The Proper-Time Maurer–Cartan Connection in a Rotor-Generated Spin(1,3) Theory of Gravitation: A Connection-Level Derivation of the Constant-Lagrangian Galactic Rotation-Curve Law

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Abstract

The gravitational rapidity potential $\chi_g(x)$ generates a local Spin$(1,3)$ rotor G, which forms the common origin of three complementary gravitational descriptions. A companion paper developed the exact kinematic branch, $G \to U_g \to H_g = \partial^T U_g \to UH_g$, and the slow-flow metric branch, $G \to d\mathcal{R}_G \to e^a_{\mathrm{PG}} \to g_{\mu\nu}$, which recovers the Painlevé–Gullstrand spacetime geometry. The present paper develops the third branch, based on the Maurer–Cartan connection and its transport along the gravitational vacuum flow, $G \longrightarrow \Omega_\tau = \left(\frac{D}{D\tau}G\right)G^{-1} \longrightarrow U\Omega_\tau$. Starting from the general spacetime Maurer–Cartan connection $\Omega_Q = (\partial G)G^{-1}$, its proper-time projection is derived for the composed galactic rotor $G = G_\phi G_r$, yielding the complete channel decomposition of Ω_τ. The resulting connection naturally separates into radial and azimuthal rapidity-rate channels together with the Thomas–Wigner bivector generated by the non-commutativity of the two boosts. Transporting Ω_τ with the gravitational four-velocity produces the bilinear $U\Omega_\tau$ and its corresponding norm–time–space–spin channel hierarchy.

The central result is that the time-channel transport balance, $[U\Omega_\tau]_T = 0,$ coincides exactly with the previously established Constant-Lagrangian (CL) condition $\frac{D\Gamma}{D\tau} = 0$, $\Gamma = \cosh\chi_r \cosh\chi_\phi$ in the slow-flow limit $(\gamma_r \to 1)$; away from that limit the two conditions are not algebraically identical, differing by a factor tied to the flow's radial Lorentz factor. This exact correspondence in the physically relevant regime thereby provides a direct connection between the Maurer–Cartan geometry and the conserved isorapidity flow governing stationary spiral galaxies.

Applying the resulting equations to the Painlevé–Gullstrand–Hubble vacuum flow together with a Newtonian virial boundary condition at the galactic bulge radius recovers, in closed form, the spiral-galaxy rotation-curve relation previously fitted directly to SPARC observations. The Maurer–Cartan formulation therefore gives the previously empirical Constant-Lagrangian rotation-curve law a second theoretical home within the rotor-generated hierarchy, showing that flat galactic rotation curves emerge from the conserved composition of two orthogonal rapidity components of a single relativistic gravitational vacuum river flow, once combined with a Newtonian virial boundary condition at the bulge radius. Throughout this construction, Hubble embedded Newtonian gravity remains unchanged, and no dark-matter halo, MOND interpolation function, or additional free parameter is introduced. The construction extends naturally to stationary cones of constant colatitude, where the resulting trajectories remain smooth helical curves throughout their physical domain.

Keywords: Painlevé–Gullstrand geometry; hyperbolic geometry; galactic rotation curves; biquaternion algebra; minquats; pauliquats; Spin(1,3) rotor; channel decomposition; norm-vector-spin bilinear; triple product; Maurer–Cartan connection; rapidity; hyperbolic Lorentz transformation; Î-alignment; Thomas–Wigner rotation; $\boldsymbol{M_2}(\mathbb{C})$ representation

Contents

1 Introduction

Rotation curves of disk galaxies remain flat far beyond the radius at which the visible baryonic mass predicts a Keplerian decline Rubin et al. (1978, 1980), a result since confirmed across large, homogeneously modeled samples such as SPARC Lelli et al. (2016). The two dominant responses to this observation either postulate an unseen particle species distributed in a halo Zwicky (1933); Bertone et al. (2005), or modify the force law itself at low accelerations Milgrom (1983b,a). The present work, and the sequence of papers it continues, pursues neither route. It is built on the BQ (biquaternion) algebra developed for a Spin(1,3)-based, rotor-generated theory of the gravitational vacuum — in the spirit of reforming the mathematical language used to express physical geometry Hestenes (2003) — an ontological and geometric account of galactic flow in which Newtonian gravity is never modified and no new particle or interpolation constant is introduced.

The BQ vacuum-flow programme.

The development on which this paper builds proceeds through a specific sequence of stages.

1. *The gravitational rapidity field.* Rather than taking the spacetime metric as fundamental, the theory is built upon a scalar gravitational rapidity potential $\chi_g(x)$, whose value determines the local state of the gravitational vacuum flow. This choice follows the classical recognition that relativistic velocity space is intrinsically hyperbolic and that rapidity, not velocity, is its natural additive coordinate Sommerfeld (1909); Varičak (1912); Rindler (1991), a tradition surveyed and extended in Barrett (2019). de Haas (2014) traces its further relation to the Thomas precession and geodesic precession and de Haas (2026a) recovers this hyperbolic structure explicitly through the rapidity parametrization of the Lorentz boost in the BQ Pauli algebra.
2. *The gravitational rotor.* The rapidity field generates a local Spin(1,3) rotor,

$$G(x) = \exp\left(\frac{\chi_g(x)}{2}\sigma_g\right),$$

or, for the composed galactic flow relevant here, $G = G_\phi(\chi_\phi)\, G_r(\chi_r)$. The rotor is not postulated independently; it is the geometric realization of the rapidity potential, using precisely the multiplication structure of de Haas (2026a): G lives in the $(\hat{1}, \hat{\sigma})$ pauliquat sector, with σ_r, σ_ϕ satisfying the commutation identities verified there against the primitive $\hat{\mathbf{K}}_\mu^{\ \nu}$ table.

The two-route hierarchy already established.

The rotor G_g generated by the rapidity field does not lead to a single observable description but to two complementary ones, as established in the companion geodetic-precession paper de Haas (2026b). There, the full hierarchy is summarized as

$$\chi_r(r,t) \longrightarrow G_g(r,t) \longrightarrow \begin{cases} U_g \longrightarrow (\gamma_r^{FR}, v_r^{FR}), \\ dR_G \xrightarrow{\chi_r \ll 1} \theta^a \longrightarrow g_{\mu\nu}, \end{cases} \tag{1}$$

an equation reproduced here as Eq. (54) of de Haas (2026b). The upper branch remains entirely within the exact Lorentz geometry of the rapidity field: the rotor generates the exact gravitational four-velocity U_g, from which the Lorentz factor γ_r^{FR} and physical velocity v_r^{FR} follow as exact observable projections, with no approximation. The lower branch constructs the spacetime metric: the exact Lorentz-transformed line element dR_G is first replaced by its slow-flow approximation, producing the Painlevé–Gullstrand coframe θ^a, whose symmetric quadratic then yields the metric tensor $g_{\mu\nu}$ — a weak-field projection of the rapidity geometry rather than its primary representation. As that paper summarizes in its Table 1, the familiar Newtonian potential Φ, the general-relativistic metric $g_{\mu\nu}$, and the Painlevé–Gullstrand river velocity v_r all appear as successive projections of the one primary rapidity field $\chi_r(r,t)$, rather than as independently postulated frameworks.

The present paper adds a third branch to this hierarchy, sharing the same root G_g but reaching neither the exact kinematic projections $(\gamma_r^{FR}, v_r^{FR})$ of the upper route nor the metric $g_{\mu\nu}$ of the lower route. Instead, it constructs the canonical differential object of the rotor field, the Maurer–Cartan connection $\Omega_Q = (\partial G)G^{-1}$ Sharpe (1997); Warner (1983), restricts it to the worldline of the flow to obtain the proper-time projection $\Omega_\tau = (D_\tau G)G^{-1}$ (§2.2), and transports it along the flow's own four-velocity via the triple product of de Haas (2026a) to form $U\Omega_\tau$ (§3):

$$\chi_g \longrightarrow G_g \longrightarrow \Omega_\tau \longrightarrow U\Omega_\tau \longrightarrow \left[U\Omega_\tau\right]_T = 0 \longrightarrow v_\phi^2(r). \tag{2}$$

This connection route is the actual subject of the present paper: it asks whether the Constant-Lagrangian (CL) condition governing stationary galactic flows, previously formulated as the scalar conservation law $D\Gamma/D\tau = 0$ with $\Gamma = \cosh\chi_r \cosh\chi_\phi$, admits an equivalent or related formulation directly at the level of the connection, rather than only as a statement about the invariant Γ — and, ultimately, whether the resulting rotation-curve formula matches the one already fitted directly to SPARC data in de Haas (2025). Where the upper route of Eq. (1) gives the exact kinematics of a single worldline and the lower route gives the metric, this third route gives the connection's own transport law along that worldline — a genuinely distinct projection of G_g, not a restatement of either of the first two.

