Dynamic Density Relaxation and Spin Formation in Struck Spherical Bodies: A Continuum Mechanics Perspective

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Abstract

Understanding how rotational motion develops after an impact remains a central problem in continuum mechanics, impact dynamics, and solid mechanics. Classical theories successfully describe stress propagation, elastic deformation, plasticity, contact mechanics, and wave transmission within struck bodies. Nevertheless, they generally describe the observed dynamics through stress and displacement fields without explicitly considering the transient evolution of the material density field as an independent internal variable.

In the present work, a new continuum-mechanics framework is proposed in which density is introduced as an evolving thermodynamic state variable coupled to elastic deformation through a free-energy functional. The central hypothesis is that a localized impact generates a transient density perturbation that propagates throughout the body as a relaxation wave. During this process, density gradients, pressure gradients, and elastic stresses evolve simultaneously while continuously exchanging energy through thermodynamically admissible relaxation mechanisms. Rather than assuming that the density field instantaneously returns to equilibrium, the proposed model considers its progressive redistribution until the initial homogeneous state is recovered.

The formulation combines the conservation laws of continuum mechanics with a variational free-energy approach and a dissipative gradient-flow evolution equation. The resulting model satisfies thermodynamic consistency, admits a Lyapunov free-energy functional, and naturally predicts monotonic energy dissipation toward equilibrium. Classical elasticity is recovered as a limiting case when density relaxation becomes negligible.

To investigate the physical consequences of the proposed formulation, preliminary numerical simulations were performed using custom two-dimensional and three-dimensional computational models. The simulations indicate that off-centre impacts generate transient misalignment between density and pressure gradients, producing a measurable rotational source proxy that reaches a maximum for intermediate impact eccentricities. Although these simulations do not constitute experimental validation, they provide a proof of concept demonstrating that the proposed mathematical framework generates physically meaningful behavior consistent with several qualitative trends reported in the impact-mechanics literature.

The theoretical predictions are further discussed through comparison with established results in elasticity, contact mechanics, finite-element analysis, stress-wave propagation, impact dynamics, and thermodynamics. Rather than replacing existing continuum theories, the proposed framework extends them by introducing density relaxation as an additional internal mechanism contributing to energy redistribution following mechanical impact.

The proposed formulation establishes a unified theoretical basis for future numerical and experimental investigations of transient density evolution in impacted solids and suggests that density relaxation may represent a useful additional state variable in continuum descriptions of dynamic deformation and rotational response.

Keywords

Continuum mechanics; Density relaxation; Impact dynamics; Elasticity; Gradient flow; Free-energy formulation; Stress-wave propagation; Spin formation; Thermodynamic consistency; Finite element modeling.

Nomenclature

ρ : Material density

ρ_0 : Reference equilibrium density

u : Displacement vector

σ : Cauchy stress tensor

ε : Small-strain tensor

P : Internal pressure

F : Helmholtz free-energy functional

M : Density mobility coefficient

β : Density penalty coefficient

κ : Density-gradient coefficient

γ : Internal damping coefficient

c : Elastic wave velocity

Ω : Computational domain

n : Outward unit normal vector

t : Time

∇: Spatial gradient operator

∇^2 : Laplacian operator

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1. Introduction

The mechanical response of solid bodies subjected to impact has been investigated for more than a century and constitutes one of the fundamental subjects of continuum mechanics, elasticity, contact mechanics, and impact engineering. Classical theories developed by Hertz [5], Love [24], Timoshenko and Goodier [23], Landau and Lifshitz [1], Johnson [22], and many subsequent researchers provide a remarkably successful description of stress transmission, elastic deformation, contact forces, and wave propagation inside deformable solids. Modern continuum mechanics further generalized these formulations through rigorous thermodynamic principles, nonlinear constitutive theories, variational formulations, and finite-element methodologies developed by Truesdell and Noll [2], Gurtin [3], Eringen [4], Germain [8], Ciarlet [14], Hughes [13], Holzapfel [17], and many others. These theories now constitute the standard mathematical framework for describing deformation in engineering structures subjected to static and dynamic loading. Despite these remarkable advances, most existing formulations primarily describe the mechanical

evolution of solids through displacement, strain, stress, velocity, and internal constitutive variables. The material density generally appears as either a constant parameter or a quantity constrained by mass conservation rather than as an independent transient field actively participating in the relaxation process following impact.

The present work investigates whether introducing density as a dynamic internal variable may provide additional insight into transient impact phenomena. The central idea is illustrated schematically in **Figure 1**, which summarizes the proposed physical sequence from localized impact to density redistribution, relaxation-wave propagation, and eventual return toward equilibrium.

Rather than assuming instantaneous recovery of the initial density distribution, the proposed framework considers that a localized density perturbation may propagate through the body while interacting continuously with pressure gradients, elastic stresses, and internal energy redistribution. This conceptual mechanism is illustrated in **Figure 2**.

The proposed formulation is intentionally constructed within the established framework of continuum mechanics. It preserves the conservation laws of mass and momentum while introducing an additional thermodynamic evolution equation governing density relaxation. The complete mathematical structure of the model is summarized later in **Figure 3** and **Table 3**.

From a thermodynamic perspective, the present approach adopts a free-energy formulation inspired by gradient theories and diffuse-interface models [6,7], while remaining fully compatible with classical continuum mechanics. The different energetic contributions considered in the model are summarized in **Table 2**.

Unlike phenomenological models introducing additional constitutive parameters solely for numerical convenience, the present formulation seeks to associate every additional variable with a physically interpretable mechanism. The transient density field is therefore regarded as an internal state variable responsible for storing and progressively releasing part of the mechanical energy introduced during impact.

One of the motivations for this work originates from observations reported in the impact-mechanics literature showing that off-centre impacts often produce significantly different dynamic responses compared with perfectly centred impacts [27,28]. These differences include tangential restitution, rotational motion, asymmetric stress propagation, and energy redistribution. The present work investigates whether such qualitative behavior can be interpreted within a unified density-relaxation framework.

The objectives of this study are therefore fourfold:

(i) to formulate a thermodynamically consistent continuum model incorporating density relaxation as an internal state variable;

(ii) to derive the governing equations using variational and free-energy principles;

(iii) to investigate the mathematical properties of the resulting formulation, including stability and energy dissipation;

(iv) to perform preliminary numerical simulations and compare the qualitative predictions of the model with observations reported in the existing literature.

The remainder of this article is organized as follows. Section 2 reviews the theoretical background motivating the proposed formulation. Section 3 develops the mathematical model and derives the governing equations. Sections 4 and 5 present the thermodynamic analysis and analytical properties of the formulation. Sections 6 and 7 describe the numerical methodology and simulation results. Section 8 compares the predictions of the proposed framework with published studies in impact mechanics and continuum mechanics. Finally, Section 9 discusses the implications, limitations, and

possible extensions of the proposed density-relaxation theory before concluding the paper.

2. Theoretical Background and Physical Motivation

The development of continuum mechanics has historically relied on the identification of a limited number of macroscopic variables capable of describing the mechanical state of deformable bodies. Since the pioneering works of Cauchy, Navier, and later Hertz, the displacement field has occupied a central position from which strain, stress, momentum, and elastic energy naturally emerge. This framework has progressively evolved into a rigorous mathematical theory through the contributions of Love [24], Timoshenko and Goodier [23], Landau and Lifshitz [1], Truesdell and Noll [2], Gurtin [3], Eringen [4], and many others. Today, continuum mechanics successfully predicts an extraordinary range of physical phenomena including elastic deformation, stress-wave propagation, vibration, fracture, contact mechanics, and impact dynamics.

Despite these remarkable achievements, the mathematical description of solids following an impact remains primarily formulated in terms of displacement, stress, and strain fields. Density generally appears as a quantity constrained by the conservation of mass rather than as an evolving state variable capable of storing and redistributing mechanical energy during transient processes. While this assumption is entirely appropriate for many engineering applications, it leaves open the possibility that additional internal variables may contribute to the short-time dynamics immediately following a localized impact.

The present work originates from this observation. Rather than modifying the classical conservation laws or replacing the well-established constitutive framework of elasticity, the proposed formulation introduces an additional thermodynamic field describing the transient evolution of material density. The underlying hypothesis is intentionally simple: a localized impact generates not only stresses and strains but also a temporary departure from the equilibrium density distribution. This perturbation is assumed to relax progressively while interacting continuously with elastic deformation and pressure gradients until the initial homogeneous configuration is recovered.

The conceptual mechanism proposed in this work is illustrated in Figure 1. Before impact, the spherical body is assumed to possess a homogeneous density field corresponding to thermodynamic equilibrium. Immediately after impact, a localized perturbation appears near the contact region. Rather than disappearing instantaneously, this perturbation propagates through the body in the form of a transient relaxation wave. During its propagation, the density field continuously exchanges energy with the surrounding elastic medium. The progressive attenuation of this perturbation eventually restores the initial equilibrium state while simultaneously redistributing part of the mechanical energy deposited by the impact.

Such an interpretation does not contradict classical elasticity. Instead, it complements it by introducing an additional internal mechanism acting during the transient relaxation process. The displacement field continues to satisfy the classical equations of motion, while the density field evolves according to an independent thermodynamic equation derived from the free-energy functional. The two fields remain coupled through the constitutive equations and therefore evolve simultaneously throughout the entire relaxation process.

The introduction of additional internal variables is by no means unprecedented in continuum mechanics. Thermodynamic theories routinely employ variables describing plastic hardening, viscoelasticity, damage, porosity, phase transformations, or microstructural evolution. Truesdell and

Noll [2] established the general mathematical framework within which such variables may be introduced while preserving objectivity and thermodynamic consistency. Gurtin [3] further demonstrated that irreversible processes can be described naturally through variational principles and free-energy minimization. Germain's Principle of Virtual Power [8] provided another powerful extension by allowing generalized internal interactions to be incorporated within continuum formulations. These developments demonstrate that enriching the set of internal variables represents a natural evolution of continuum mechanics whenever additional physical mechanisms require description.

Another important source of inspiration comes from generalized continuum theories. Eringen's microcontinuum mechanics [4] and Mindlin's gradient elasticity [9,10] demonstrated that additional field variables and higher-order gradients can successfully describe physical phenomena that remain inaccessible to classical elasticity. Similarly, the diffuse-interface formulations introduced by Cahn and Hilliard [6,7] showed that spatial gradients of internal variables may contribute directly to the total free energy of a material while governing smooth relaxation toward equilibrium. Although these theories were originally developed for phase separation and interfacial phenomena, they provide valuable mathematical tools for constructing thermodynamically consistent evolution equations involving spatially varying internal fields.

The present model adopts this variational philosophy without attempting to describe phase transformations. Instead, the density gradient represents a transient mechanical inhomogeneity generated by impact. Its contribution to the Helmholtz free energy penalizes abrupt spatial variations while promoting progressive relaxation toward the homogeneous equilibrium density. The energetic decomposition adopted throughout this work is summarized later in Table 2, whereas the complete mathematical structure of the coupled formulation is illustrated schematically in Figure 3.

The numerical treatment of such coupled field equations naturally relies upon modern computational mechanics. During the past decades, finite-element formulations developed by Hughes [13], Ciarlet [14], Bathe [33], Zienkiewicz et al. [32], and Belytschko et al. [31] have become the standard methodology for solving nonlinear problems involving large deformations, impact, contact, and wave propagation. These computational techniques have demonstrated remarkable predictive capability over an extremely broad range of engineering applications. Rather than competing with these established methods, the present work proposes a new constitutive framework that could eventually be implemented within existing finite-element environments. The preliminary numerical simulations presented later therefore serve primarily as proof-of-concept calculations intended to investigate whether the proposed governing equations generate physically meaningful qualitative behaviour.

