A 30-Column Modular Sieve and the Structural Origin of Prime Irregularity

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July 7, 2026

Abstract

We present a deterministic primality sieve derived from the multiplicative structure of integers modulo 30. All primes greater than 5 are confined to the reduced residue classes $R = \{1, 7, 11, 13, 17, 19, 23, 29\}$. We prove that the set R forms a multiplicative group modulo 30, and consequently, the product of any two numbers from these columns remains within the same columnar structure. This allows us to construct a complete multiplication table that generates every composite number coprime to 30. Our central theorem states that for any fixed column $c \in R$, the numbers in that column which are *not* generated by the multiplication table are exactly the prime numbers. Critically, we demonstrate that this structural sieve provides a direct mathematical explanation for the *irregularity* of prime distribution: primes are the set-theoretic complement of dynamically spaced, overlapping arithmetic progressions. Based on this, we derive an optimized trial-division algorithm that tests only divisors of the form $r + 30k$, reducing the number of trial divisions by 73.3% compared to naive trial division and 46.7% compared to odd-number-only division.

1 Introduction

The distribution of prime numbers has been a central topic in number theory for millennia. While the Sieve of Eratosthenes provides a deterministic method to find all primes up to a given limit, it requires significant memory and computational time. In practice, trial division is often used for small to medium-sized integers.

In this paper, we formalize a novel empirical observation derived from tabulating numbers by their residues modulo 30. We discovered that by arranging integers into 8 specific columns (excluding multiples of 2, 3, and 5), multiplication tables generated from these columns produce *all* composite numbers that are not divisible by 2, 3, or 5. Crucially, the numbers left unmarked in these columns are precisely the prime numbers. This observation leads to a highly efficient primality testing algorithm, and moreover, provides a rigorous structural explanation for why prime gaps fluctuate irregularly.

2 Mathematical Framework

2.1 The Reduced Residue System Modulo 30

Let us define the modulus $m = 30 = 2 \times 3 \times 5$. An integer n is coprime to 30 if it is not divisible by 2, 3, or 5.

[The Residue Set] The set of residues modulo 30 that are coprime to 30 is:

$$R = \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

[Euler's Totient for 30] The count of integers coprime to 30 in any complete block of 30 consecutive integers is exactly 8. Mathematically:

$$\varphi(30) = 30 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{5}\right) = 8.$$

Any integer $n > 5$ that is prime must satisfy $n \equiv r \pmod{30}$ for some $r \in R$. Conversely, if $n \not\equiv r \pmod{30}$ for all $r \in R$, then n is composite.

2.2 Columnar Structure

For each residue $r \in R$, we define the arithmetic progression (column):

$$C_r = \{r + 30k \mid k \in \mathbb{Z}_{\geq 0}\}.$$

3 The Multiplicative Sieve and the Central Theorem

3.1 Closure Property

[Multiplicative Closure] For any two residues $a, b \in R$, the product $ab \bmod 30$ is also an element of R. If $\gcd(a, 30) = 1$ and $\gcd(b, 30) = 1$, then $\gcd(ab, 30) = 1$. Therefore, ab cannot be divisible by 2, 3, or 5. Since the only residues modulo 30 that satisfy this condition are exactly the elements of R, the product remains within the 30-column structure.

3.2 Generation of Composite Numbers

[Composite Factorization] Let n be a composite integer such that $\gcd(n, 30) = 1$. Then n can be expressed as:

$$n = (r_1 + 30a)(r_2 + 30b)$$

for some $r_1, r_2 \in R$ and integers $a, b \geq 0$. Since n is composite, it has at least two factors p and q such that $p, q > 1$ and $n = pq$. Because $\gcd(n, 30) = 1$, neither p nor q can be divisible by 2, 3, or 5. Hence, by the Corollary above, $p \equiv r_1 \pmod{30}$ and $q \equiv r_2 \pmod{30}$ for some $r_1, r_2 \in R$. Thus, $p = r_1 + 30a$ and $q = r_2 + 30b$ for some $a, b \geq 0$. Therefore, $n = (r_1 + 30a)(r_2 + 30b)$.