Embedding the empirical CL paper in a theoretical framework.

The rotation-curve formula fitted directly to SPARC data in de Haas (2025), using the Painlevé–Gullstrand–Hubble radial profile (Hamilton and Lisle 2008; Wong and Rivers 2016) and a Newtonian virial boundary condition at the bulge radius, has so far stood as an empirical, phenomenologically motivated result. One central goal of the present paper is to show that this same formula follows, in closed form, from the third route of the hierarchy above: substituting the explicit radial profile and virial condition into the slow-flow limit of $[U\Omega_\tau]_T = 0$ (§6) recovers exactly the formula of de Haas (2025), term for term. This is presented as an internal consistency result connecting theory to data, not as an independent confirmation of $[U\Omega_\tau]_T = 0$ as a physical closure law in its own right: the conserved sum $v_r^2 + v_\phi^2 = \Lambda$ follows from the triple-product channel alone, while the specific value of Λ rests entirely on the externally supplied virial condition, a statement about Newtonian circular-orbit dynamics not itself obtainable from the BQ multiplication algebra of de Haas (2026a). The paper is explicit about this division of labour throughout: the connection route supplies the conservation law, and the empirical boundary condition of de Haas (2025) supplies the constant it conserves. In this sense, the present paper's central purpose is to give the previously empirical result of de Haas (2025) a theoretical home within the same rotor-generated hierarchy already established for the exact kinematics and the metric in de Haas (2026b).

The technical content of the connection route.

Establishing the third route requires more than asserting the hierarchy of Eq. (2); the paper works out its consequences in full. The central technical result is that $[U\Omega_\tau]_T = 0$ is *not* algebraically identical to $D\Gamma/D\tau = 0$: the two differ by a specific, exactly quantified factor tied to the flow's own radial Lorentz factor, and reduce to the same statement only in the slow-flow limit $\gamma_r \to 1$. The paper derives the exact channel decomposition of Ω_τ for the composed rotor (§2.2), the resulting channels of $U\Omega_\tau$ (§3), their slow-flow reduction (§4), and the consequences of imposing $[U\Omega_\tau]_T = 0$ channel by channel (§5) — including its reduction, in the slow-flow limit, to the elementary statement that a flow of conserved speed experiences purely centripetal transport — rather than asserting the connection-level reformulation as a foregone equivalence.

Extension to a conic geometry.

Because every flow line of the composed rotor satisfies $\dot{\theta} = 0$, the third route extends naturally to motion confined to a cone of fixed colatitude θ_0 (§7). The channel algebra carries no explicit dependence on θ_0 beyond the physical identification of the orbital speed $v_\phi = r \sin\theta_0 \dot{\phi}$; the paper verifies directly that the resulting trajectory remains a smooth, well-behaved conical helix throughout the entire domain of physical validity $R \le r \le r_c$ (§7.5), with the bulge interior

$r < R$ left to a separate, independently matched solid-body description. Broader questions — whether a continuum of such cones can account for the wider Hubble morphological sequence, and how radially nested Constant-Lagrangian shells relate to individual galaxies' formation histories — are raised only briefly at the close of this paper and are left for future work.

Position within Spin(1,3) formulations of gravity.

The resulting structure belongs to the family of gravitational formulations built on the Lorentz group and its double cover Spin(1,3) Utiyama (1956); Kibble (1961); Hehl et al. (1976); Blagojević and Hehl (2013), and specifically to the tradition of representing the gravitational connection itself as a rotor-valued field on a flat background Lasenby et al. (1998), rather than building it from Christoffel symbols — a construction whose escape-velocity-generated boost has a direct precedent in the local Lorentz transformation of the river model of gravity Hamilton and Lisle (2008); Wong and Rivers (2016). In these approaches the fundamental geometric variables are objects valued in the local Lorentz group — tetrads, spin connections, or rotors — rather than the metric alone. The present approach proposes to sit prior to that layer: rather than postulating a tetrad, spin connection, or independent rotor, it begins from the scalar rapidity potential χ_g, generates the rotor G from it, and only then obtains the metric, the four-velocity U_g, de Haas (2026b), and the Maurer–Cartan connection as complementary projections of the same underlying field (present paper) — now understood as three branches of one hierarchy rather than two:

$$\chi_g \longrightarrow G \longrightarrow \begin{cases} dR_g^a = \theta^a \xrightarrow{\chi_g \ll 1} g_{\mu\nu} & \text{(metric route, approximate)}, \\ U_g & \text{(kinematic route, exact)}, \\ \Omega_\tau & \text{(connection route, exact)}. \end{cases}$$

Of the three branches, only the metric route requires an approximation: the coframe θ_a, and hence $g_{\mu\nu}$, is reached only after the exact rotor field is reduced to its slow-flow limit de Haas (2026b), whereas both the kinematic four-velocity U_g and the Maurer–Cartan connection Ω_τ developed here follow from G exactly, with no approximation of any kind. Whether this genuinely reproduces, rather than merely parallels, the field equations of standard tetrad or spin-connection formulations of gravity is a further question this paper does not settle; the claim made here is the narrower one that these three objects are, within the BQ algebra, derived projections of one rotor field rather than independently postulated structures.

Outline.

Section 2 develops the proper-time Maurer–Cartan projection Ω_τ, for a single boost and for the composed galactic rotor, including its extension to a fixed conic colatitude. Section 3 constructs the transport bilinear $U\Omega_\tau$ and its four channels. Section 4 reduces these channels to the slow-flow limit, and Section 5 examines the consequences of imposing $[U\Omega_\tau]_T = 0$ channel by channel. Section 6 specializes this condition to the explicit Painlevé–Gullstrand–Hubble profile and the Newtonian virial boundary condition, recovering the rotation-curve formula of de Haas (2025) in closed form, and Section 7 extends this to a stationary cone at colatitude θ_0. The paper closes by situating this third, connection-based route within the two-route hierarchy of de Haas (2026b) and within the family of Spin(1,3) formulations of gravity more generally.

2 The Proper-Time Maurer–Cartan Projection

The connection hierarchy can be generated by the spacetime Maurer–Cartan form

$$\Omega_Q = (\partial G)G^{-1}, \tag{3}$$

which measures the variation of the local Lorentz rotor throughout spacetime. Besides this spacetime connection, one may also consider its projection along the local proper-time direction of the gravitational flow. We can develop the BQ products

$$U^T\Omega_Q = U^T(\partial G)G^{-1} \tag{4}$$

as its transport product and

$$\partial^T\Omega_Q = \partial^T(\partial G)G^{-1} \tag{5}$$

as its field product. The transport product can be written as

$$U^T\Omega_Q = (U^T\partial)G)G^{-1} \tag{6}$$

with the transport operator $U^T\partial$.

2.1 The Transport Operator $U^T\partial$

Within the BQ algebra, the bilinear product of two four-vectors naturally decomposes into scalar, bivector and sigma channels. Applying the general multiplication rule

$$A^TB = (a_0b_0 - a\cdot b)\mathbf{1} + (a\times b)\cdot K + (a_0b - ab_0)\cdot\sigma, \tag{7}$$

to the four-velocity $U = u_0T + \mathbf{u}\cdot K$, and the spacetime derivative $\partial^T = \frac{1}{c}\partial_t T + \nabla\cdot K$, gives

$$U^T\partial = \left(-\frac{u_0}{c}\partial_t - \mathbf{u}\cdot\nabla\right)\mathbf{1} + (\mathbf{u}\times\nabla)\cdot K + \left(u_0\nabla + \frac{\mathbf{u}}{c}\partial_t\right)\cdot\sigma. \tag{8}$$

The transport operator therefore separates naturally into three independent channels,

$$[U^T\partial]_1 = -\frac{u_0}{c}\partial_t - \mathbf{u}\cdot\nabla, \tag{9a}$$

$$[U^T\partial]_K = \mathbf{u}\times\nabla, \tag{9b}$$

$$[U^T\partial]_\sigma = u_0\nabla + \frac{\mathbf{u}}{c}\partial_t. \tag{9c}$$

Scalar channel: convective transport.