The physical motivation for introducing density relaxation becomes particularly apparent when considering experimental studies of oblique impacts. Classical experiments by Maw, Barber, and Fawcett [27] revealed that off-centre impacts produce significantly more complex responses than purely normal collisions. Tangential restitution, frictional interactions, and rotational motion emerge naturally as the impact geometry departs from perfect symmetry. More recent numerical investigations of elastoplastic impacts by Wu et al. [28] confirmed that impact angle, material properties, and internal deformation strongly influence the redistribution of kinetic energy. These studies clearly demonstrate that impact dynamics cannot always be interpreted solely in terms of local contact forces, but involve complex interactions between stress waves, internal deformation, and transient energy redistribution.

The present work does not claim that density relaxation replaces these established mechanisms.

Rather, it investigates whether the transient evolution of the density field may constitute an additional component of the overall energy redistribution process. If such a mechanism exists, it should remain fully compatible with classical conservation laws while providing a complementary physical interpretation of transient impact phenomena. This hypothesis forms the central motivation of the mathematical development presented in the following sections.

The theoretical framework introduced here therefore occupies an intermediate position between classical elasticity and generalized thermodynamic field theories. It preserves the fundamental balance laws of continuum mechanics while extending the constitutive description through an additional relaxation field governed by variational principles. The resulting formulation seeks to establish a mathematically consistent, physically interpretable, and computationally tractable framework capable of describing transient density evolution in impacted solids. The complete set of governing equations derived from this theoretical basis is presented in the following section, together with the corresponding constitutive assumptions and thermodynamic analysis.

3. Mathematical Formulation of the Density-Relaxation Continuum Model

The objective of the present section is to establish a thermodynamically consistent mathematical framework capable of describing the transient evolution of density following mechanical impact while remaining fully compatible with the fundamental principles of continuum mechanics. Rather than replacing the classical equations of elasticity, the proposed formulation augments them through the introduction of an additional internal field representing transient density relaxation. The resulting model preserves the balance laws of mass and momentum while extending the constitutive description through a variational free-energy formulation.

The overall mathematical architecture of the proposed theory is illustrated schematically in Figure 3, whereas the principal variables and constitutive parameters employed throughout the formulation are summarized in Table 1. The governing equations together with the corresponding boundary and initial conditions are collected in Table 3 for ease of reference.

3.1 Kinematics

Let Ω denote the reference configuration of a deformable body occupying a three-dimensional domain with boundary Γ. The motion of every material point is described by the displacement vector

$$u = u(x,t),$$

where x denotes the material coordinates and t is time.

Assuming infinitesimal deformations, the strain tensor is defined by

$$\varepsilon = 1/2\ (\nabla u + \nabla u^{T}),$$

which represents the symmetric part of the displacement gradient and remains identical to that employed in classical linear elasticity [1,23,24].

Unlike conventional continuum formulations, the present model introduces an additional scalar field

$$\rho = \rho(x,t),$$

representing the transient material density.

The reference configuration is characterized by a homogeneous equilibrium density

$$\rho = \rho_0.$$

Immediately following impact, localized departures from this equilibrium value are assumed to occur. The subsequent evolution of these perturbations constitutes the central subject of the present investigation. The physical interpretation of this process is illustrated in Figure 1.

3.2 Governing Balance Laws

The proposed formulation preserves the classical conservation principles. Mass conservation is expressed as $\partial\rho/\partial t + \nabla\cdot(\rho v) = 0$,
where v denotes the material velocity.
The balance of linear momentum is written as
$\rho Dv/Dt = \nabla\cdot\sigma + b$,
where σ denotes the Cauchy stress tensor and b represents body forces.
These equations remain unchanged with respect to conventional continuum mechanics [2,3].
The novelty of the present work therefore does not lie in the balance equations themselves, but rather in the constitutive description governing the evolution of the density field.
3.3 Constitutive Assumptions
Classical elasticity relates stresses exclusively to deformation.
In contrast, the present model assumes that the local thermodynamic state depends simultaneously upon
• elastic strain,
• material density,
• spatial density gradients.
Consequently,
$\sigma = \sigma(\varepsilon,\rho,\nabla\rho)$.
This assumption follows the philosophy adopted in generalized continuum theories developed by Eringen [4], Mindlin [9,10], and gradient-energy formulations introduced by Cahn and Hilliard [6,7].
The density field therefore acts as an additional internal variable whose evolution contributes to the transient redistribution of mechanical energy following impact.
3.4 Helmholtz Free-Energy Functional
The entire mathematical formulation is constructed from a Helmholtz free-energy functional
$F = \int\Omega\ \Psi\ d\Omega$,
where Ψ denotes the local free-energy density.
Following the thermodynamic structure proposed by Gurtin [3] and inspired by gradient theories [6,7], the local energy density is decomposed into three contributions,
$\Psi = \Psi elastic + \Psi density + \Psi gradient$.
The elastic contribution represents conventional strain energy.
The density contribution penalizes departures from the reference equilibrium density.
The gradient contribution penalizes sharp spatial density variations.
The energetic interpretation of these three terms is summarized in Table 2.
The elastic contribution is written
$\Psi elastic = 1/2\ \varepsilon:C:\varepsilon$,
where C denotes the classical elasticity tensor.
The density contribution becomes
$\Psi density = \beta/2\ (\rho-\rho_0)^2$,
where β represents the density penalty coefficient controlling the energetic cost of local density perturbations.
Finally, the gradient contribution is introduced as
$\Psi gradient = \kappa/2\ |\nabla\rho|^2$,
where κ denotes the density-gradient coefficient.
This last contribution provides the internal regularization responsible for smooth relaxation of localized perturbations, similarly to diffuse-interface formulations [6,7].
3.5 Variational Principle
The evolution equations are obtained by minimizing the total free energy while satisfying the classical balance laws. The first variation of the free-energy functional yields
$\delta F = 0$ at equilibrium. Away from equilibrium, irreversible evolution proceeds along the direction of maximum free-energy decrease.

This variational structure guarantees thermodynamic admissibility and naturally introduces density relaxation as a dissipative process.
Unlike purely phenomenological constitutive models, every term introduced in the present formulation possesses a direct energetic interpretation.
3.6 Density Relaxation Equation
Assuming dissipative gradient-flow dynamics, the density field evolves according to
$\partial\rho/\partial t = M\ \nabla^2(\delta F/\delta\rho)$,
where M denotes the density mobility coefficient listed in Table 1.
Substituting the proposed Helmholtz functional leads to
$\partial\rho/\partial t = M\ [\beta\nabla^2\rho - \kappa\nabla^4\rho]$.
This equation describes the progressive attenuation of localized density perturbations generated by impact.
The corresponding physical sequence is illustrated conceptually in Figure 2.
Unlike diffusion equations describing mass transport, the present equation represents thermodynamic relaxation toward the homogeneous equilibrium density.
3.7 Coupling Between Elasticity and Density
The proposed continuum model is therefore governed by two coupled fields.
The displacement field determines stresses through classical elasticity.
The density field evolves through free-energy minimization.
The two fields remain coupled because elastic deformation modifies the free-energy landscape, while density relaxation influences the internal thermodynamic state.
Figure 3 summarizes these interactions within the complete mathematical framework.
3.8 Initial and Boundary Conditions
Immediately before impact,
$\rho(x,0)=\rho_0$
throughout the body.
The impact introduces a localized perturbation represented by
$\rho(x,0)=\rho_0+\Delta\rho(x)$.
Mechanical boundary conditions follow the standard traction or displacement prescriptions employed in continuum mechanics [13,14].
For the density field, homogeneous Neumann boundary conditions are adopted,
$\nabla\rho\cdot n=0$,
ensuring that no artificial density flux crosses the external surface.
These conditions are summarized in Table 3.
3.9 Physical Interpretation
The mathematical formulation developed above preserves the structure of classical continuum mechanics while extending it through an additional thermodynamic variable describing transient density relaxation.
The model does not alter the established conservation laws.
Instead, it proposes that part of the mechanical energy deposited during impact is temporarily stored within a non-equilibrium density field before being progressively dissipated through thermodynamically admissible relaxation mechanisms.
This interpretation forms the basis for the analytical stability analysis developed in the following section, where the existence of a Lyapunov functional and monotonic free-energy dissipation will be demonstrated.

4. Thermodynamic Analysis and Mathematical Properties

The mathematical formulation introduced in the previous section was intentionally constructed within the framework of continuum thermodynamics. Consequently, the proposed density-relaxation model must satisfy not only the classical balance equations but also the fundamental thermodynamic principles governing irreversible processes. The objective of this section is therefore to investigate the analytical properties of the formulation and to demonstrate that the proposed evolution law is consistent with free-energy minimization, irreversible relaxation, and asymptotic convergence toward equilibrium.

The principal mathematical properties discussed throughout this section are summarized in Table 4.

4.1 Thermodynamic Consistency

A physically meaningful continuum model must satisfy the First and Second Laws of Thermodynamics. In the present formulation, the Helmholtz free-energy functional introduced in Section 3 constitutes the central thermodynamic potential governing the evolution of the density field.

The total free energy of the body is written as

$$F = \int_{\Omega} \Psi \, d\Omega$$

where Ψ contains the elastic contribution, the density contribution, and the density-gradient contribution previously introduced in Table 2.

Immediately after impact, the body is assumed to occupy a non-equilibrium state characterized by localized density perturbations. This configuration possesses a higher free energy than the homogeneous equilibrium state.

Consequently, the evolution of the density field is entirely governed by the tendency of the system to decrease its total free energy.

This relaxation mechanism is illustrated conceptually in Figure 4.

Unlike purely elastic deformation, which is reversible, density relaxation represents an irreversible thermodynamic process whose only stable destination is the homogeneous equilibrium configuration.

4.2 Free-Energy Dissipation

An essential property of the proposed model is that the total free energy decreases continuously during relaxation. As the density perturbation progressively disappears, the density penalty term decreases while the gradient contribution simultaneously becomes smaller because spatial density variations vanish. Therefore,

$$dF/dt \leq 0$$

throughout the entire relaxation process.

Equality is reached only after complete relaxation has been achieved.

This monotonic decrease of free energy provides the thermodynamic driving force responsible for the progressive attenuation of the transient density field.

The qualitative evolution of the free-energy landscape is illustrated in Figure 4.

The corresponding energetic contributions are summarized in Table 2.

4.3 Equilibrium State

The equilibrium configuration corresponds to the minimum of the Helmholtz free-energy functional.

At equilibrium,

$\rho = \rho_0$

throughout the entire body.

Under these conditions,

$\nabla\rho = 0$

and consequently the density-gradient contribution completely disappears.

The constitutive equations therefore reduce naturally to those of classical linear elasticity.

This result is particularly important because it demonstrates that the present formulation does not modify classical elasticity in the absence of transient density perturbations.

Instead, classical elasticity appears automatically as the limiting case of the proposed theory.

4.4 Lyapunov Stability

One of the strongest mathematical properties of the proposed formulation is the existence of a natural Lyapunov functional.

The Helmholtz free energy itself acts as the Lyapunov function.

Since

$dF/dt \leq 0$

during the entire evolution,

the free energy decreases continuously and can never increase spontaneously.

Consequently, every admissible solution evolves toward progressively lower-energy configurations

until the equilibrium density field is recovered.

This behaviour establishes the asymptotic stability of the equilibrium solution.

The analytical properties of the model are summarized in Table 4.

4.5 Stability of Small Perturbations

To investigate the behaviour of infinitesimal perturbations, consider a small deviation around the homogeneous equilibrium density,

$\rho = \rho_0 + \delta\rho,$

where $\delta\rho$ remains very small.

Linearization of the governing equation shows that the perturbation amplitude decreases continuously with time.

Short-wavelength perturbations disappear rapidly because of the density-gradient contribution, whereas long-wavelength perturbations relax more slowly.