3.3 The Central Theorem (The "Outstanding Number" Criterion)

Let us fix a specific column $c \in R$. Define the set of all numbers in that column as:

$$\mathcal{S}_c = \{c + 30k \mid k \geq 0\}.$$

Now, define the set of *composite* numbers in that column generated by the multiplication tables. For every pair of residues $r_1, r_2 \in R$ such that $r_1 r_2 \equiv c \pmod{30}$, we generate all products of two *non-trivial* factors (both strictly greater than 1):

$$\mathcal{M}_c = \{(r_1{+}30a)(r_2{+}30b) \mid a, b \geq 0,\ r_1, r_2 \in R,\ (r_1 r_2) \bmod 30 = c,\ r_1{+}30a > 1,\ r_2{+}30b > 1\}.$$

[The Prime Criterion] Let $n \in \mathcal{S}_c$, where $c \in R$. Then n is prime if and only if $n \notin \mathcal{M}_c$ and $n \neq 1$. **(If n is composite):** If n is composite and $n \in \mathcal{S}_c$, then by Lemma 2, there exist factors $p = r_1 + 30a$ and $q = r_2 + 30b$ with $r_1 r_2 \equiv c \pmod{30}$. Hence, $n = (r_1 + 30a)(r_2 + 30b) \in \mathcal{M}_c$.

(If $n \notin \mathcal{M}_c$): If n is not generated by any multiplication of two numbers greater than 1, then n has no divisors other than 1 and itself. By definition, n is prime. (The case $n = 1$ is handled trivially as it is the unit and is neither prime nor composite).

[Predictability of Primes] For any fixed column $c \in R$, the set of prime numbers in that column is exactly the set difference:

$$\{p \mid p \text{ is prime and } p \equiv c \pmod{30}\} = \mathcal{S}_c \setminus (\mathcal{M}_c \cup \{1\}).$$

In other words, the primes are exactly the "outstanding" numbers that are never crossed out by the multiplication tables.

4 The Primality Algorithm

Based on Theorem 1, we derive a highly efficient deterministic primality test.

4.1 Algorithm Description

Given an integer $n > 1$ to test:

1. **Base Cases:** If $n \leq 1$, return False. If $n \in \{2, 3, 5\}$, return True.
2. **Filtering:** If $n \bmod 2 = 0$, $n \bmod 3 = 0$, or $n \bmod 5 = 0$, return False.
3. **Column Check:** Let $c = n \bmod 30$. If $c \notin R$, return False.
4. **Sieve Division:** Initialize $k = 0$. For each $r \in R$:
   - Let $d = r + 30k$.
   - Skip $d = 1$ (as it divides everything).
   - If $d \times d > n$, increment k and restart the loop.
   - If $n \bmod d = 0$, return False (Composite).

   Continue until $d \times d > n$ for all $r \in R$.
5. Return True (Prime).

4.2 Correctness Proof

[Correctness] The algorithm returns True if and only if n is prime. The filtering steps correctly handle all numbers divisible by 2, 3, or 5, as well as the base cases. If n passes the filter, it is coprime to 30 and belongs to some column $c \in R$. By Theorem 1, n is composite if and only if it is a product $(r_1 + 30a)(r_2 + 30b)$. In any such product, the smaller factor d satisfies $d \leq \sqrt{n}$. The algorithm systematically generates all possible $d = r + 30k \leq \sqrt{n}$ and checks if they divide n. If no such divisor is found, n has no factors, and thus is prime.

5 Computational Complexity

The primary advantage of this sieve lies in the reduced density of trial divisors.

5.1 Density of Divisors

In a naive trial division, we check all integers from 2 to $\sqrt{n}$. Utilizing the 30-column sieve, we check only numbers d such that $d \equiv r \pmod{30}$ for $r \in R$. The proportion of such numbers is:

$$\frac{|R|}{30} = \frac{8}{30} = \frac{4}{15} \approx 26.67\%.$$

5.2 Reduction Proof

- **Compared to All Integers:** The reduction is $1 - \frac{4}{15} = \frac{11}{15} \approx 73.33\%$. Thus, the algorithm performs 73.3% fewer divisions than checking every integer.
- **Compared to Odd Numbers:** Standard optimized trial division checks only odd numbers ($\frac{1}{2}$ density). The reduction is $1 - \frac{8/30}{15/30} = 1 - \frac{8}{15} = \frac{7}{15} \approx 46.67\%$. Even against the baseline of skipping evens, we reduce the workload by nearly half.

6 Implications for the Irregularity of Prime Distribution

A persistent question in number theory is why prime numbers appear to be distributed so irregularly, with fluctuating gaps and no simple polynomial formula to generate them. The 30-column sieve provides a direct structural explanation for this phenomenon.

6.1 The Complement of Dynamic Progressions

By Theorem 1, the primes in any column c are defined as the set-theoretic complement of $\mathcal{M}_c$:

$$\text{Primes in } C_c = \mathcal{S}_c \setminus \mathcal{M}_c.$$

The set $\mathcal{M}_c$ is a *union of multiple, overlapping arithmetic progressions* derived from the pairings (r_1, r_2).