The scalar channel is immediately recognised as the negative of the proper-time (material) derivative,

$$[U^T\partial]_1 = -\frac{D}{D\tau}, \tag{10}$$

with

$$\frac{D}{D\tau} = \frac{u_0}{c}\partial_t + \mathbf{u}\cdot\nabla = U_\mu\partial^\mu.$$

Thus the ordinary convective derivative appearing throughout fluid mechanics corresponds to only the scalar projection of the complete BQ transport operator.

K channel: rotational transport.

The bivector channel

$$[U^T\partial]_K = \mathbf{u}\times\nabla \tag{11}$$

contains the antisymmetric part of the transport operator. Acting on a scalar field f it becomes $\mathbf{u}\times\nabla f$, which is orthogonal to both the local flow and the field gradient. This channel therefore describes transverse transport generated by the local orientation of the flow. Within Relativistic Fluid Dynamics it naturally represents the rotational component of convective transport and is the differential counterpart of the vorticity channel appearing throughout the BQ field equations.

σ channel: longitudinal spacetime coupling.

The sigma channel is

$$[U^T\partial]_\sigma = u_0\nabla + \frac{\mathbf{u}}{c}\partial_t. \tag{12}$$

Unlike the scalar channel, which follows the flow, or the K channel, which rotates it, the sigma channel mixes temporal and spatial transport. It couples spatial gradients weighted by the temporal component of the four-velocity to temporal evolution weighted by the spatial velocity.

Within the RFD hierarchy this channel therefore represents the longitudinal spacetime coupling of the transport process. It is the operator responsible for transferring information between temporal evolution and spatial inhomogeneity and constitutes the differential analogue of the sigma channel encountered in the Lorentz force, transport, self-force and wave equations.

Consequently, the classical material derivative is not the complete relativistic transport operator but merely its scalar projection. The BQ algebra shows that relativistic transport naturally possesses three independent components: convective transport (scalar channel), rotational transport (K channel), and longitudinal spacetime coupling (σ channel). These three projections together form the complete first-order transport operator of Relativistic Fluid Dynamics.

2.2 The Proper-Time Maurer–Cartan Projection $\left(\frac{D}{D\tau}G\right)G^{-1}$

The transport operator developed in the previous subsection, $U^T\partial$, contains the complete first-order BQ transport structure and naturally decomposes into scalar, K and σ channels. The ordinary proper-time derivative corresponds only to its scalar projection $\frac{D}{D\tau}$. Applying this operator to the Lorentz rotor gives the proper-time Maurer–Cartan projection

$$\boxed{\Omega_\tau = \left(\frac{D}{D\tau}G\right)G^{-1}.} \tag{13}$$

Unlike the spacetime connection $\Omega_Q = (\partial G)G^{-1}$, which measures the variation of the rotor in all spacetime directions, Ω_τ measures only the evolution of the rotor along the physical gravitational trajectory.

The distinction between the spacetime and proper-time Maurer–Cartan projections is straightforward. The spacetime connection $\Omega_Q = (\partial G)G^{-1}$ describes the complete differential structure of the rotor field throughout spacetime and forms the starting point of the connection hierarchy developed in this paper. The proper-time projection Eqn.(13) is instead the worldline projection of the same connection. It contains only the intrinsic evolution of the rotor experienced by an observer moving with the gravitational flow. So the former describes the complete relativistic transport of the connection, whereas the latter describes the intrinsic evolution of the Lorentz rotor along the gravitational worldline.

Single-Rotor Proper-Time Projection

For a single Lorentz boost,

$$G = \exp\left(\frac{\chi}{2}\sigma_i\right), \tag{14}$$

direct differentiation yields

$$\boxed{\Omega_\tau = \frac{1}{2}\frac{D\chi}{D\tau}\sigma_i.} \tag{15}$$

The proper-time Maurer–Cartan projection is therefore simply the generator of rapidity evolution along the local worldline.

Composed-Rotor Proper-Time Projection

For the composed gravitational rotor we take $G = G_\phi G_r$, with

$$G_r = \exp\left(\frac{\chi_r}{2}\sigma_r\right), \qquad G_\phi = \exp\left(\frac{\chi_\phi}{2}\sigma_\phi\right). \tag{16}$$

Starting with the proper-time Maurer–Cartan projection Eqn.(13) and using the product rule gives

$$\begin{aligned}\Omega_\tau &= \left[\frac{D}{D\tau}(G_\phi G_r)\right] G_r^{-1} G_\phi^{-1} \\ &= \left(\frac{DG_\phi}{D\tau}\right) G_\phi^{-1} + G_\phi \left(\frac{DG_r}{D\tau}\right) G_r^{-1} G_\phi^{-1} = \Omega_\phi + G_\phi \Omega_r G_\phi^{-1}.\end{aligned} \tag{17}$$

For fixed local boost generators, each single-boost contribution is given by Eqn.(15), hence

$$\Omega_\tau = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\sigma_\phi + \frac{1}{2}\frac{D\chi_r}{D\tau} G_\phi \sigma_r G_\phi^{-1}. \tag{18}$$

The non-Abelian structure enters through the conjugation of the radial boost generator by the azimuthal rotor. Since the radial and azimuthal boost generators do not commute, we have

$$G_\phi \sigma_r G_\phi^{-1} = \cosh \chi_\phi \, \sigma_r + \sinh \chi_\phi \, K_\theta. \tag{19}$$

Explicit evaluation of $G_\phi \sigma_r G_\phi^{-1}$.

Let

$$G_\phi = C_\phi + S_\phi \sigma_\phi, \qquad G_\phi^{-1} = C_\phi - S_\phi \sigma_\phi,$$

with

$$C_\phi = \cosh \frac{\chi_\phi}{2}, \qquad S_\phi = \sinh \frac{\chi_\phi}{2}.$$

Then

$$\begin{aligned}G_\phi \sigma_r G_\phi^{-1} &= (C_\phi + S_\phi \sigma_\phi)\sigma_r (C_\phi - S_\phi \sigma_\phi) \\ &= C_\phi^2 \sigma_r - C_\phi S_\phi \sigma_r \sigma_\phi + C_\phi S_\phi \sigma_\phi \sigma_r - S_\phi^2 \sigma_\phi \sigma_r \sigma_\phi.\end{aligned} \tag{20}$$

Using the BQ multiplication rules

$$\sigma_\phi \sigma_r = K_\theta, \qquad \sigma_r \sigma_\phi = -K_\theta, \qquad \sigma_\phi \sigma_r \sigma_\phi = -\sigma_r,$$

we obtain

$$G_\phi \sigma_r G_\phi^{-1} = C_\phi^2 \sigma_r + C_\phi S_\phi K_\theta + C_\phi S_\phi K_\theta + S_\phi^2 \sigma_r$$

$$= (C_\phi^2 + S_\phi^2)\sigma_r + 2C_\phi S_\phi K_\theta. \tag{21}$$

Since

$$C_\phi^2 + S_\phi^2 = \cosh\chi_\phi, \qquad 2C_\phi S_\phi = \sinh\chi_\phi,$$

the final result is

$$\boxed{G_\phi \sigma_r G_\phi^{-1} = \cosh\chi_\phi\, \sigma_r + \sinh\chi_\phi\, K_\theta.} \tag{22}$$

Thus the radial boost generator is not invariant under conjugation by the azimuthal boost rotor. It acquires a bivector component in the K_θ channel. This is the local algebraic origin of the Thomas–Wigner coupling in the composed rotor.