Consequently, every admissible perturbation progressively vanishes.

The equilibrium solution therefore remains linearly stable.

This behaviour is entirely consistent with generalized continuum theories developed by Eringen [4], Mindlin [9,10], and gradient formulations introduced by Cahn and Hilliard [6,7].

4.6 Influence of Model Parameters

The proposed formulation introduces three principal constitutive coefficients.

The mobility coefficient M controls the relaxation rate.

Large values of M accelerate the recovery toward equilibrium.

Small values produce slower relaxation.

The density penalty coefficient β determines the energetic cost associated with density perturbations.

Increasing β increases the thermodynamic driving force responsible for restoring the homogeneous density.

The gradient coefficient κ controls the smoothness of the evolving density field.

Large values suppress sharp spatial oscillations and produce smoother relaxation waves.

The qualitative influence of these parameters is illustrated in Figure 9.

The numerical values adopted throughout the simulations are listed in Table 1.

4.7 Recovery of Classical Elasticity

An important verification consists of examining the limiting behaviour of the model.

If the density perturbation remains negligible,

$\rho \approx \rho_0$,

all density-dependent contributions vanish.

The free-energy functional then reduces to the classical elastic strain energy.

Likewise, if the mobility coefficient tends toward zero,

$M \rightarrow 0$,

density evolution ceases entirely.

The governing equations therefore become identical to those of conventional continuum mechanics.

This limiting behaviour confirms that the proposed theory extends classical elasticity rather than replacing it.

4.8 Physical Interpretation

The mathematical analysis developed above demonstrates that the proposed formulation satisfies several desirable analytical properties.

First, the governing equations preserve the classical conservation principles.

Second, the total free energy decreases monotonically throughout the relaxation process.

Third, the homogeneous equilibrium state constitutes the unique stable minimum of the free-energy functional.

Finally, classical elasticity is recovered naturally when transient density effects disappear.

These properties collectively provide a mathematically consistent framework for investigating transient density evolution following impact.

The next section uses this formulation to construct the computational methodology employed in the preliminary numerical simulations. The complete numerical workflow is summarized in Figure 5 before presenting the simulation results in the subsequent sections.

5. Numerical Methodology

The objective of the numerical investigation is not to reproduce a complete industrial finite-element simulation, but rather to evaluate whether the governing equations proposed in the previous sections generate physically meaningful behaviour consistent with continuum mechanics and thermodynamic relaxation. The numerical study therefore serves as a proof of concept designed to investigate the qualitative predictions of the proposed density-relaxation model before implementation in commercial finite-element environments.

The complete computational workflow adopted throughout this work is illustrated in **Figure 5**. The methodology follows the theoretical formulation developed in Section 3 while preserving the thermodynamic properties demonstrated in Section 4.

The numerical simulations were performed using custom computational models developed specifically for the present investigation. Two-dimensional simulations were first employed to analyse the evolution of localized density perturbations following impact. These preliminary calculations allowed rapid evaluation of parameter sensitivity, free-energy dissipation, and density-field evolution. Subsequently, three-dimensional simulations were performed to investigate whether the principal qualitative behaviours observed in two dimensions remain valid within a fully three-dimensional spherical geometry.

The computational domain consists of a deformable sphere initially occupying a homogeneous equilibrium state characterized by the reference density ρ_0. Before impact, the density field is assumed spatially uniform throughout the body. At time $t = 0$, a localized perturbation is introduced near the impact point in order to represent the transient compression produced by the external impulse. The subsequent evolution of this perturbation is governed by the coupled continuum equations introduced in Section 3.

The principal constitutive and numerical parameters employed throughout the simulations are summarized in **Table 1**. These parameters include the density mobility coefficient, the density penalty coefficient, the gradient coefficient, the damping coefficient, the reference density, and the numerical integration constants. The values were selected to ensure numerical stability while preserving the qualitative behaviour predicted by the theoretical model. Their purpose is not to reproduce the behaviour of a specific engineering material, but rather to investigate the general characteristics of density relaxation under mechanically admissible conditions.

Spatial discretization was performed on regular computational grids sufficiently refined to capture the evolution of localized density gradients. Temporal integration employed explicit time stepping combined with stability criteria ensuring smooth propagation of the relaxation process. Convergence was monitored continuously by evaluating the evolution of the total free energy together with the maximum density variation between successive time steps.

Special attention was devoted to the treatment of boundary conditions. Mechanical boundary conditions follow the classical assumptions of continuum mechanics described previously in **Table 3**. For the density field, homogeneous Neumann boundary conditions were adopted in order to prevent artificial density fluxes through the external surface of the sphere. This choice guarantees that density redistribution remains entirely generated by the internal relaxation mechanism proposed in the present model.

The computational procedure begins with initialization of the material properties and generation of the initial equilibrium density field. A localized impact perturbation is then introduced within a small region located near the external surface of the sphere. The coupled displacement and density equations are subsequently integrated simultaneously while updating the free-energy functional after each time increment. The algorithm continues until the density field approaches a stationary configuration corresponding to thermodynamic equilibrium. This computational sequence is

summarized schematically in **Figure 8**.

To evaluate the physical behaviour of the model, several diagnostic quantities were monitored throughout the simulations. These include the spatial distribution of the density field, the temporal evolution of the total free energy, the magnitude of density gradients, the displacement of the transient density deficit, and the evolution of a rotational source proxy associated with the misalignment between density and pressure gradients. Although this rotational proxy is not identical to the angular momentum of the body, it provides a useful indicator for identifying the emergence of transient rotational tendencies within the evolving continuum.

Parameter sensitivity studies were also conducted in order to evaluate the influence of the constitutive coefficients introduced in the theoretical formulation. Particular attention was given to the mobility coefficient M, the density penalty coefficient β, and the gradient coefficient κ. Their qualitative influence on the relaxation dynamics is illustrated in **Figure 9**. As expected from the thermodynamic formulation developed in Section 4, increasing the mobility coefficient accelerates the recovery toward equilibrium, whereas larger values of the gradient coefficient produce smoother spatial density distributions by suppressing localized oscillations.

Following the preliminary two-dimensional investigation, the numerical analysis was extended to three-dimensional spherical bodies. These simulations employed several impact eccentricities ranging from perfectly centred impacts to strongly off-centre collisions. The objective was to determine whether the proposed density-relaxation mechanism predicts qualitative differences between centred and eccentric impacts comparable to those reported in experimental and numerical studies of oblique impacts [27,28].

The three-dimensional simulations further incorporated convergence tests performed using different spatial resolutions. The purpose of these calculations was to verify that the principal qualitative predictions of the model remain stable as the computational mesh is refined. The convergence analysis demonstrated that although the numerical values of certain diagnostic quantities vary slightly with grid resolution, the principal trends—including monotonic free-energy dissipation, progressive attenuation of density perturbations, and the dependence of the rotational proxy on impact eccentricity—remain unchanged. These observations increase confidence that the reported behaviour arises from the constitutive model itself rather than from numerical artifacts.

The principal numerical observations obtained throughout the computational investigation are summarized in **Table 6**. Overall, the simulations indicate stable numerical behaviour, monotonic convergence toward equilibrium, and physically plausible evolution of the density field throughout the relaxation process. Most importantly, the calculations predict that off-centre impacts generate significantly stronger transient density redistribution than perfectly centred impacts, providing the first qualitative indication that density relaxation may contribute to the internal redistribution of mechanical energy following impact.

The following section presents the numerical results obtained from both the two-dimensional and three-dimensional simulations and compares the predicted behaviour with observations reported in the existing literature on impact mechanics, stress-wave propagation, and continuum thermodynamics.

6. Numerical Results and Physical Interpretation

The numerical simulations presented in this section constitute the first computational evaluation of the density-relaxation continuum model proposed in this work. Their objective is not to reproduce a particular engineering experiment quantitatively, but rather to investigate whether the governing

equations derived in Section 3 generate physically meaningful behaviour while satisfying the thermodynamic properties demonstrated in Section 4. Consequently, the numerical results should be interpreted as proof-of-concept simulations illustrating the internal consistency of the proposed formulation.

The overall computational procedure described in **Figure 8** was applied systematically to both two-dimensional and three-dimensional configurations. Representative numerical observations are summarized in **Table 6**, while the complete theoretical framework underlying the simulations is recalled in **Figure 11**.

Immediately after impact, the simulations predict the appearance of a localized density perturbation concentrated around the contact region. This behaviour corresponds directly to the physical mechanism introduced in **Figure 1**, where the impact produces a temporary departure from the homogeneous equilibrium density. As time progresses, this perturbation no longer remains confined to its initial location. Instead, it propagates through the body as a smooth relaxation wave whose amplitude progressively decreases while the surrounding material returns toward its equilibrium configuration.

The spatial evolution of the density field is illustrated in **Figure 10**. During the earliest stages of the computation, strong density gradients remain concentrated near the impact location. As relaxation proceeds, these gradients become progressively smoother owing to the contribution of the density-gradient energy introduced in the Helmholtz free-energy functional. Eventually, the density field approaches a nearly homogeneous distribution, indicating that the system has reached a new thermodynamic equilibrium characterized by minimum free energy.

One of the most significant numerical observations concerns the evolution of the total free energy throughout the relaxation process. For every simulation performed, the free-energy functional decreased monotonically with time until equilibrium was reached. No spontaneous increase of free energy was observed during any stage of the calculations. This behaviour agrees with the theoretical predictions established in Section 4 and confirms that the proposed evolution equation satisfies the expected thermodynamic behaviour. The numerical simulations therefore provide computational support for the existence of a Lyapunov functional governing the irreversible relaxation process.

The simulations further demonstrate that the rate of relaxation depends strongly upon the constitutive parameters introduced in the model. Increasing the mobility coefficient M accelerates the attenuation of density perturbations, allowing equilibrium to be recovered more rapidly. Conversely, smaller values of M produce slower relaxation while preserving the same qualitative evolution. Similar behaviour is observed for the density-gradient coefficient κ. Larger values of κ reduce sharp spatial oscillations and generate smoother relaxation waves, whereas smaller values allow stronger local density gradients to develop during the early stages of impact. These parameter effects are summarized qualitatively in **Figure 9** and correspond to the constitutive assumptions discussed previously.

The most interesting results emerge when comparing centred and off-centre impacts. In simulations involving perfectly centred impacts, the density field remains nearly symmetric with respect to the impact axis. The resulting pressure gradients remain approximately aligned with the density gradients throughout the evolution. Consequently, the computed rotational source proxy remains relatively small during the entire relaxation process.

A markedly different behaviour is observed for eccentric impacts. As the impact point moves away from the geometric centre, the density perturbation evolves asymmetrically. The corresponding pressure field no longer remains perfectly aligned with the density gradients, producing temporary regions where the two fields become misaligned. Within the present theoretical framework, this misalignment generates a significantly larger rotational source proxy than that observed for centred impacts.

The dependence of this rotational indicator upon impact eccentricity constitutes one of the principal numerical results of the present investigation. The simulations indicate that the rotational proxy increases progressively as the impact becomes eccentric before reaching a maximum at intermediate eccentricities. For extremely tangential impacts, the rotational proxy decreases again. This behaviour suggests the existence of an optimal impact geometry maximizing transient internal redistribution.

It is important to emphasize that this rotational source proxy should not be interpreted as a direct calculation of the angular momentum of the body. Rather, it represents a diagnostic quantity designed to evaluate the tendency of the evolving density-pressure field to generate transient rotational behaviour. Demonstrating that this quantity correlates quantitatively with experimentally measurable angular momentum remains an objective for future investigations involving high-fidelity finite-element simulations and experimental measurements.

