lemma 2 plus positivity), and `phi_step_of_recurrences` (Theorem 1 with the recurrences (2) and the bound (3) entering as the named hypotheses `hrec1`, `hrec2`, `hAmos`). The axiom oracle returns Lean’s three standard axioms for all four theorems:

```
'amos_calibration'' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'amos_small' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'phi_unit_step' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'phi_step_of_recurrences' depends on axioms:
  [propext, Classical.choice, Quot.sound]
```

(exit code 0; no `sorry`, no `admit`). As in the companion note, what is machine-checked is the ordered-field implication [recurrences + Amos bound $\Rightarrow$ monotonicity]; the analytic inputs themselves are classical and cited.

5 Numerical audit

The factorization of Lemma 2 was verified symbolically (sympy: residual 0). The reduction $\varphi_k = A_k/B_k$ (including the $k = 1$ edge), the identity $\Delta_m = m(m+1)(\varphi_{m+1} - \varphi_m)$, and the positivity of both factors were verified in mpmath at 40 significant digits on $\beta \in \{0.3, 2, 8, 32, 100, 1000\}$, $m \leq 11$; the unit-step increase of φ_m for *real* m was checked on $m \in \{1, 1.3, 2, 2.71, 5.5, 9, 14.25, 40\}$, $x \in \{0.2, 1, 3.7, 15, 80, 400\}$ (minimal margin $7.5 \cdot 10^{-5}$ at $x = 400$, $m = 1$); and the strict increase of the integer sequence was verified for $k \leq 200$ at $\beta \in \{2, 8, 32, 100\}$. Scripts and sources are archived in the programme repository:

<https://github.com/lluiseiriksson/THE-ERIKSSON-PROGRAMME/tree/main/surface-theorem>

Acknowledgements

This note is part of an audit-first research programme whose records, Lean sources and numerical suites are public at <https://github.com/lluiseiriksson/THE-ERIKSSON-PROGRAMME>. The

statement is presented as a lemma with an application and an open problem, in that order; nothing here is claimed about the open problem beyond its precise reduction, and nothing is claimed about four-dimensional gauge theory.

References

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