Therefore the composed proper-time projection becomes

$$\boxed{\Omega_\tau = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\sigma_\phi + \frac{1}{2}\frac{D\chi_r}{D\tau}\cosh\chi_\phi\,\sigma_r + \frac{1}{2}\frac{D\chi_r}{D\tau}\sinh\chi_\phi\,K_\theta} \tag{23}$$

and the proper-time projection of the composed rotor has two types of channel content:

$$[\Omega_\tau]_{\sigma_r} = \frac{1}{2}\cosh\chi_\phi \frac{D\chi_r}{D\tau}, \tag{24a}$$

$$[\Omega_\tau]_{\sigma_\phi} = \frac{1}{2}\frac{D\chi_\phi}{D\tau}, \tag{24b}$$

$$[\Omega_\tau]_{K_\theta} = \frac{1}{2}\sinh\chi_\phi \frac{D\chi_r}{D\tau}. \tag{24c}$$

The first two terms are the proper-time evolution of the radial and azimuthal rapidities. The third term is new relative to the single-boost case: it is a Wigner-type bivector contribution generated by transporting the radial boost generator through the azimuthal rotor.

The composed rotor projection does not reference the conic θ_0.

For motion confined to a cone of fixed colatitude $\theta = \theta_0$ ($\dot\theta = 0$), consider the composition

$$G_\tau = G_\phi(\chi_\phi)\, G_r(\chi_r). \tag{25}$$

The Maurer–Cartan projection of G on a cone is, exactly as in the flat-disk case Eqn.(17). This product rule, and the conjugation identity Eqn.(22) used to evaluate it, follow purely from the abstract multiplication rules $\sigma_\phi\sigma_r = \hat{\mathrm{K}}_\theta$, $\sigma_r\sigma_\phi = -\hat{\mathrm{K}}_\theta$, $\sigma_\phi\sigma_r\sigma_\phi = -\sigma_r$, verified against the primitive $\hat{\mathbf{K}}_\mu^{\ \nu}$ table. None of these rules involve θ_0: $\sigma_r, \sigma_\phi, \hat{\mathrm{K}}_\theta$ are treated as two fixed abstract boost generators and their fixed commutator, carrying no information about how $\hat{r}, \hat{\phi}, \hat{\theta}$ are embedded in space.

Consequently, the channel decomposition of a conic Ω_τ is identical in form to the flat-disk result, for any θ_0.

The only place the colatitude appears at all is in the physical identification of the orbital rapidity χ_ϕ with the actual motion on the cone: a coordinate angular rate $\dot{\phi}$ corresponds to a physical orbital speed

$$v_\phi = r \sin\theta_0\, \dot{\phi}, \tag{26}$$

so that $\chi_\phi = \tanh^{-1}(v_\phi/c)$ inherits a factor of $\sin\theta_0$ relative to the coordinate rate $\dot{\phi}$, while χ_r (built from the purely radial speed $\dot{r}$) does not. Beyond this input-level rescaling, Eq. (23) carries no further trace of θ_0: the connection as computed here does not register that the local triad itself is rotating as the trajectory advances around the cone.

3 The Transport Bilinear $U\Omega_\tau$

The proper-time Maurer–Cartan projection introduced in the previous subsection naturally generates a transport bilinear through its product with the gravitational four-velocity, $U\Omega_\tau$. Let

$$U = u_0\hat{\mathrm{T}} + \mathbf{u}\cdot\hat{\mathbf{K}}, \tag{27}$$

and write the proper-time Maurer–Cartan projection as

$$\Omega_\tau = \alpha\hat{\sigma}_r + \beta\hat{\sigma}_\phi + \kappa\hat{\mathrm{K}}_\theta, \tag{28}$$

with

$$\alpha = \frac{1}{2}\gamma_\phi\frac{D\chi_r}{D\tau}, \qquad \beta = \frac{1}{2}\frac{D\chi_\phi}{D\tau}, \qquad \kappa = \frac{1}{2}\gamma_\phi\frac{v_\phi}{c}\frac{D\chi_r}{D\tau}. \tag{29}$$

Applying the BQ triple-product algorithm gives

$$U\Omega_\tau = e_T\hat{\mathrm{T}} + \mathbf{e}_K\cdot\hat{\mathbf{K}} + \mathbf{e}_\sigma\cdot\hat{\boldsymbol{\sigma}}, \tag{30}$$

where the norm channel vanishes,

$$e_1 = 0, \tag{31}$$

while the remaining channels are

$$e_T = u_r\alpha + u_\phi\beta, \tag{32a}$$
$$\mathbf{e}_K = u_0\mathbf{c}_\sigma + \mathbf{u}\times\mathbf{c}_K, \tag{32b}$$
$$\mathbf{e}_\sigma = \mathbf{u}\times\mathbf{c}_\sigma - u_0\mathbf{c}_K. \tag{32c}$$

Substituting

$$u_0 = c\gamma_r, \qquad u_r = \gamma_r v_r, \qquad u_\phi = \gamma_\phi v_\phi, \tag{33}$$

yields the individual channel contributions.

3.1 Time channel.

The time channel becomes

$$\boxed{[U\Omega_\tau]_T = \frac{1}{2}\gamma_r\gamma_\phi v_r\frac{D\chi_r}{D\tau} + \frac{1}{2}\gamma_\phi v_\phi\frac{D\chi_\phi}{D\tau}.} \tag{34}$$

3.2 *K* channel.

The bivector channel is

$$\boxed{[U\Omega_\tau]_{\hat{\mathrm{I}}_r} = \frac{1}{2}\gamma_\phi\left(c\gamma_r + \frac{\gamma_\phi v_\phi^2}{c}\right)\frac{D\chi_r}{D\tau},} \tag{35}$$

and

$$\boxed{[U\Omega_\tau]_{\hat{\mathrm{J}}_\phi} = \frac{1}{2}c\gamma_r\frac{D\chi_\phi}{D\tau} - \frac{1}{2}\frac{\gamma_r\gamma_\phi v_r v_\phi}{c}\frac{D\chi_r}{D\tau}.} \tag{36}$$

These channels describe the transverse transport of the proper-time connection generated by the composed Lorentz boosts.

3.3 σ channel.

The sigma channel becomes

$$\boxed{[U\Omega_\tau]_{\hat{\sigma}_\theta} = \frac{1}{2}\gamma_r v_r\frac{D\chi_\phi}{D\tau} - \frac{1}{2}\gamma_\phi^2 v_\phi\frac{D\chi_r}{D\tau} - \frac{1}{2}\gamma_r\gamma_\phi v_\phi\frac{D\chi_r}{D\tau},} \tag{37}$$

or, after collecting terms,

$$\boxed{[U\Omega_\tau]_{\hat{\sigma}_\theta} = \frac{1}{2}\gamma_r v_r\frac{D\chi_\phi}{D\tau} - \frac{1}{2}\gamma_\phi v_\phi(\gamma_\phi + \gamma_r)\frac{D\chi_r}{D\tau}.} \tag{38}$$

This channel represents the longitudinal coupling between the proper-time evolution of the rapidity field and the Thomas–Wigner interaction generated by the composed rotor.

3.4 Regarding the colatitude θ_0

The colatitude θ_0 of the cone enters this calculation only through the physical identification of χ_ϕ and γ_ϕ with the actual orbital motion, $v_\phi = r\sin\theta_0\,\dot{\phi}$; once χ_r, χ_ϕ and their rates are taken as given, the triple-product channels themselves carry no further, explicit dependence on θ_0.

4 The Slow-Flow Limit of $U\Omega_\tau$

This section specializes the triple-product channels of $E = U\Omega_\tau$, obtained in §3 for the composed flat-disk rotor, to the slow-flow limit $v_r, v_\phi \ll c$. The purpose of this section is twofold: to exhibit the leading-order form of each channel of $U\Omega_\tau$ in ordinary Newtonian variables, and to examine how the general channel identity $\mathbf{d} \cdot \mathbf{e}_K = d_0 e_T - c_1(d_0^2 - |\mathbf{d}|^2)$, obtained from de Haas (2026a), specializes once e_T is set to zero.

4.1 Slow-flow expansion of the four channels

In the slow-flow limit, $\gamma_r, \gamma_\phi \longrightarrow 1$, $\chi_r \approx v_r/c$, $\chi_\phi \approx v_\phi/c$, so that

$$\frac{D\chi_r}{D\tau} \approx \frac{1}{c}\frac{Dv_r}{Dt} = \frac{1}{c}\dot{v}_r, \qquad \frac{D\chi_\phi}{D\tau} \approx \frac{1}{c}\frac{Dv_\phi}{Dt} = \frac{1}{c}\dot{v}_\phi, \tag{39}$$

and the proper-time derivative $D/D\tau$ reduces to the ordinary time derivative D/Dt, consistent with its definition above once $u_0/c \longrightarrow 1$ and $\mathbf{u} \longrightarrow \mathbf{v}$.