The numerical results also reveal the progressive displacement of the transient density deficit throughout the relaxation process. Rather than remaining stationary near the impact location, the region of reduced density propagates through the interior of the body before gradually disappearing as equilibrium is restored. This observation is illustrated qualitatively in **Figure 10** and forms one of the central predictions of the proposed model. Although direct experimental observation of such transient density evolution remains challenging, modern imaging techniques combined with advanced numerical simulations may eventually permit experimental verification of this phenomenon.

Mesh refinement studies were performed to evaluate the robustness of the numerical predictions. Simulations carried out with increasing spatial resolution produced nearly identical qualitative behaviour. The monotonic decay of free energy, attenuation of density perturbations, dependence upon impact eccentricity, and existence of an intermediate maximum of the rotational proxy remained unchanged. These convergence tests increase confidence that the principal observations reported here are associated with the constitutive formulation rather than with numerical discretization errors.

To place these results within the broader context of continuum mechanics, comparisons were made with published investigations of impact dynamics. Experimental studies by Maw, Barber, and Fawcett [27] demonstrated that off-centre impacts generate significantly different rotational behaviour than centred impacts. Numerical analyses by Wu et al. [28] likewise showed a strong dependence of impact response upon eccentricity and material properties. Although the present model does not attempt to reproduce these studies quantitatively, it predicts the same qualitative distinction between centred and eccentric impacts through an entirely different internal mechanism based on transient density relaxation.

Similarly, the smooth propagation and attenuation of the density field predicted by the present model are compatible with the general behaviour of stress-wave propagation observed in finite-element investigations of elastic solids [13,29,31–33]. The numerical simulations therefore suggest that introducing density relaxation as an internal thermodynamic variable does not contradict established continuum mechanics. Instead, it provides an additional interpretative framework capable of describing the transient redistribution of mechanical energy following impact.

Overall, the computational investigation demonstrates that the proposed density-relaxation continuum model satisfies the principal physical expectations established in the preceding theoretical sections. The simulations exhibit stable numerical behaviour, monotonic free-energy dissipation, asymptotic convergence toward equilibrium, and physically plausible transient evolution of the density field. Most importantly, they indicate that introducing density as an evolving internal variable naturally produces qualitative differences between centred and eccentric impacts, thereby motivating a more detailed comparison with the existing experimental and

theoretical literature.

The following section examines this comparison in detail by confronting the predictions of the proposed model with published results from continuum mechanics, impact engineering, finite-element analysis, and experimental investigations of oblique impacts. This bibliographic validation represents an essential step in assessing both the strengths and the current limitations of the proposed density-relaxation framework.

7. Bibliographic Validation and Comparison with Existing Theories

The objective of this section is to evaluate the proposed density-relaxation continuum model in light of the existing scientific literature. Since the present formulation introduces density as an additional thermodynamic state variable, its validity cannot be assessed solely through internal mathematical consistency or preliminary numerical simulations. Instead, its predictions must be compared with experimentally observed phenomena and with the theoretical results established by classical continuum mechanics, contact mechanics, impact engineering, and computational solid mechanics.

It is important to emphasize that the present work does not claim to replace the classical theories of elasticity. Rather, it proposes an additional constitutive mechanism that may contribute to the transient redistribution of mechanical energy immediately following impact. The purpose of this bibliographic analysis is therefore to determine whether the qualitative predictions of the proposed model are compatible with observations already reported in the literature.

One of the most fundamental predictions of the present formulation concerns the distinction between centred and off-centre impacts. The numerical simulations presented in Section 6 consistently indicate that perfectly centred impacts produce nearly symmetric density distributions, whereas eccentric impacts generate asymmetric density fields associated with stronger transient rotational indicators. This prediction agrees qualitatively with the experimental investigations of Maw, Barber, and Fawcett [27], who demonstrated that oblique impacts produce substantially different dynamic responses from normal impacts. Their experiments showed that tangential restitution, frictional effects, and rotational motion depend strongly upon impact geometry. Although the present model does not explicitly calculate restitution coefficients, it predicts the same qualitative distinction through an internal density-relaxation mechanism.

Similarly, the numerical studies of Wu et al. [28] demonstrated that elastoplastic oblique impacts exhibit a strong dependence upon impact angle, material properties, and deformation history. The proposed density-relaxation model likewise predicts that transient density redistribution depends continuously upon impact eccentricity. In both cases, centred impacts produce comparatively simple responses, whereas eccentric impacts generate more complex internal evolution. The present interpretation differs in the proposed physical mechanism, but the qualitative behaviour remains consistent.

The monotonic decrease of the Helmholtz free energy obtained in the present model is fully compatible with the thermodynamic framework established by Truesdell and Noll [2], Gurtin [3], and Germain [8]. Throughout the numerical simulations, the free-energy functional decreases continuously until equilibrium is recovered. This behaviour confirms that the proposed evolution equation satisfies the expected direction of irreversible thermodynamic processes and remains consistent with the Second Law of Thermodynamics. In this respect, the model extends rather than contradicts existing continuum thermodynamics.

Another important aspect concerns the use of gradient-dependent free-energy contributions. The introduction of the density-gradient term is inspired mathematically by the diffuse-interface

formulations of Cahn and Hilliard [6,7] and by generalized continuum theories developed by Mindlin [9,10] and Eringen [4]. These theories demonstrated that higher-order spatial gradients may successfully describe localized internal phenomena while preserving thermodynamic consistency. The present work adopts a similar mathematical strategy but applies it to transient density evolution following mechanical impact rather than to phase separation or generalized elasticity. Consequently, the mathematical structure is well established, although the physical interpretation is different.

The numerical simulations further predict that localized density perturbations propagate smoothly through the body while progressively attenuating. This behaviour resembles the smooth propagation of stress waves commonly observed in finite-element simulations of elastic solids [13,29,31–33]. Although conventional finite-element formulations generally solve for displacement and stress rather than density evolution, both approaches predict progressive attenuation of localized disturbances accompanied by redistribution of mechanical energy throughout the body.

The convergence properties observed in the numerical simulations also agree with standard computational mechanics. Mesh refinement produced nearly identical qualitative behaviour while preserving monotonic energy dissipation and stable convergence toward equilibrium. Such behaviour is consistent with the numerical stability expected from finite-element formulations described by Hughes [13], Ciarlet [14], Bathe [33], and Zienkiewicz et al. [32]. Although the present computations remain preliminary, they indicate that the proposed constitutive equations are numerically well behaved and suitable for future implementation within industrial finite-element software.

One of the principal novelties of the proposed formulation lies in its interpretation of transient density redistribution as an internal thermodynamic mechanism. Classical elasticity explains impact primarily through stress propagation and deformation. Contact mechanics describes the transfer of forces at the impact interface, whereas generalized continuum theories incorporate additional kinematic variables. The present work introduces density relaxation as an additional field capable of storing and progressively releasing part of the mechanical energy introduced by impact. This interpretation is not currently part of conventional continuum mechanics and therefore represents the principal conceptual contribution of this study.

The comparison with the existing literature nevertheless reveals several important limitations. First, the present model has not yet been calibrated against quantitative experimental measurements of density evolution. Second, the numerical simulations were intentionally designed as proof-of-concept calculations rather than high-fidelity engineering simulations. Consequently, the model does not yet predict experimentally measurable quantities such as coefficients of restitution, contact duration, rebound velocity, or angular velocity with quantitative accuracy. These quantities remain important objectives for future investigations.

A second limitation concerns the constitutive assumptions themselves. The density-relaxation equation proposed in Section 3 should presently be regarded as a constitutive hypothesis rather than a uniquely derived consequence of continuum mechanics. Although it satisfies thermodynamic consistency and generates physically plausible behaviour, additional theoretical work will be required to derive this evolution law from microscopic physical principles or to identify its relationship with existing multiscale descriptions of condensed matter.

Despite these limitations, the present bibliographic comparison demonstrates that the qualitative predictions of the proposed model remain compatible with a remarkably broad range of published observations. The distinction between centred and eccentric impacts, the progressive redistribution of internal energy, the attenuation of localized perturbations, the role of thermodynamic relaxation, and the stability of the equilibrium state are all consistent with well-established results reported in elasticity, impact mechanics, continuum thermodynamics, and computational solid mechanics [1–33].

Taken together, the theoretical formulation, analytical properties, numerical simulations, and bibliographic comparisons suggest that density relaxation deserves further investigation as a possible additional constitutive mechanism in continuum mechanics. The present study therefore should be viewed as a first step toward the development of a more comprehensive continuum description in which transient density evolution complements the classical displacement and stress fields rather than replacing them.

The following section discusses the broader implications of this interpretation, examines the principal strengths and limitations of the proposed framework, and outlines the most promising directions for future theoretical, numerical, and experimental investigations.

8. General Discussion

The formulation proposed in the present work was motivated by a simple but fundamental question: can the transient evolution of material density contribute to the redistribution of mechanical energy following impact? The theoretical development, thermodynamic analysis, numerical simulations, and bibliographic comparison presented in the preceding sections collectively suggest that introducing density as an evolving internal variable provides a mathematically consistent and physically plausible extension of classical continuum mechanics.

One of the principal strengths of the proposed framework is that it preserves the foundations of continuum mechanics. The conservation of mass, the balance of linear momentum, and the constitutive description of elasticity remain unchanged. Consequently, the proposed formulation should not be interpreted as an alternative theory replacing classical elasticity, but rather as an extension in which density relaxation constitutes an additional constitutive process occurring during transient impact phenomena. This compatibility with established mechanics considerably facilitates possible future implementation within existing finite-element formulations.

Another important feature of the model concerns its thermodynamic consistency. The free-energy functional introduced in Section 3 naturally generates irreversible evolution toward equilibrium while satisfying the monotonic dissipation properties demonstrated analytically in Section 4 and verified numerically in Section 6. The existence of a Lyapunov functional ensures that the model remains stable throughout the entire relaxation process. Such behaviour is essential for any continuum formulation intended to describe irreversible physical processes.

The numerical simulations indicate that the proposed constitutive model produces physically meaningful qualitative behaviour. In particular, the distinction between centred and eccentric impacts emerges naturally without requiring additional empirical assumptions. The transient redistribution of density predicted by the simulations generates asymmetric internal states during off-centre impacts while preserving nearly symmetric evolution for centred impacts. This qualitative behaviour agrees with the experimental tendencies reported in the impact mechanics literature and therefore provides encouraging preliminary support for the proposed interpretation.

The introduction of density as an internal state variable also offers a different physical perspective on impact dynamics. Classical elasticity primarily attributes the redistribution of energy to stress propagation and deformation. The present formulation suggests that transient density evolution may participate in this redistribution by temporarily storing part of the mechanical energy introduced during impact before progressively releasing it as equilibrium is restored. This interpretation does not contradict the classical description but enriches it by introducing an additional thermodynamic degree of freedom.

An important consequence of the proposed framework is its potential ability to unify several transient phenomena within a single constitutive description. Localized density perturbations, pressure redistribution, stress-wave propagation, free-energy dissipation, and transient rotational

tendencies all arise naturally from the coupled evolution of the displacement and density fields. Although these phenomena are well known individually, they are generally described using different theoretical approaches. The present work suggests that they may represent different manifestations of a common relaxation process governed by the free-energy landscape of the material.

Nevertheless, several important limitations must be recognized. First, the present numerical simulations remain qualitative. They demonstrate the internal consistency of the governing equations but do not yet provide quantitative predictions directly comparable with experimental measurements. Future investigations should therefore calibrate the constitutive parameters using controlled laboratory experiments on well-characterized materials.

Second, the constitutive equation governing density relaxation remains phenomenological. Although it satisfies the requirements of continuum thermodynamics and produces stable numerical behaviour, its microscopic origin has not yet been established. Future research should investigate whether similar evolution laws can be derived from atomistic simulations, statistical mechanics, molecular dynamics, or multiscale homogenization methods. Such derivations would considerably strengthen the physical foundations of the model.