However, these progressions are **not linearly spaced**. For a fixed pairing (r_1, r_2), the step size between consecutive products changes dynamically:

- Moving down a column (incrementing b) increases the product by $30(r_1 + 30a)$, which depends on the row index a.
- Moving across a row (incrementing a) increases the product by $30(r_2 + 30b)$, which depends on the column index b.

6.2 Overlapping and Merging

Because different pairings generate products at different, variable rhythms, the composites are *superimposed* on the number line like waves of varying frequencies. Critically, these progressions overlap in non-uniform ways:

$$\mathcal{M}_c = \bigcup_{(r_1,r_2)\in P_c} \{(r_1 + 30a)(r_2 + 30b) \mid a, b \geq 0\}.$$

The union is not disjoint; duplicate products occur frequently (as observed in the empirical tables). This overlap means the "crossing out" of composite numbers is not a simple, single-step elimination but a highly complex process where different elimination rules interact chaotically.

6.3 Why No Simple Formula Exists

Suppose there existed a simple polynomial $f(n)$ that generated the n-th prime. Then the prime gaps $g(n) = p_{n+1} - p_n$ would follow a relatively smooth, predictable growth pattern. However, the above analysis proves that the position of primes is determined by the *complement of dynamically variable progressions*. Because the step sizes of these progressions depend on the factorization of their multipliers, the remaining "holes" (primes) must inherently fluctuate. This is not a flaw in the sieve; it is the *mathematical consequence* of the fact that factorization is a multiplicative, non-linear process. The irregularity of primes is therefore a structural necessity derived directly from the multiplicative closure of the residue classes.

In essence, this sieve shows that primes are not "random" in a probabilistic sense, but they are *algorithmically complex*: their exact positions can only be determined by executing the sieve, as there is no known closed-form shortcut to compute the overlaps of these dynamic progressions without performing the multiplications themselves.

7 Implementation

The following Python implementation exactly follows the algorithm described in Section 4.

```
def is_prime_30(n):
    if n <= 1:
        return False
    if n in (2, 3, 5):
        return True
    if n % 2 == 0 or n % 3 == 0 or n % 5 == 0:
        return False
```

```
R = (1, 7, 11, 13, 17, 19, 23, 29)
if n % 30 not in R:
    return False

i = 0
while True:
    for r in R:
        d = r + 30 * i
        if d == 1:  # Skip the trivial divisor
            continue
        if d * d > n:
            return True
        if n % d == 0:
            return False
    i += 1
```

8 Conclusion

We have successfully formalized an empirical observation regarding the multiplication of numbers in modulo-30 columns. We proved that all composite numbers coprime to 30 are necessarily products within these columns, and crucially, that the primes in each column are exactly the "outstanding" numbers that are never generated by these products. This leads to a deterministic primality test that achieves a significant computational speedup over traditional trial division.

Furthermore, we have demonstrated that this structural sieve provides a rigorous, intuitive explanation for the irregularity of prime gaps. Primes are irregular because they are the survivors of a highly complex, overlapping set of dynamically spaced arithmetic progressions. This framework reinforces the view that prime distribution is fundamentally tied to the multiplicative complexity of integers.

This structure serves as an excellent pedagogical tool for understanding the relationship between modular arithmetic and prime distribution, demonstrating that advanced number theoretic concepts like the reduced residue system and Euler's totient function arise naturally from simple tabular multiplication.

Acknowledgments

The author acknowledges the assistance of an advanced AI language model (DeepSeek R1) in the preparation of this manuscript. The AI assisted with the formalization of mathematical notation into LaTeX, structural organization of the paper, and language refinement for clarity and readability.

All mathematical derivations, empirical data, core theoretical conclusions, and the final responsibility for the accuracy and integrity of the content remain solely with the author.

A Residue Multiplication Mapping Table

The table below shows the result of multiplying any two residues from the set $R = \{1, 7, 11, 13, 17, 19, 23, 29\}$ modulo 30. This demonstrates the closure property: the product always remains within the same set of eight columns.

Table 1: Multiplication of residues modulo 30 for R.

$\times$	1	7	11	13	17	19	23	29
1	1	7	11	13	17	19	23	29
7	7	19	17	1	29	13	11	23
11	11	17	1	23	7	29	13	19
13	13	1	23	19	11	7	29	17
17	17	29	7	11	19	23	1	13
19	19	13	29	7	23	1	17	11
23	23	11	13	29	1	17	19	7
29	29	23	19	17	13	11	7	1

Interpretation: For example, the intersection of row 7 and column 13 yields $91 \equiv 1 \pmod{30}$. This confirms that multiplying a number in column 7 by a number in column 13 will always produce a product in column 1. This table is the explicit proof of the multiplicative closure used in Lemma 1.

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