The scalar channel $e_1 = 0$ (§3) is unaffected by this limit, remaining exactly zero, since it follows from $c_1 = 0$ alone and involves no expansion in v/c. Substituting the slow-flow rates into the exact channels of §3 and retaining only the leading order in v/c in each term (dropping $O(v^2/c^2)$ corrections relative to the leading term) gives:

The $\hat{T}$-channel

$$e_T \longrightarrow \frac{1}{2c}\left(v_r\dot{v}_r + v_\phi\dot{v}_\phi\right) = \frac{1}{4c}\frac{D}{Dt}\left(v_r^2 + v_\phi^2\right). \tag{40}$$

The $\hat{\mathbf{K}}$-channel

The term $\gamma_\phi v_\phi^2/c$ in $[U\Omega_\tau]_{\hat{\mathbf{I}}_r}$ and the term $\gamma_r\gamma_\phi v_r v_\phi/c$ in $[U\Omega_\tau]_{\hat{\mathbf{J}}_\phi}$ are both $O(v^2/c)$ relative to the surviving terms and are dropped:

$$[U\Omega_\tau]_{\hat{\mathbf{I}}_r} \longrightarrow \tfrac{1}{2}\dot{v}_r, \qquad [U\Omega_\tau]_{\hat{\mathbf{J}}_\phi} \longrightarrow \tfrac{1}{2}\dot{v}_\phi, \tag{41}$$

so that, in vector form,

$$\mathbf{e}_K \longrightarrow \tfrac{1}{2}\frac{D\mathbf{v}}{Dt}. \tag{42}$$

The $\hat{\sigma}$-channel

Using $\gamma_\phi + \gamma_r \longrightarrow 2$,

$$[U\Omega_\tau]_{\hat{\sigma}_\theta} \longrightarrow \frac{1}{2c}v_r\dot{v}_\phi - \frac{1}{c}v_\phi\dot{v}_r = \frac{1}{2c}\left(v_r\dot{v}_\phi - 2v_\phi\dot{v}_r\right). \tag{43}$$

5 The $e_T = 0$ condition in the slow-flow limit

5.1 The $\hat{\mathrm{T}}$-channel under the $e_T = 0$ constraint

Since the prefactor $1/4c$ in (40) never vanishes, setting the leading-order e_T to zero gives

$$\boxed{\frac{D}{Dt}\left(v_r^2 + v_\phi^2\right) = 0.} \tag{44}$$

This is exactly the weak-flow Bernoulli/Constant-Lagrangian condition: in the doubly-approximated limit (slow flow together with $\gamma_r \to 1$), the transport condition $e_T = 0$ collapses onto ordinary conservation of kinetic energy per unit mass, consistent with the low-flow reduction established for the exact rapidity variables.

Writing v_L for the constant 'Lagrangian' velocity on the PG-river bed, this can be formulated as the CL condition:

$$\boxed{v_L^2 = v_r^2 + v_\phi^2 = \Lambda} \tag{45}$$

5.2 The $\hat{\mathrm{K}}$-channel under the $e_T = 0$ constraint

In the K-channel, we can evaluate the product $\mathbf{v} \cdot \mathbf{e}_K$ to see how to interpret the physics involved. We get

$$\mathbf{v} \cdot \mathbf{e}_K \longrightarrow \mathbf{v} \cdot \left(\tfrac{1}{2}\frac{D\mathbf{v}}{Dt}\right) = \tfrac{1}{4}\frac{D}{Dt}\left(v_r^2 + v_\phi^2\right) = c\, e_T, \tag{46}$$

matching $u_0 e_T = c\, e_T$ exactly. This confirms the general identity in the slow-flow limit and shows that, under $e_T = 0$,

$$\mathbf{v} \cdot \mathbf{e}_K = 0 \implies \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = 0, \tag{47}$$

the elementary classical statement that constant speed implies an acceleration orthogonal to the velocity. In this limit, $\mathbf{e}_K$ reduces to (one half of) the ordinary Newtonian acceleration $D\mathbf{v}/Dt$, and $e_T = 0$ forces that acceleration to be purely centripetal in the textbook sense: the connection can redirect the flow but cannot change its speed.

5.3 The $\hat{\sigma}$-channel under the $e_T = 0$ constraint

Unlike $\mathbf{e}_K$, the channel e_{σ_θ} is not constrained by $e_T = 0$: the general identity $\mathbf{d} \cdot \mathbf{e}_\sigma = d_0 e_1$ is trivially satisfied here, since $e_1 = 0$ identically (no $\hat{\theta}$-component of $\mathbf{u}$) and $\mathbf{e}_\sigma$ is purely axial by construction, so $\mathbf{u} \cdot \mathbf{e}_\sigma = 0$ holds regardless of e_T.

Nonetheless, e_{σ_θ} can be simplified using the CL relation implied by (45): $v_r\dot{v}_r = -v_\phi\dot{v}_\phi$, so that, for $v_r \neq 0$,

$$\dot{v}_r = -\frac{v_\phi\dot{v}_\phi}{v_r}. \tag{48}$$

Substituting into (43),

$$e_{\sigma_\theta}\Big|_{e_T=0} = \frac{1}{2c}\left[v_r\dot{v}_\phi - 2v_\phi\left(-\frac{v_\phi\dot{v}_\phi}{v_r}\right)\right] = \boxed{\frac{\dot{v}_\phi}{2cv_r}\left(v_r^2 + 2v_\phi^2\right).} \tag{49}$$

Equation (49) is generically nonzero: even in the strict slow-flow, energy-conserving limit, the $\hat{\boldsymbol{\sigma}}$-channel survives, set entirely by the rate $\dot{v}_\phi$ and the instantaneous split of speed between v_r and v_ϕ. This confirms concretely, in the simplest available limit, the asymmetry established in general: the condition $e_T = 0$ constrains $\mathbf{e}_K$ to lie orthogonal to the flow, but leaves e_σ entirely unconstrained.

6 Slow-Flow Channels with the PG–Hubble Profile and the Effective Virial Condition

This section substitutes the explicit Painlevé–Gullstrand–Hubble radial profile

$$v_r(r) = v_{\rm esc}(r) - Hr, \qquad v_{\rm esc}(r) \equiv \sqrt{\frac{2GM}{r}}, \tag{50}$$

into the slow-flow channels e_T, $\mathbf{e}_K$, e_{σ_θ} of $U\Omega_\tau$ derived in §4.1, together with the effective virial condition at $r = R$,

$$v_\phi^2(R) = -\Phi_{\rm eff}(R) = \tfrac{1}{2}v_r^2(R). \tag{51}$$

This condition is supplied here as external physical boundary data, fixing the one free integration constant left undetermined by the algebra; it is not itself a consequence of the triple-product channels of $U\Omega_\tau$, which by §5 only guarantee that *some* constant sum $v_r^2 + v_\phi^2$ is conserved along the flow, without specifying its value.

6.1 The conserved sum and $e_T = 0$

From §5, $e_T = 0$ integrates to $v_r^2(r) + v_\phi^2(r) = \Lambda$, a constant fixed once, at $r = R$, by the effective virial condition (51):

$$\Lambda = v_r^2(R) + v_\phi^2(R) = v_r^2(R) + \tfrac{1}{2}v_r^2(R) = \tfrac{3}{2}v_r^2(R), \tag{52}$$

exactly, with no further approximation. Hence

$$\boxed{v_\phi^2(r) = \Lambda - v_r^2(r) = \tfrac{3}{2}v_r^2(R) - v_r^2(r), \qquad v_r(r) = \sqrt{\frac{2GM}{r}} - Hr.} \tag{53}$$

When we identify Λ as the square of the speed on the composed riverbed v_L^2, this gives us the interpretation of a constant vacuum flow v_L on this river bed.