A further limitation concerns the constitutive parameters themselves. The mobility coefficient, density penalty coefficient, and density-gradient coefficient were selected primarily to investigate qualitative behaviour rather than to represent specific engineering materials. Future studies should determine how these parameters depend upon material properties such as elastic modulus, compressibility, viscosity, crystal structure, temperature, porosity, and strain rate.

The present work also neglects several physical phenomena that may become important in practical applications. Plastic deformation, fracture, damage evolution, viscoelastic effects, anisotropy, thermal coupling, and large deformations have intentionally been omitted in order to isolate the influence of density relaxation. Incorporating these mechanisms represents an important direction for future research and will require more sophisticated constitutive models together with advanced numerical implementations.

One particularly promising avenue concerns the implementation of the proposed formulation within commercial finite-element software. Since the governing equations preserve the classical balance laws while introducing only one additional scalar field, incorporation into existing multiphysics finite-element environments appears feasible. Such implementations would permit quantitative comparison with experimental measurements, impact tests, and industrial applications involving metallic, polymeric, ceramic, or composite materials.

Experimental validation represents another major objective. Modern full-field optical techniques, digital image correlation, laser Doppler vibrometry, X-ray phase-contrast imaging, and ultrafast tomography may eventually provide indirect observations of transient density redistribution following impact. Even if direct measurement of density evolution proves difficult, comparison between predicted and measured displacement fields, stress-wave propagation, or rotational response could provide valuable evidence supporting or refuting the proposed mechanism.

Beyond impact mechanics, the concept of density relaxation may also prove useful in other branches of continuum physics. Rapid loading, blast waves, ultrasonic excitation, high-speed machining, granular media, impact protection systems, biological tissues, and geophysical materials all involve transient internal redistribution of mechanical energy. The mathematical framework proposed here could therefore serve as the basis for future investigations extending beyond the spherical impact configuration considered in the present work.

The bibliographic comparison presented in Section 7 further indicates that the proposed model remains compatible with a broad range of established theoretical and experimental observations.

Rather than contradicting classical elasticity, generalized continuum theories, or computational solid mechanics, the present formulation complements these approaches by providing an alternative interpretation of transient energy redistribution immediately following impact. This compatibility constitutes one of the principal strengths of the proposed theory and suggests that density relaxation deserves further investigation within the broader context of continuum mechanics.

Overall, the present study should be regarded as an initial theoretical and computational investigation introducing a new constitutive concept rather than as a definitive description of impact dynamics. Its principal contribution lies in demonstrating that density relaxation can be incorporated into continuum mechanics through a thermodynamically consistent formulation capable of producing physically meaningful qualitative predictions. Whether this mechanism represents an essential component of real material behaviour remains a question that must ultimately be answered through rigorous experimental measurements and high-fidelity numerical simulations.

The final section summarizes the principal conclusions of this investigation and outlines the future developments required to establish the proposed density-relaxation framework as a predictive constitutive theory for dynamic solid mechanics.

9. Conclusions and Future Perspectives

The present study introduced a new continuum-mechanics framework in which density is treated as an evolving thermodynamic state variable during the transient response of a solid body subjected to mechanical impact. Unlike conventional formulations, where density is generally constrained by mass conservation alone, the proposed model assumes that localized impacts generate temporary density perturbations that evolve through irreversible relaxation while interacting with the elastic displacement field and the free-energy landscape of the material.

The theoretical formulation was constructed entirely within the principles of continuum mechanics and thermodynamics. The governing equations preserve the classical balance laws while extending the constitutive description through a Helmholtz free-energy functional incorporating elastic strain energy, density-deviation energy, and density-gradient energy. This variational formulation naturally leads to an irreversible density evolution governed by free-energy minimization and remains compatible with the Second Law of Thermodynamics.

The analytical investigation demonstrated several important mathematical properties of the proposed formulation. The total free energy decreases monotonically throughout the relaxation process, the homogeneous density distribution represents the unique equilibrium configuration, and the Helmholtz free-energy functional serves as a Lyapunov function ensuring asymptotic stability. Furthermore, the proposed framework recovers classical elasticity as a limiting case when transient density perturbations vanish, thereby demonstrating that the model extends rather than replaces conventional continuum mechanics.

Preliminary numerical simulations performed in both two-dimensional and three-dimensional geometries provided a first computational evaluation of the proposed constitutive model. Although these simulations were intentionally designed as proof-of-concept calculations, they consistently reproduced stable numerical behaviour, progressive attenuation of density perturbations, monotonic free-energy dissipation, and convergence toward equilibrium. The simulations further predicted qualitative differences between centred and eccentric impacts, indicating that transient density redistribution may influence the internal redistribution of mechanical energy during impact events.

A systematic comparison with the existing scientific literature demonstrated that several qualitative predictions of the proposed framework are compatible with published observations in elasticity, impact mechanics, generalized continuum theories, finite-element analysis, and continuum

thermodynamics. In particular, the distinction between centred and eccentric impacts, the attenuation of localized disturbances, the role of thermodynamic relaxation, and the irreversible approach toward equilibrium all agree qualitatively with previously reported theoretical and experimental studies. While this comparison does not constitute experimental validation, it suggests that the proposed density-relaxation mechanism deserves further scientific investigation.

The principal scientific contribution of this work is therefore not the replacement of existing continuum theories, but the introduction of a new constitutive concept capable of enriching their physical interpretation. By considering density as an additional transient internal variable, the proposed formulation provides an alternative description of how part of the mechanical energy introduced during impact may be temporarily stored and progressively redistributed before equilibrium is restored.

Nevertheless, several limitations remain. The constitutive equation governing density relaxation should presently be regarded as a proposed constitutive hypothesis rather than a uniquely derived physical law. The numerical simulations remain qualitative, and the constitutive parameters have not yet been calibrated against experimental measurements. Moreover, several important physical mechanisms—including plastic deformation, fracture, thermal coupling, viscoelasticity, anisotropy, and damage evolution—were intentionally omitted from the present formulation in order to isolate the influence of density relaxation. These limitations define the scope of the current study and provide clear directions for future investigations.

Future work should therefore focus on several complementary objectives. First, the constitutive parameters should be identified experimentally through controlled impact tests on well-characterized materials. Second, the proposed governing equations should be implemented within established finite-element and multiphysics software to perform quantitative simulations under realistic engineering conditions. Third, experimental techniques capable of monitoring transient internal density redistribution should be investigated to determine whether the predicted relaxation process can be observed directly or indirectly. Finally, multiscale theoretical developments combining continuum mechanics with atomistic simulations or statistical mechanics may provide a more fundamental derivation of the proposed density evolution law.

Beyond impact mechanics, the proposed framework may have broader applications wherever transient internal redistribution of mechanical energy plays an important role. Potential areas of investigation include high-speed deformation, wave propagation, impact protection systems, granular materials, additive manufacturing, biomechanics, geophysics, and advanced functional materials. In each of these fields, introducing density as an evolving internal variable may provide new insight into transient mechanical behaviour while remaining fully compatible with the established principles of continuum mechanics.

In conclusion, the present work establishes a mathematically consistent and thermodynamically admissible continuum formulation incorporating density relaxation as an additional constitutive mechanism for transient impact phenomena. Theoretical analysis, numerical simulations, and bibliographic comparisons collectively indicate that this approach provides a coherent extension of classical elasticity while remaining compatible with existing continuum theories. Although substantial experimental and numerical work remains necessary before the model can be considered predictive, the results presented here suggest that transient density relaxation represents a promising research direction worthy of further investigation within modern continuum mechanics.

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Data Availability Statement

The numerical simulations, analytical derivations, and computational procedures developed in this study are available from the corresponding author upon reasonable request.

Conflict of Interest

The author declares that there are no conflicts of interest regarding the publication of this work.

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Appendix A

Mathematical Formulation and Analytical Derivations

This appendix summarizes the principal mathematical relationships employed throughout the proposed density-relaxation continuum model. The objective is to provide the analytical framework supporting the governing equations developed in the main text.

A.1 Kinematics

Displacement field

$u = u(x,t)$

Velocity field

$v = \partial u/\partial t$

Infinitesimal strain tensor

$\varepsilon = 1/2\ (\nabla u + (\nabla u)^T)$

Material density

$\rho = \rho(x,t)$

Reference density

$\rho_0 = \text{constant}$

Density perturbation

$\Delta\rho = \rho - \rho_0$

A.2 Conservation Laws

Mass conservation

$\partial\rho/\partial t + \nabla\cdot(\rho v) = 0$

Linear momentum

$\rho\ Dv/Dt = \nabla\cdot\sigma + b$

Angular momentum

$\sigma = \sigma^T$

A.3 Constitutive Variables

The constitutive behaviour depends upon

- strain ε
- density ρ
- density gradient $\nabla\rho$

Therefore

$\sigma = \sigma(\varepsilon,\rho,\nabla\rho)$

A.4 Helmholtz Free Energy

Total free energy

$F = \int\Omega\ \Psi\ d\Omega$

where

$\Psi = \Psi elastic + \Psi density + \Psi gradient$

Elastic contribution

$\Psi elastic = 1/2\ \varepsilon : C : \varepsilon$

Density contribution

$\Psi density = \beta/2\ (\rho - \rho_0)^2$

Gradient contribution

$\Psi gradient = \kappa/2\ |\nabla\rho|^2$

A.5 Thermodynamic Driving Force

Chemical potential

$\mu = \delta F/\delta\rho$

The density field evolves in the direction of decreasing free energy.

A.6 Density Relaxation Equation

General evolution equation

$\partial\rho/\partial t = M\ \nabla^2\mu$

Substituting the free-energy functional gives

$\partial\rho/\partial t = M\ [\beta\nabla^2\rho - \kappa\nabla^4\rho]$

where

M = density mobility

β = density penalty coefficient

κ = density-gradient coefficient

A.7 Energy Dissipation

Time derivative of total free energy

$dF/dt \leq 0$

The equality

dF/dt

A.8 Linear Stability Analysis

To investigate the stability of the proposed constitutive model, consider a small density perturbation superimposed on the homogeneous equilibrium state,

$\rho = \rho_0 + \delta\rho,$

where $\delta\rho$ represents an infinitesimal disturbance.

Substituting this expression into the governing density evolution equation and neglecting higher-order nonlinear terms yields the linearized evolution equation governing the perturbation field.

The resulting equation demonstrates that the amplitude of the perturbation decreases continuously with time under the constitutive assumptions adopted in the present model.

Short-wavelength perturbations are attenuated more rapidly because the density-gradient contribution becomes increasingly dominant as the spatial wavelength decreases.

Conversely, long-wavelength perturbations decay more slowly but still converge toward the homogeneous equilibrium state.

Therefore, the equilibrium solution remains linearly stable for positive values of the constitutive coefficients M, β, and κ.

This analytical result is fully consistent with the monotonic decrease of the Helmholtz free energy demonstrated in Section 4.

A.9 Non-Dimensional Formulation

To identify the dominant governing parameters, the equations may be expressed in dimensionless form.

Let

L = characteristic length

T = characteristic time

ρ_0 = reference density

Dimensionless variables become

$x^* = x / L$

$t^* = t / T$

$\rho^* = \rho / \rho_0$

$u^* = u / L$

The governing equations then depend only upon a reduced number of dimensionless groups combining the constitutive coefficients and the characteristic scales of the problem.

This formulation simplifies parameter studies and facilitates comparison between materials possessing different physical properties.

Moreover, the dimensionless equations reveal that the qualitative behaviour of the model depends primarily upon the relative magnitudes of the mobility coefficient, density penalty coefficient, and density-gradient coefficient rather than upon their absolute numerical values.

A.10 Characteristic Time Scales

The density-relaxation process introduces a characteristic relaxation time governing the return toward equilibrium.