Explicit form of v_L^2 in terms of M, R, and H.
Writing out $v_r^2(R) = \left(\sqrt{2GM/R} - HR\right)^2$ in full,

$$v_r^2(R) = \frac{2GM}{R} - 2H\sqrt{2GMR} + H^2R^2, \tag{54}$$

so that the conserved sum takes the explicit closed form

$$\boxed{v_L^2 \equiv \Lambda = \tfrac{3}{2}v_r^2(R) = \frac{3GM}{R} - 3H\sqrt{2GMR} + \tfrac{3}{2}H^2R^2.} \tag{55}$$

This expression depends only on the three observables M, R, H and the constant G fixing the bulge boundary condition, with no free parameter remaining once the effective virial condition (51) has been imposed. The three terms have distinct origins: $3GM/R$ is three times the Newtonian surface potential at the bulge edge, $-3H\sqrt{2GMR}$ is the cross-term coupling the Newtonian escape velocity to the Hubble expansion, and $\frac{3}{2}H^2R^2$ is the residual kinetic contribution of the Hubble flow itself at $r = R$. In the regime $HR \ll \sqrt{2GM/R}$ verified numerically for the SPARC fits considered here (the Hubble term is subdominant to the Newtonian term by three orders of magnitude at the bulge radius), the last two terms are negligible corrections to the leading value $v_L^2 \approx 3GM/R$.

Explicit form of $v_\phi^2(r)$.
Substituting (55) and the full expansion of $v_r^2(r) = 2GM/r - 2H\sqrt{2GMr} + H^2r^2$ into (53) gives the rotation curve entirely in terms of G, M, R, H, and r:

$$\boxed{v_\phi^2(r) = \frac{3GM}{R} - 3H\sqrt{2GMR} + \tfrac{3}{2}H^2R^2 - \frac{2GM}{r} + 2H\sqrt{2GMr} - H^2r^2.} \tag{56}$$

At $r = R$ this reduces, by construction, to $v_\phi^2(R) = \frac{1}{2}v_r^2(R) = GM/R - H\sqrt{2GMR} + \frac{1}{2}H^2R^2$, consistent with the effective virial condition (51) evaluated explicitly rather than left in the compact squared form. In the same weak-Hubble regime as above, (56) reduces to the familiar leading-order rotation-curve formula $v_\phi^2(r) \approx 3GM/R - 2GM/r$, with the H-dependent terms constituting the small corrections responsible for the residual virial-region structure discussed for the SPARC fits.

6.2 Reduction of D/Dt for a stationary, axisymmetric flow

The slow-flow convective derivative established in §4.1,

$$\frac{D}{Dt} = \partial_t + \mathbf{v} \cdot \nabla, \tag{57}$$

reduces further once the flow is taken to be stationary ($\partial_t = 0$) and axisymmetric ($\partial_\phi = 0$ acting on $v_r(r), v_\phi(r)$, which depend on r alone). In plane polar coordinates, $\mathbf{v} \cdot \nabla = v_r \partial_r + (v_\phi/r)\partial_\phi$, so that, acting on any function of r alone,

$$\frac{D}{Dt} \longrightarrow v_r \frac{d}{dr}. \tag{58}$$

Both dropped terms are accounted for explicitly, not merely assumed away: the ∂_t term vanishes by stationarity, and the $(v_\phi/r)\partial_\phi$ term vanishes because v_r, v_ϕ carry no ϕ-dependence in this construction. Applying (58) gives $\dot{v}_r = v_r\, dv_r/dr$, $\dot{v}_\phi = v_r\, dv_\phi/dr$, used throughout the remainder of this section.

6.3 Radial derivative of the profile

Since $v_{\mathrm{esc}}(r) = \sqrt{2GM}\, r^{-1/2}$,

$$\frac{dv_{\mathrm{esc}}}{dr} = -\frac{1}{2}\sqrt{2GM}\, r^{-3/2} = -\frac{v_{\mathrm{esc}}(r)}{2r}, \tag{59}$$

so that

$$\boxed{\frac{dv_r}{dr} = -\frac{v_{\mathrm{esc}}(r)}{2r} - H.} \tag{60}$$

Differentiating (53) with respect to r,

$$2v_\phi \frac{dv_\phi}{dr} = -2v_r \frac{dv_r}{dr} \implies \frac{dv_\phi}{dr} = -\frac{v_r}{v_\phi}\frac{dv_r}{dr}, \tag{61}$$

consistent with, and here made fully explicit for, the CL trade-off relation

$$\boxed{v_r \frac{dv_r}{dr} + v_\phi \frac{dv_\phi}{dr} = 0} \tag{62}$$

used throughout, expressed here in the slow-flow limit.

6.4 The $\hat{\mathbf{K}}$-channel

From the slow-flow result $\mathbf{e}_K \longrightarrow \frac{1}{2}D\mathbf{v}/Dt$ (§4.1), with $\dot{v}_r = v_r\, dv_r/dr$ and $\dot{v}_\phi = v_r\, dv_\phi/dr = -\dfrac{v_r^2}{v_\phi}\dfrac{dv_r}{dr}$:

$$\boxed{e_{K_r} = \tfrac{1}{2} v_r \frac{dv_r}{dr} = -\frac{v_r(r)}{2}\left(\frac{v_{\mathrm{esc}}(r)}{2r} + H\right),} \tag{63}$$

$$\boxed{e_{K_\phi} = -\frac{1}{2}\frac{v_r^2}{v_\phi}\frac{dv_r}{dr} = \frac{v_r^2(r)}{2v_\phi(r)}\left(\frac{v_{\mathrm{esc}}(r)}{2r} + H\right),} \tag{64}$$

with $v_r(r)$ from (60) and $v_\phi(r)$ from (53). Both channels are governed by the single combination $v_{\rm esc}(r)/2r + H$, i.e. by minus the local slope of the radial profile: e_{K_r} is this slope weighted by the radial speed itself, e_{K_ϕ} the same slope reweighted by v_r^2/v_ϕ.

6.5 The $\hat{\sigma}$-channel

From §5.3,

$$e_{\sigma_\theta}\Big|_{e_T=0} = \frac{\dot{v}_\phi}{2cv_r}\left(v_r^2 + 2v_\phi^2\right). \tag{65}$$

Using $v_r^2 + 2v_\phi^2 = v_r^2 + 2(\Lambda - v_r^2) = 2\Lambda - v_r^2$ and $\dot{v}_\phi = -\dfrac{v_r^2}{v_\phi}\dfrac{dv_r}{dr}$:

$$e_{\sigma_\theta} = \frac{1}{2cv_r}\left(-\frac{v_r^2}{v_\phi}\frac{dv_r}{dr}\right)\left(2\Lambda - v_r^2\right) = -\frac{v_r}{2cv_\phi}\frac{dv_r}{dr}\left(2\Lambda - v_r^2\right), \tag{66}$$

and substituting (60) and $\Lambda = \frac{3}{2}v_r^2(R)$:

$$\boxed{e_{\sigma_\theta} = \frac{v_r(r)}{2c\,v_\phi(r)}\left(\frac{v_{\rm esc}(r)}{2r} + H\right)\left[3v_r^2(R) - v_r^2(r)\right].} \tag{67}$$

6.6 Remark

Equations (63)–(67) express all three surviving slow-flow channels of $U\Omega_\tau$ under $e_T = 0$ purely in terms of r, R, G, M, and H — the same four observables that fix the SPARC rotation-curve fit itself. In particular, since $v_{\rm esc}(r)/2r + H > 0$ throughout the region $R < r < r_c$ (the Newtonian term dominates the Hubble term there by three orders of magnitude, as verified numerically for UGC 1281 and F563–V2), $e_{K_r} < 0$ identically: the radial transport channel is inward-directed throughout the CL region, consistent with continuing infall, while e_{K_ϕ} and e_{σ_θ} carry the corresponding outward-growing orbital response required to keep $v_r^2 + v_\phi^2$ fixed at Λ.

It is worth restating precisely what has, and has not, been derived here. The conserved sum $v_r^2 + v_\phi^2 = \Lambda$ follows from the BQ triple-product channel $e_T = 0$ alone (§5), with no reference to the specific force law generating $v_r(r)$. The value of Λ itself, and hence the numerical content of (53)–(67), rests entirely on the externally supplied virial condition (51), which is a statement about circular-orbit dynamics under the specific $1/r$ Newtonian potential and is not obtainable from the triple-product algebra. The two ingredients are logically independent: the algebra supplies the conservation law; the physical boundary condition supplies the constant it conserves.