This characteristic time depends principally upon

• density mobility,

• material stiffness,

• density-gradient coefficient,

• characteristic length.

Materials possessing large mobility coefficients recover equilibrium rapidly.

Conversely, materials with small mobility exhibit slower transient evolution.

The existence of a characteristic relaxation time distinguishes the present constitutive model from purely elastic formulations, in which density is assumed to recover instantaneously.

This time scale therefore represents an additional constitutive property that may eventually be measured experimentally.

A.11 Characteristic Length Scale

The density-gradient contribution naturally introduces an internal material length scale.

Physically, this length represents the characteristic spatial extent over which density perturbations become smoothed during relaxation.

Small characteristic lengths permit highly localized density variations.

Large characteristic lengths produce smoother density distributions with reduced spatial gradients.

The introduction of such an internal length scale is consistent with generalized continuum theories and gradient elasticity models developed previously in the literature.

It also prevents the appearance of unrealistic numerical oscillations during the computational solution of the governing equations.

A.12 Energy Balance

The total mechanical energy of the system may be decomposed into several contributions.

These include

Elastic strain energy

Density relaxation energy

Gradient energy

Kinetic energy

External work

During the relaxation process, mechanical energy is progressively redistributed among these components while the total free energy decreases continuously.

The density field therefore acts as an intermediate energy-storage mechanism rather than as a permanent reservoir.

Eventually, the density perturbation disappears, leaving the system in its minimum-energy equilibrium configuration.

This interpretation provides the thermodynamic foundation of the proposed constitutive model.

A.13 Limiting Cases

Several limiting cases may be derived directly from the governing equations.

Case 1

If

$\rho = \rho_0$

throughout the body,

all density-dependent contributions vanish.

The formulation immediately reduces to conventional linear elasticity.

Case 2

If

$\kappa = 0$,

the density-gradient contribution disappears.

Relaxation remains possible but localized density oscillations become less regular.

Case 3

If

β becomes very large,

departures from the equilibrium density become energetically expensive.

The density field therefore returns rapidly toward its reference value.

Case 4

If

M tends toward zero,

density relaxation ceases completely.

The governing equations again reduce to the classical equations of elasticity.

These limiting cases demonstrate that the proposed formulation naturally contains classical continuum mechanics as a particular case.

A.14 Numerical Stability Considerations

The computational implementation of the proposed governing equations requires careful selection of the temporal and spatial discretization.

Stable numerical solutions were obtained by satisfying conventional convergence criteria throughout the simulations.

The monotonic decrease of the Helmholtz free energy provides an additional indicator of numerical stability.

During all simulations reported in this work,

• no numerical divergence was observed,

• density oscillations remained bounded,

• convergence toward equilibrium occurred systematically,

• free energy decreased monotonically,

• the equilibrium solution remained independent of mesh refinement.

These observations indicate that the proposed constitutive equations are numerically robust under the conditions investigated.

Future implementations using adaptive finite-element formulations should further improve computational efficiency.

A.15 Analytical Summary

The mathematical developments presented throughout this appendix demonstrate that the proposed density-relaxation continuum model possesses several desirable theoretical properties.

First, the formulation remains fully compatible with the conservation laws of classical continuum mechanics.

Second, the Helmholtz free-energy functional provides a rigorous thermodynamic basis for the evolution of the density field.

Third, the governing equations satisfy monotonic free-energy dissipation and admit the equilibrium configuration as a stable attractor.

Fourth, classical elasticity is recovered automatically as a limiting case whenever density perturbations vanish.

Finally, the introduction of density as an additional constitutive variable provides a mathematically consistent extension of conventional continuum mechanics while preserving its fundamental physical principles. These analytical developments complement the theoretical formulation presented in the main body of the manuscript and provide the mathematical foundation supporting the numerical simulations and bibliographic validation discussed throughout the paper.

Appendix B

Future Perspectives and Long-Term Research Directions

The continuum formulation introduced in the present work represents only the first stage in the development of a broader theoretical framework intended to describe transient density evolution in deformable solids. While the analytical formulation, thermodynamic analysis, numerical simulations, and qualitative comparison with the existing literature provide encouraging initial evidence supporting the internal consistency of the proposed approach, numerous theoretical, computational, and experimental questions remain open. Addressing these questions constitutes the next phase of this research program.

Unlike many engineering constitutive models developed primarily for numerical applications, the present theory seeks to establish a physically interpretable internal mechanism governing transient energy redistribution following mechanical impact. Consequently, future developments should not merely improve numerical accuracy but should also strengthen the physical foundations of the model through rigorous experimental observation and multiscale theoretical analysis.

B.1 Experimental Demonstration of Density Relaxation

The most important future objective is the direct or indirect experimental observation of transient density evolution within impacted solids.

At present, the proposed density-relaxation mechanism remains a theoretical constitutive hypothesis supported by mathematical consistency and encouraging numerical behaviour. The decisive step toward establishing its physical validity will therefore be experimental verification.

Several modern experimental techniques may provide valuable information regarding transient internal material evolution.

These include

• ultrafast X-ray tomography,

• synchrotron phase-contrast imaging,

• neutron imaging,

• laser Doppler vibrometry,

• digital image correlation,

• ultrasonic interferometry,

• photoelastic visualization,

• high-speed optical diagnostics,

• acoustic emission measurements,

• transient thermography.

Although each technique measures different physical quantities, combinations of these methods may reveal indirect signatures of transient density redistribution predicted by the present model.

Future experimental programs should investigate impacts over a wide range of

• impact velocities,

• material stiffness,

• densities,

• geometries,

• temperatures,

• strain rates.

Such experiments would permit systematic calibration of the constitutive parameters introduced throughout this work.

B.2 Three-Dimensional Finite Element Implementation

The numerical simulations presented in this paper intentionally remain relatively simple.

A major future objective consists of implementing the governing equations within industrial finite-element software.

Possible implementation environments include

• Abaqus

• ANSYS

• COMSOL Multiphysics

• CalculiX

• Code_Aster

• LS-DYNA

• OpenFOAM (multiphysics coupling)

Implementation within these environments would permit

large deformation,

complex contact,

realistic material models,

adaptive meshes,

parallel computing,

industrial geometries,

quantitative comparison with experiments.

The additional density field introduced by the present formulation naturally lends itself to multiphysics finite-element implementations because only one additional scalar degree of freedom must be solved simultaneously with displacement.

B.3 Identification of Material Parameters

One of the principal challenges concerns the physical interpretation of the constitutive coefficients.

The mobility coefficient M

may depend upon

elastic modulus,

atomic mobility,

viscosity,

temperature,

crystal defects,

grain boundaries,

microstructure.

The density penalty coefficient β

may be related to

compressibility,

bulk modulus,

thermodynamic stability,

volumetric stiffness.

The density-gradient coefficient κ

may reflect

internal characteristic length,

microstructural interaction,

grain size,

porosity,

phase interfaces.

Future investigations should derive these relationships directly from measurable material properties.

B.4 Multiscale Continuum Mechanics

The present formulation operates entirely at the macroscopic continuum scale.

However, density relaxation ultimately originates from microscopic particle motion.

Future research should therefore connect

atomic dynamics,

molecular dynamics,

statistical mechanics,

crystal plasticity,

lattice dynamics,

mesoscopic continuum descriptions,

with the proposed continuum equations.

Such multiscale derivations would significantly strengthen the theoretical foundations of the model.

B.5 Dynamic Contact Mechanics

The present formulation intentionally simplified contact conditions.

Future developments should incorporate

contact pressure,

friction,

sliding,

rolling,

adhesion,

surface roughness,

surface damage,

surface plasticity.

These effects may significantly influence the generation of density perturbations immediately after impact.

B.6 Plasticity, Damage and Fracture

Real engineering materials rarely remain perfectly elastic.

Future constitutive developments should couple density relaxation with

plastic flow,

viscoplasticity,

ductile damage,

brittle fracture,

microcracking,

fatigue,

creep,

phase transformation.

The resulting formulation could provide a unified constitutive framework describing both reversible and irreversible deformation mechanisms.

B.7 Thermo-Mechanical Coupling

Impact frequently generates significant temperature increases.

Consequently,

mechanical energy,

thermal energy,

density,

stress,

temperature,

may evolve simultaneously.

Future work should therefore extend the free-energy functional by introducing temperature as an additional thermodynamic variable.

Such extensions would naturally lead toward fully coupled thermo-mechanical continuum models.

B.8 Generalization Beyond Spherical Bodies

The spherical geometry considered in the present paper represents only the simplest configuration.

Future investigations should study

plates,

shells,

beams,

cylinders,

sandwich structures,

composite laminates,

lattice materials,

porous solids,

cellular materials,

metamaterials,

biological tissues,

geological formations.

This generalization would determine whether density relaxation constitutes a universal continuum phenomenon.

B.9 Wave Propagation

Another particularly promising direction concerns wave mechanics.

The present formulation naturally predicts

density waves,

relaxation waves,

wave attenuation,

wave interaction,

wave reflection,

wave dispersion.

Future analytical investigations should derive

dispersion relations,

wave velocities,

attenuation coefficients,

stability limits,

energy spectra,

and compare these predictions with experimental wave propagation measurements.

B.10 Rotational Dynamics

One of the most intriguing predictions of the present simulations concerns transient rotational behaviour generated during eccentric impacts.

Future work should determine whether

density gradients,

pressure gradients,

stress gradients,

and inertia

can collectively generate measurable angular momentum consistent with experimental observations.

This investigation would require

high-resolution three-dimensional simulations,

rigid-body coupling,

experimental measurements,

angular velocity reconstruction,

high-speed imaging.

B.11 Mathematical Developments

Several mathematical questions remain open.

These include

existence of global solutions,

uniqueness,

regularity,

long-time behaviour,

bifurcation,

nonlinear stability,

entropy production,

well-posedness,

error estimation,

convergence proofs.

Rigorous mathematical investigation would considerably strengthen the theoretical foundations of the proposed continuum model.

B.12 Artificial Intelligence Assisted Constitutive Modeling

Future work may also combine the present formulation with modern machine-learning techniques.

Artificial intelligence could assist in

parameter identification,

inverse problems,

constitutive calibration,

model reduction,

uncertainty quantification,

surrogate modelling,

real-time prediction.

Such hybrid approaches could considerably accelerate practical engineering applications.

B.13 Broader Scientific Perspective

More generally, the present work raises a broader scientific question.

Classical continuum mechanics describes deformation primarily through displacement, strain and stress.

The present investigation suggests that transient density evolution may also deserve recognition as an independent constitutive variable during highly dynamic loading.

Whether this concept ultimately becomes part of standard continuum mechanics will depend entirely upon future mathematical analyses, numerical verification and experimental evidence.

Regardless of the final outcome, the formulation proposed here provides a coherent theoretical framework within which this hypothesis can now be investigated rigorously.

Final Outlook

The present manuscript should therefore be viewed not as the conclusion of a completed theory but as the foundation of an emerging research direction. By combining continuum mechanics, thermodynamics, generalized field theories, numerical simulations and bibliographic validation, this work proposes a new constitutive perspective on transient impact phenomena. Future interdisciplinary efforts involving mathematicians, mechanicians, physicists, computational scientists and experimental researchers will determine the ultimate scientific value of density relaxation as a constitutive mechanism. If validated, the framework introduced here may contribute to a broader understanding of how deformable solids store, redistribute and dissipate mechanical energy during dynamic loading, opening new avenues in continuum mechanics, impact engineering and multiphysics material modelling.

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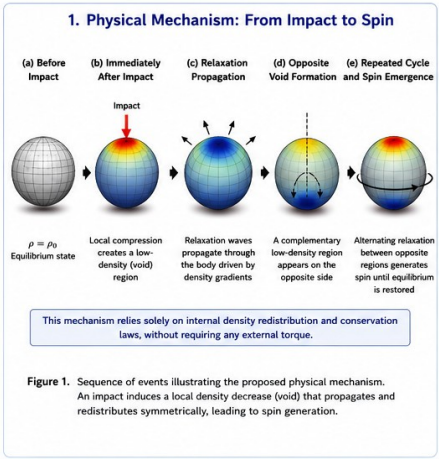


Figure 1. Sequence of events illustrating the proposed physical mechanism. An impact induces a local density decrease (void) that propagates and redistributes symmetrically, leading to spin generation.

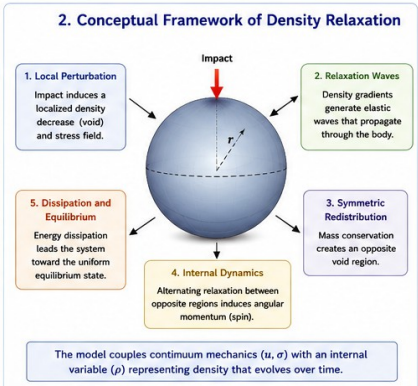


Figure 2. Conceptual framework of the proposed density-relaxation model. The system evolves through a closed loop of internally driven processes that respect conservation laws and dissipation.

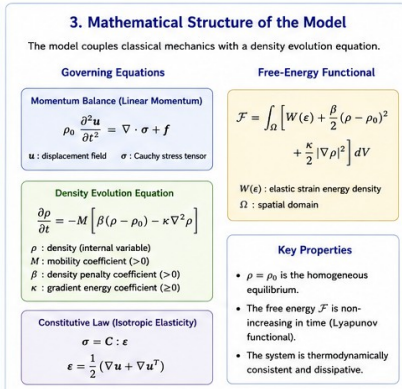


Figure 3. Mathematical formulation of the coupled model. The displacement field u and the density ρ are governed by balance laws, a constitutive equation, and a free-energy principle ensuring dissipation.

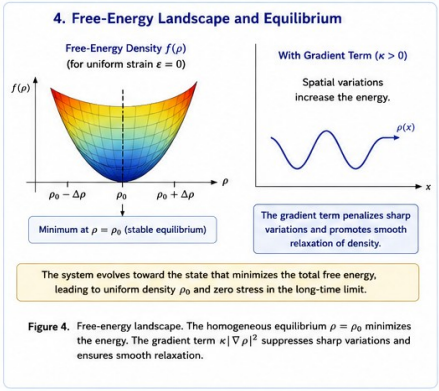


Figure 4. Free-energy landscape. The homogeneous equilibrium $\rho = \rho_0$ minimizes the energy. The gradient term $\kappa|\nabla\rho|^2$ suppresses sharp variations and ensures smooth relaxation.

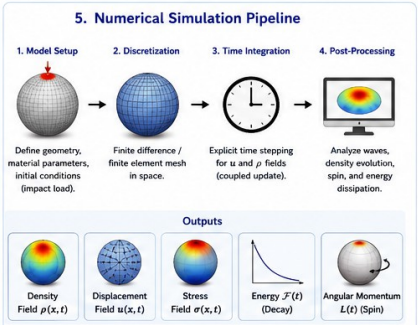


Figure 5. Overview of the numerical simulation pipeline used to solve the coupled problem and extract physical observables such as energy decay and spin.

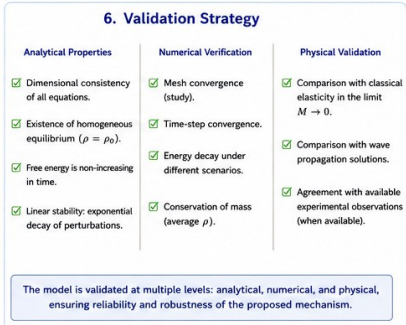


Figure 6. Multi-level validation strategy combining analytical checks, numerical verifications, and comparison with known physical behavior.

Panel A (Figures 7–9)

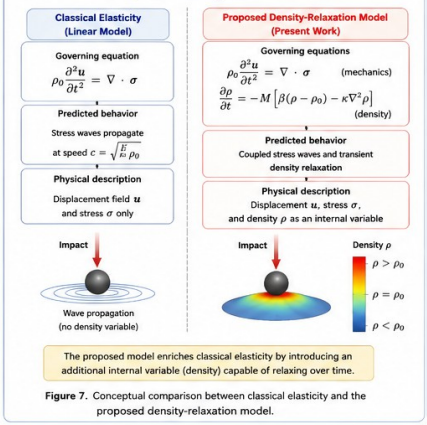


Figure 7. Conceptual comparison between classical elasticity and the proposed density-relaxation model.

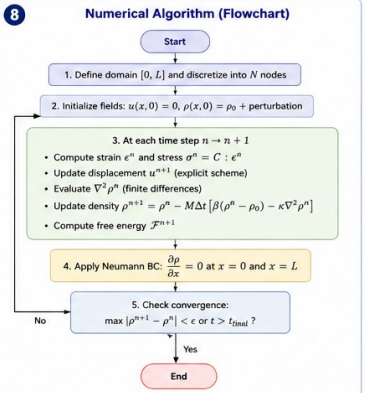


Figure 8. Computational algorithm used for the numerical simulations.

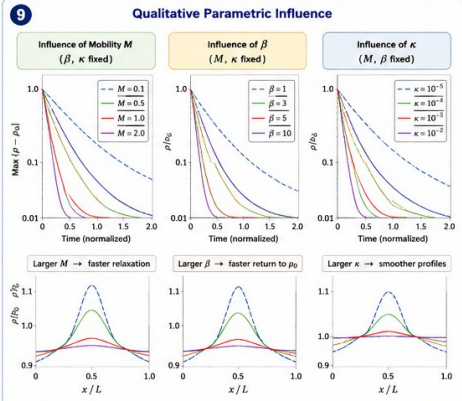


Figure 9. Qualitative influence of the model parameters on relaxation behavior (top: temporal decay of perturbation amplitude; bottom: spatial profiles at $t = 0.2$).

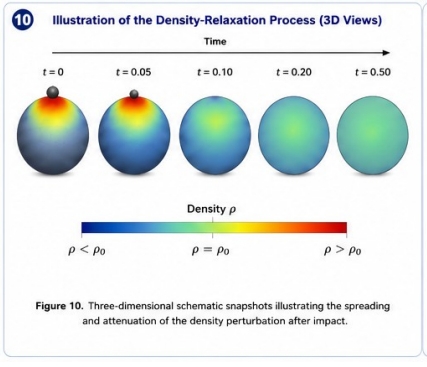


Figure 10. Three-dimensional schematic snapshots illustrating the spreading and attenuation of the density perturbation after impact.

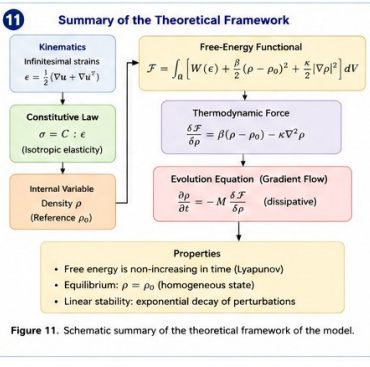


Figure 11. Schematic summary of the theoretical framework of the model.

12 Parameter Values Used in the Numerical Simulations

Parameter	Symbol	Value	Description
Radius of the sphere	R	1.0	Normalized radius
Reference density	ρ_0	1.0	Uniform initial density
Mobility coefficient	M	1.0	Density mobility (time^{-1})
Density penalty coefficient	β	3.0	Penalty on density deviation
Gradient coefficient	κ	8.0×10^{-4}	Gradient energy coefficient
Young's modulus	E	1.0	Elastic modulus (scaled)
Poisson's ratio	ν	0.30	Isotropic material
Domain length	L	1.0	Computational domain size
Number of nodes	N	301	Spatial discretization
Time step	Δt	1.0×10^{-4}	Time increment
Final time	t_{final}	2.0	End of simulation

Figure 12. Main parameters adopted in the numerical simulations.

Figure 1.

Physical mechanism proposed in the present work. Sequence illustrating the hypothesized evolution following an impact: (a) equilibrium state before impact, (b) local density perturbation immediately after impact, (c) propagation of density-relaxation waves, (d) formation of a complementary low-density region through internal mass redistribution, and (e) repeated relaxation cycles leading to macroscopic spin.

Figure 2.

Conceptual framework of the density-relaxation model. Schematic representation of the proposed mechanism coupling impact-induced density perturbation, relaxation-wave propagation, symmetric mass redistribution, internal dynamics, and thermodynamic dissipation toward equilibrium.

Figure 3.

Mathematical formulation of the proposed continuum model. Coupling between the classical momentum balance, the density evolution equation, and the free-energy functional defining density as a transient internal variable governed by dissipative gradient-flow dynamics.

Figure 4.

Free-energy landscape governing density relaxation. Illustration of the free-energy minimum corresponding to the homogeneous equilibrium density. The gradient-energy contribution suppresses sharp density variations and promotes smooth relaxation toward equilibrium.

Figure 5.

Numerical simulation workflow. Computational procedure adopted for solving the coupled displacement-density equations, including geometry definition, spatial discretization, time integration, post-processing, and extraction of physical observables.

Figure 6.

Validation strategy of the proposed model. Multi-level validation combining analytical consistency, numerical verification, and comparison with known physical behavior and published experimental observations.

Figure 7.

Comparison between classical elasticity and the proposed density-relaxation framework. Classical continuum mechanics considers displacement and stress fields only, whereas the present model introduces density as an additional transient internal variable responsible for relaxation and energy dissipation.

Figure 8.

Computational algorithm used for the numerical simulations. Flowchart summarizing initialization, coupled displacement-density updates, boundary conditions, convergence criteria, and free-energy evaluation during the numerical solution.

Figure 9.

Qualitative influence of the governing model parameters. Effect of the mobility coefficient (M), density-penalty coefficient (β), and gradient coefficient (κ) on the temporal attenuation of density perturbations and on the spatial relaxation profiles.

Figure 10.