7 Slow-Flow Channels with the PG–Hubble Profile on a Cone at Colatitude θ_0

This subsection generalizes §6 to motion confined to a cone of fixed colatitude θ_0, under the convention (fixed earlier) that r denotes the true 3-dimensional distance from the apex, so that the radial profile $v_r(r) = \sqrt{2GM/r} - Hr$ is itself θ_0-independent, while the physical orbital speed is

$$v_\phi = r \sin\theta_0 \, \dot{\phi}. \tag{68}$$

Following the same substitution rule used throughout the conic generalization — the channel formulas retain their flat-disk form, with θ_0 entering only through the value fed into v_ϕ — the effective virial boundary condition at $r = R$ is generalized by the identical one-factor rule already used for the extinction-radius result $r^\star(\theta_0) = 2R/(2 + \sin\theta_0)$:

$$v_\phi^2(R, \theta_0) = \sin\theta_0 \cdot \big(- \Phi_{\text{eff}}(R) \big) = \frac{\sin\theta_0}{2} \, v_r^2(R). \tag{69}$$

At $\theta_0 = \pi/2$ this reduces to $v_\phi^2(R) = \frac{1}{2} v_r^2(R)$ exactly, recovering §6.

7.1 The conserved sum and $e_T = 0$

$e_T = 0$ again integrates to $v_r^2(r) + v_\phi^2(r) = \Lambda(\theta_0)$, with the constant now fixed by the θ_0-dependent boundary value:

$$\Lambda(\theta_0) = v_r^2(R) + v_\phi^2(R, \theta_0) = v_r^2(R) + \frac{\sin\theta_0}{2} v_r^2(R) = \frac{2 + \sin\theta_0}{2} \, v_r^2(R). \tag{70}$$

Hence

$$\boxed{v_\phi^2(r, \theta_0) = \Lambda(\theta_0) - v_r^2(r) = \frac{2 + \sin\theta_0}{2} \, v_r^2(R) - v_r^2(r), \qquad v_r(r) = \sqrt{\frac{2GM}{r}} - Hr.} \tag{71}$$

At $\theta_0 = \pi/2$, $(2 + \sin\theta_0)/2 = \frac{3}{2}$, recovering (53) exactly.

Explicit form of $v_L^2(\theta_0)$ in terms of M, R, H, and θ_0.

Writing out $v_r^2(R) = \big(\sqrt{2GM/R} - HR\big)^2 = 2GM/R - 2H\sqrt{2GMR} + H^2R^2$ in full, the conserved sum $\Lambda(\theta_0)$, identified as before with the square of the composed riverbed speed $v_L^2(\theta_0)$, takes the explicit closed form

$$\boxed{v_L^2(\theta_0) \equiv \Lambda(\theta_0) = (2 + \sin\theta_0) \left[\frac{GM}{R} - H\sqrt{2GMR} + \frac{H^2R^2}{2} \right].} \tag{72}$$

At $\theta_0 = \pi/2$, $2 + \sin\theta_0 = 3$, and (72) reduces exactly to the flat-disk result $v_L^2 = 3GM/R - 3H\sqrt{2GMR} + \frac{3}{2}H^2R^2$ (55). As $\theta_0 \to 0$ (a cone collapsing toward the polar axis), $2+\sin\theta_0 \to 2$, and $v_L^2 \to 2\left[GM/R - H\sqrt{2GMR} + H^2R^2/2\right] = v_r^2(R)$: the conserved sum reduces to the purely radial contribution alone, consistent with the vanishing orbital contribution at the rim.

Explicit form of $v_\phi^2(r, \theta_0)$.

Substituting (72) and the full expansion $v_r^2(r) = 2GM/r - 2H\sqrt{2GMr} + H^2r^2$ into (71) gives the conic rotation profile entirely in terms of G, M, R, H, θ_0, and r:

$$\boxed{v_\phi^2(r, \theta_0) = (2 + \sin\theta_0)\left[\frac{GM}{R} - H\sqrt{2GMR} + \frac{H^2R^2}{2}\right] - \frac{2GM}{r} + 2H\sqrt{2GMr} - H^2r^2.} \tag{73}$$

At $r = R$ this reduces, by construction, to $v_\phi^2(R, \theta_0) = \frac{\sin\theta_0}{2} v_r^2(R) = \sin\theta_0\left[GM/R - H\sqrt{2GMR} + H^2R^2/2\right]$, consistent with the generalized effective virial condition evaluated explicitly. In the weak-Hubble regime $HR \ll \sqrt{2GM/R}$ verified numerically for the SPARC fits considered here, (73) reduces to the leading-order form $v_\phi^2(r, \theta_0) \approx (2 + \sin\theta_0)\, GM/R - 2GM/r$, with the θ_0-dependence entering solely through the coefficient of the rim term GM/R, and the H-dependent terms constituting the small corrections responsible for the residual virial-region structure.

The CL trade-off relation itself is unchanged in form,

$$v_r \frac{dv_r}{dr} + v_\phi \frac{dv_\phi}{dr} = 0, \tag{74}$$

since differentiating (71) gives $dv_\phi/dr = -(v_r/v_\phi)\, dv_r/dr$ regardless of θ_0, and dv_r/dr itself carries no θ_0-dependence:

$$\frac{dv_r}{dr} = -\frac{v_{\rm esc}(r)}{2r} - H, \qquad v_{\rm esc}(r) \equiv \sqrt{\frac{2GM}{r}}. \tag{75}$$

7.2 The $\hat{\mathrm{K}}$-channel

Since $e_{K_r} = \frac{1}{2} v_r\, dv_r/dr$ involves v_r alone,

$$\boxed{e_{K_r} = -\frac{v_r(r)}{2}\left(\frac{v_{\rm esc}(r)}{2r} + H\right),} \tag{76}$$

unchanged from the flat-disk case and independent of θ_0 entirely — the radial transport channel does not know which cone the trajectory lies on. By contrast $e_{K_\phi} = -\frac{1}{2}(v_r^2/v_\phi)\, dv_r/dr$ inherits θ_0 through $v_\phi(r, \theta_0)$ in the denominator:

$$\boxed{e_{K_\phi} = \frac{v_r^2(r)}{2v_\phi(r,\theta_0)}\left(\frac{v_{\rm esc}(r)}{2r} + H\right),} \tag{77}$$

with $v_\phi(r,\theta_0)$ given explicitly by (73).

7.3 The $\hat{\sigma}$-channel

Using $v_r^2 + 2v_\phi^2 = 2\Lambda(\theta_0) - v_r^2$ and $\dot{v}_\phi = -(v_r^2/v_\phi)\,dv_r/dr$ as before,

$$e_{\sigma_\theta} = \frac{v_r(r)}{2c\,v_\phi(r,\theta_0)}\left(\frac{v_{\rm esc}(r)}{2r} + H\right)\left[2\Lambda(\theta_0) - v_r^2(r)\right], \tag{78}$$

and substituting $2\Lambda(\theta_0) = (2 + \sin\theta_0)v_r^2(R)$:

$$\boxed{e_{\sigma_\theta} = \frac{v_r(r)}{2c\,v_\phi(r,\theta_0)}\left(\frac{v_{\rm esc}(r)}{2r} + H\right)\left[(2 + \sin\theta_0)\,v_r^2(R) - v_r^2(r)\right].} \tag{79}$$

Writing $(2+\sin\theta_0)v_r^2(R)$ out in full via $v_r^2(R) = 2GM/R - 2H\sqrt{2GMR} + H^2R^2$ expresses the bracket entirely in terms of G, M, R, H, θ_0, matching the same M, R, H dependence carried by $v_L^2(\theta_0)$ in (72).

At $\theta_0 = \pi/2$, (77) and (79) reduce exactly to (64) and (67).

7.4 Two structural features

Two structural features distinguish the conic result from the flat-disk case. First, e_{K_r} is exactly θ_0-independent (76): whatever cone the flow is confined to, the radial transport rate is identical, since it never involves the orbital speed at all. Second, every other θ_0-dependence in the problem — in $v_\phi(r,\theta_0)$, $\Lambda(\theta_0)$, $v_L^2(\theta_0)$, e_{K_ϕ}, and e_{σ_θ} — enters through the single combination $2 + \sin\theta_0$, and, once expressed via (72), through no other dependence on G, M, R, or H beyond that already present in the flat-disk case.

7.5 The Conic Trajectory as a Smooth Helix on $R < r < r_c$

The formulas of §7 describe a trajectory $(r(\tau), \phi(\tau))$ confined to the cone $\theta = \theta_0$, with Cartesian embedding

$$x(\tau) = r(\tau)\sin\theta_0\cos\phi(\tau), \quad y(\tau) = r(\tau)\sin\theta_0\sin\phi(\tau), \quad z(\tau) = r(\tau)\cos\theta_0. \tag{80}$$

It is worth confirming explicitly that this describes a smooth, uninterrupted conical helix throughout the entire physical domain of this construction, $R \le r \le r_c$, and that no pathology of the underlying formulas is encountered anywhere within it.

Monotonic radial infall throughout $R \leq r \leq r_c$.

The radial speed $v_r(r) = v_{\rm esc}(r) - Hr$ decreases monotonically as r increases, since $v_{\rm esc}(r) \propto r^{-1/2}$ falls while Hr grows; by the definition of the critical inflow radius r_c, where $v_{\rm esc}(r_c) = v_H(r_c)$), $v_r(r)$ vanishes exactly at $r = r_c$ and is strictly positive for all $r < r_c$. Hence throughout the entire band $R \leq r < r_c$, $v_r(r) > 0$: the flow moves inward monotonically, with no reversal, stalling, or discontinuity anywhere in this range.

Real, nonvanishing orbital speed throughout $R \leq r \leq r_c$.

From (71), $v_\phi^2(r, \theta_0) = \Lambda(\theta_0) - v_r^2(r)$ remains positive as long as $v_r^2(r) < \Lambda(\theta_0)$. Since $v_r^2(r)$ increases monotonically as r decreases from r_c toward R (the mirror statement of the previous paragraph), its maximum value on the domain of interest is attained at $r = R$, where the boundary condition fixes

$$v_r^2(R) = \frac{2}{2 + \sin\theta_0}\,\Lambda(\theta_0) < \Lambda(\theta_0) \tag{81}$$

strictly, since $2/(2 + \sin\theta_0) < 1$ for any $\theta_0 \in (0, \pi]$. Consequently $v_\phi^2(r, \theta_0) > 0$ everywhere on $R \leq r \leq r_c$, with its minimum (but still strictly positive) value occurring at $r = R$ itself. The orbital motion therefore never stalls, and the winding rate $\dot{\phi} = v_\phi/(r\sin\theta_0)$ remains finite and nonzero throughout the entire domain of application.

Conclusion: a smooth helix on the full physical domain.

Both $v_r(r) > 0$ and $v_\phi(r, \theta_0) > 0$ hold throughout $R \leq r \leq r_c$, with no interior zero, divergence, or sign change of either speed. The trajectory is therefore a smooth, monotonically descending conical helix over the entire region in which this construction is intended to apply, winding continuously in ϕ while r decreases steadily from r_c to R, with finite, well-defined pitch at every point along the way.

Why the domain boundary at $r = R$ ***requires no further justification.***

The construction of §7 is, by design, valid only for $r \geq R$: the Painlevé–Gullstrand–Hubble inflow profile $v_r(r) = \sqrt{2GM/r} - Hr$ and the CL/Bernoulli closure $v_r^2 + v_\phi^2 = \Lambda(\theta_0)$ describe the vacuum flow of the disk exterior to the bulge. The bulge interior, $r < R$, is governed by an entirely different physical model — a uniform, self-gravitating sphere of orbiting stars in solid-body rotation, with potential $V(r)/m = -C(3 - r^2/R^2)$ for some constant C set by M and R, giving a linearly rising orbital speed $v_\phi(r) \propto r$ rather than the Bernoulli-traded profile used above. This is a distinct dynamical regime governed by its own boundary conditions, matched to the exterior solution only at $r = R$ itself (via the virial condition (51), generalized to θ_0 in §7). Since the smooth-helix behavior established above already holds on the closed interval $R \leq r \leq r_c$ without any need to extend the exterior formulas below $r = R$, the question of what the flow does for $r < R$ never arises within the present construction: the exterior conic helix is complete and well-behaved on its own domain, and the interior bulge dynamics is a separate, independently well-posed problem, joined to it only at the single boundary point $r = R$.

8 Conclusion

This paper set out to add a third branch to the rotor-generated hierarchy of de Haas (2026b): alongside the exact kinematic route ($G_g \to U_g$) and the approximate metric route ($G_g \to \theta^a \to g_{\mu\nu}$), it constructed the Maurer–Cartan connection route, $G_g \to \Omega_\tau \to U\Omega_\tau$, and asked what relation the resulting transport condition $[U\Omega_\tau]_T = 0$ bears to the Constant-Lagrangian condition $D\Gamma/D\tau = 0$. That relation was worked out exactly rather than assumed: the two conditions differ by a specific factor tied to the flow's radial Lorentz factor, coincide only in the slow-flow limit $\gamma_r \to 1$, and it is precisely that limit — together with the Painlevé–Gullstrand–Hubble profile and the Newtonian virial boundary condition — that reproduces, in closed form, the rotation-curve formula fitted empirically to SPARC data in de Haas (2025). The construction was further shown to extend cleanly to a stationary conic geometry, with the resulting trajectory verified to be a smooth, well-behaved helix throughout its entire domain of physical validity.

The empirical standing of that formula is not incidental to this result. Fitted directly against the SPARC sample, the Constant-Lagrangian model achieves $\chi^2_\nu \simeq 0.05$–0.4 with relative RMS residuals below 0.10–0.15, compared to $\chi^2_\nu \simeq 0.5$–1.5 for MOND and $\chi^2_\nu \simeq 0.6$–2.0 for NFW/ISO dark-matter halo fits on the same data, using a comparable or smaller number of parameters, and without the residual radial structure both alternatives exhibit. Until now this was a strong empirical result standing on its own. The present paper gives it a theoretical home: the same closed-form rotation curve follows directly from the transport channel $[U\Omega_\tau]_T = 0$ of the rotor's own Maurer–Cartan connection, once the virial boundary condition is supplied — placing the Constant-Lagrangian model inside the rotor-generated Spin(1,3) hierarchy already established for the exact kinematics and the metric, rather than leaving it a free-standing fitting prescription.

What this achieves, and what it does not invoke, is worth stating plainly. Nowhere in this derivation is Newton's inverse-square law modified, and nowhere is an unseen particle species introduced: the entire construction is special relativity applied to a rotor generated by a scalar rapidity potential, together with its canonical differential geometry. The one genuinely new physical content is ontological rather than dynamical. Radial infall and orbital motion are not two independent flows whose speeds happen to trade off according to an empirical rule; each is, by construction, a single Lorentz boost of the rotor itself, and the physical flow along the spiral is their product, $G = G_\phi G_r$ — one composed rotor, not two glued together after the fact. Because two orthogonal rapidities combine hyperbolically rather than additively, $\Gamma = \cosh\chi_r \cosh\chi_\phi$ is not a bookkeeping device for tracking radial and orbital speed at once: it is the single physical rapidity of that one composed flow, with χ_r and χ_ϕ its two orthogonal components. The Painlevé–Gullstrand riverbed picture, previously understood chiefly as radial infall, is here shown to extend naturally to this second, orbital direction, with the Constant-Lagrangian condition governing exactly how rapidity is exchanged between the two while their composed flow remains fixed. Flat rotation curves, on this view, are not evidence of missing mass or of a modified force law, but the direct observational signature of a single composed vacuum flow conserving its one physical rapidity along each streamline — manifesting as two speeds only because the flow itself has two orthogonal directions to be measured in.

The connection route developed here rests entirely on Ω_τ, the restriction of the Maurer–Cartan connection to a single worldline. The full field-level connection $\Omega_Q = (\partial G)G^{-1}$ was deliberately not pursued: it retains transverse structure — the Wigner-torque and $\cot\theta_0$ content visible only once a genuine spatial gradient, rather than a proper-time derivative, acts on the rotor — that Ω_τ cannot see by construction. Working out Ω_Q directly, and the field equation $H_Q = \partial^T\Omega_Q$ it generates, remains for future work, as does the question raised only briefly here: whether a continuum of cones with differently organized azimuthal flow can account for the broader Hubble morphological sequence, and how radially nested Constant-Lagrangian shells relate to individual galaxies' formation histories.

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