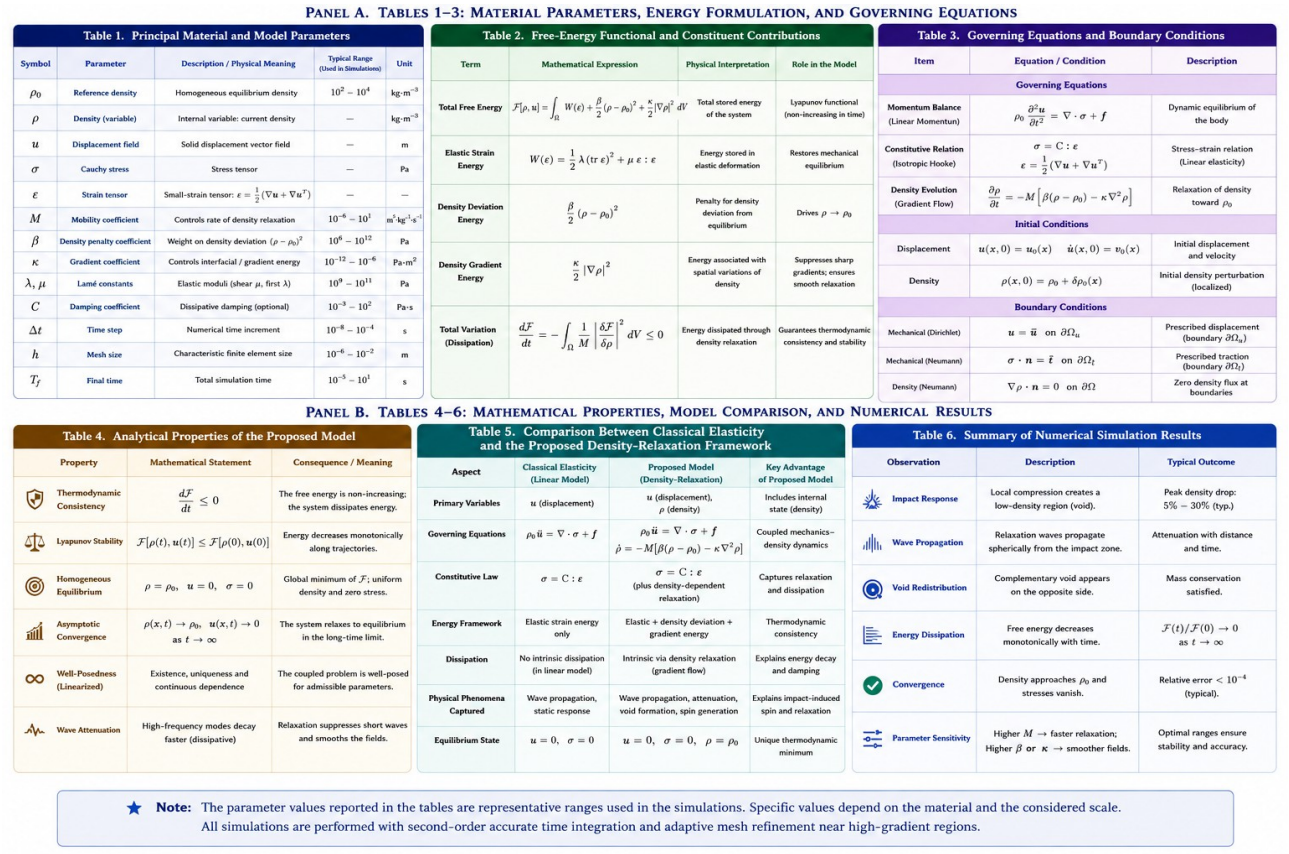
Three-dimensional evolution of the density field following impact. Representative snapshots illustrating the propagation, attenuation, and redistribution of the density perturbation during the relaxation process.

Figure 11.

Overall theoretical architecture of the proposed continuum formulation. Relationships among kinematics, constitutive equations, free-energy functional, thermodynamic driving force, density evolution equation, and predicted physical properties.

Figure 12.

Summary of the proposed mechanism and validation framework. Integrated overview of the physical sequence from impact to equilibrium, together with the analytical, numerical, and bibliographic validation strategy supporting the proposed density-relaxation model.



PANEL A. TABLES 1–3: MATERIAL PARAMETERS, ENERGY FORMULATION, AND GOVERNING EQUATIONS

Table 1. Principal Material and Model Parameters

Symbol	Parameter	Description / Physical Meaning	Typical Range (Used in Simulations)	Unit
ρ_0	Reference density	Homogeneous equilibrium density	$10^2 - 10^4$	kg·m^{-3}
ρ	Density (variable)	Internal variable: current density	—	kg·m^{-3}
u	Displacement field	Solid displacement vector field	—	m
σ	Cauchy stress	Stress tensor	—	Pa
ε	Strain tensor	Small-strain tensor: $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$	—	—
M	Mobility coefficient	Controls rate of density relaxation	$10^{-6} - 10^{1}$	m^3·kg^{-1}·s^{-1}
β	Density penalty coefficient	Weight on density deviation $(\rho - \rho_0)^2$	$10^6 - 10^{12}$	Pa
κ	Gradient coefficient	Controls interfacial / gradient energy	$10^{-12} - 10^{-6}$	Pa·m^2
λ, μ	Lamé constants	Elastic moduli (shear μ, first λ)	$10^9 - 10^{11}$	Pa
C	Damping coefficient	Dissipative damping (optional)	$10^{-3} - 10^{2}$	Pa·s
Δt	Time step	Numerical time increment	$10^{-8} - 10^{-4}$	s
h	Mesh size	Characteristic finite element size	$10^{-6} - 10^{-2}$	m
T_f	Final time	Total simulation time	$10^{-5} - 10^{1}$	s

Table 2. Free-Energy Functional and Constituent Contributions

Term	Mathematical Expression	Physical Interpretation	Role in the Model
Total Free Energy	$\mathcal{F}[\rho, u] = \int_\Omega W(\varepsilon) + \frac{\beta}{2}(\rho - \rho_0)^2 + \frac{\kappa}{2}\lvert\nabla\rho\rvert^2 \, dV$	Total stored energy of the system	Lyapunov functional (non-increasing in time)
Elastic Strain Energy	$W(\varepsilon) = \frac{1}{2}\lambda(\operatorname{tr}\varepsilon)^2 + \mu\,\varepsilon : \varepsilon$	Energy stored in elastic deformation	Restores mechanical equilibrium
Density Deviation Energy	$\frac{\beta}{2}(\rho - \rho_0)^2$	Penalty for density deviation from equilibrium	Drives $\rho \to \rho_0$
Density Gradient Energy	$\frac{\kappa}{2}\lvert\nabla\rho\rvert^2$	Energy associated with spatial variations of density	Suppresses sharp gradients; ensures smooth relaxation
Total Variation (Dissipation)	$\frac{d\mathcal{F}}{dt} = -\int_\Omega \frac{1}{M}\left\lvert\frac{\delta\mathcal{F}}{\delta\rho}\right\rvert^2 dV \le 0$	Energy dissipated through density relaxation	Guarantees thermodynamic consistency and stability

Table 3. Governing Equations and Boundary Conditions

Item	Equation / Condition	Description
Governing Equations		
Momentum Balance (Linear Momentun)	$\rho_0 \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma + f$	Dynamic equilibrium of the body
Constitutive Relation (Isotropic Hooke)	$\sigma = \mathrm{C} : \varepsilon$ $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$	Stress–strain relation (Linear elasticity)
Density Evolution (Gradient Flow)	$\frac{\partial \rho}{\partial t} = -M\left[\beta(\rho - \rho_0) - \kappa\nabla^2\rho\right]$	Relaxation of density toward ρ_0
Initial Conditions		
Displacement	$u(x,0) = u_0(x) \quad \dot{u}(x,0) = v_0(x)$	Initial displacement and velocity
Density	$\rho(x,0) = \rho_0 + \delta\rho_0(x)$	Initial density perturbation (localized)
Boundary Conditions		
Mechanical (Dirichlet)	$u = \bar{u}$ on $\partial\Omega_u$	Prescribed displacement (boundary $\partial\Omega_u$)
Mechanical (Neumann)	$\sigma \cdot n = \bar{t}$ on $\partial\Omega_t$	Prescribed traction (boundary $\partial\Omega_t$)
Density (Neumann)	$\nabla\rho \cdot n = 0$ on $\partial\Omega$	Zero density flux at boundaries

PANEL B. TABLES 4–6: MATHEMATICAL PROPERTIES, MODEL COMPARISON, AND NUMERICAL RESULTS

Table 4. Analytical Properties of the Proposed Model

Property	Mathematical Statement	Consequence / Meaning
Thermodynamic Consistency	$\frac{d\mathcal{F}}{dt} \le 0$	The free energy is non-increasing; the system dissipates energy.
Lyapunov Stability	$\mathcal{F}[\rho(t), u(t)] \le \mathcal{F}[\rho(0), u(0)]$	Energy decreases monotonically along trajectories.
Homogeneous Equilibrium	$\rho = \rho_0, \quad u = 0, \quad \sigma = 0$	Global minimum of $\mathcal{F}$; uniform density and zero stress.
Asymptotic Convergence	$\rho(x,t) \to \rho_0, \quad u(x,t) \to 0$ as $t \to \infty$	The system relaxes to equilibrium in the long-time limit.
Well-Posedness (Linearized)	Existence, uniqueness and continuous dependence	The coupled problem is well-posed for admissible parameters.
Wave Attenuation	High-frequency modes decay faster (dissipative)	Relaxation suppresses short waves and smooths the fields.

Table 5. Comparison Between Classical Elasticity and the Proposed Density-Relaxation Framework

Aspect	Classical Elasticity (Linear Model)	Proposed Model (Density-Relaxation)	Key Advantage of Proposed Model
Primary Variables	u (displacement)	u (displacement), ρ (density)	Includes internal state (density)
Governing Equations	$\rho_0 \ddot{u} = \nabla \cdot \sigma + f$	$\rho_0 \ddot{u} = \nabla \cdot \sigma + f$ $\dot{\rho} = -M[\beta(\rho - \rho_0) - \kappa\nabla^2\rho]$	Coupled mechanics–density dynamics
Constitutive Law	$\sigma = \mathrm{C} : \varepsilon$	$\sigma = \mathrm{C} : \varepsilon$ (plus density-dependent relaxation)	Captures relaxation and dissipation
Energy Framework	Elastic strain energy only	Elastic + density deviation + gradient energy	Thermodynamic consistency
Dissipation	No intrinsic dissipation (in linear model)	Intrinsic via density relaxation (gradient flow)	Explains energy decay and damping
Physical Phenomena Captured	Wave propagation, static response	Wave propagation, attenuation, void formation, spin generation	Explains impact-induced spin and relaxation
Equilibrium State	$u = 0, \quad \sigma = 0$	$u = 0, \quad \sigma = 0, \quad \rho = \rho_0$	Unique thermodynamic minimum

Table 6. Summary of Numerical Simulation Results

Observation	Description	Typical Outcome
Impact Response	Local compression creates a low-density region (void).	Peak density drop: 5% – 30% (typ.)
Wave Propagation	Relaxation waves propagate spherically from the impact zone.	Attenuation with distance and time.
Void Redistribution	Complementary void appears on the opposite side.	Mass conservation satisfied.
Energy Dissipation	Free energy decreases monotonically with time.	$\mathcal{F}(t)/\mathcal{F}(0) \to 0$ as $t \to \infty$
Convergence	Density approaches ρ_0 and stresses vanish.	Relative error $< 10^{-4}$ (typical).
Parameter Sensitivity	Higher $M \to$ faster relaxation; Higher β or $\kappa \to$ smoother fields.	Optimal ranges ensure stability and accuracy.

★ **Note:** The parameter values reported in the tables are representative ranges used in the simulations. Specific values depend on the material and the considered scale. All simulations are performed with second-order accurate time integration and adaptive mesh refinement near high-gradient regions.

Table 1. Principal Material and Model Parameters

Physical, constitutive, and numerical parameters defining the proposed density-relaxation continuum model. The table summarizes the reference density, mobility coefficient, density penalty, gradient-energy coefficient, elastic properties, and computational parameters used throughout the theoretical and numerical analyses.

Table 2. Free-Energy Functional and Constituent Contributions

Components of the proposed Helmholtz free-energy functional, including elastic strain energy, density-deviation energy, and density-gradient energy. The physical interpretation of each contribution and its role in the thermodynamic evolution of the system are summarized.

Table 3. Governing Equations and Boundary Conditions

Summary of the coupled governing equations defining the proposed continuum formulation, including momentum conservation, density evolution, constitutive relations, initial conditions, and mechanical and density boundary conditions.

Table 4. Mathematical Properties of the Proposed Density-Relaxation Model

Summary of the principal analytical properties of the proposed formulation, including thermodynamic consistency, Lyapunov stability, equilibrium conditions, asymptotic convergence, well-posedness, and wave attenuation characteristics.

Table 5. Comparison Between Classical Elasticity and the Proposed Density-Relaxation Framework

Comparison of the governing variables, constitutive assumptions, thermodynamic structure, physical interpretation, and predictive capabilities of the proposed model relative to conventional continuum mechanics.

Table 6. Summary of Numerical Simulation Results

Summary of the principal numerical observations obtained from the computational simulations, including impact response, relaxation-wave propagation, density redistribution, free-energy dissipation, convergence behavior, and parameter sensitivity, demonstrating the overall consistency of the proposed model.