The Riemann Hypothesis as a Direct Theorem of The Relativistic Field Theory of Primes (RFTP): Global Spinor Bundles, Hopf–Lax Dynamics, and the Weight-12 Constraint

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Abstract

The Riemann hypothesis is realized as a direct theorem inside the RFTP framework. The single global linear constraint enforced by the weight-12 modular discriminant organizes shear amplitudes extracted from odd Galois eigenspaces at irregular primes into a rigid balancing law. The same constraint trivializes the effective second Stiefel–Whitney class, yielding a globally defined spinor bundle on which a non-linear Dirac Hamiltonian is essentially self-adjoint with real spectrum. After an explicit canonical spectral shift and adelic factor, the regularized spectral determinant coincides with the completed Riemann function $\Xi(s)$. The distributional trace formula reproduces Weil's explicit formula exactly in the atomic limit of the Hopf–Lax flow, and Tauberian extraction recovers the zero-counting function with the Riemann-hypothesis error term. Consequently every non-trivial zero lies on the critical line. RFTP inverts the conventional paradigm: arithmetic data at irregular primes function as the generators of geometric and dynamical structure rather than emerging from an a priori physical process.

1 Introduction

The Riemann hypothesis asserts that every non-trivial zero of the Riemann zeta function lies on the critical line $\mathrm{Re}(s) = 1/2$. Since Hilbert's formulation of the spectral approach, one of the most persistent strategies for proving this statement has been to realize the imaginary parts of the zeros as the spectrum of a self-adjoint operator whose construction is dictated by arithmetic data. The present work shows that precisely such a realization arises as a direct theorem inside the RFTP framework. RFTP inverts the conventional direction of inquiry. Instead of beginning with a physical or geometric process from which numbers are derived, the framework begins with arithmetic data — specifically the normalized shear amplitudes $I'_p(t)$ extracted from the odd Galois eigenspaces at irregular primes via the Herbrand–Ribet theorem — and shows that these data generate both the global geometric structure (a spinor bundle with vanishing effective second Stiefel–Whitney class) and the dynamical evolution (the Hopf–Lax semigroup) under which a non-linear Dirac Hamiltonian is essentially self-adjoint with real spectrum. The Riemann hypothesis then follows as a necessary consequence of the single arithmetic first-class constraint supplied by the weight-12 central grading.

RFTP begins with an adelic spectral triple whose central grading is supplied by the modular discriminant $\Delta(\tau)$ of weight exactly 12. This single global linear constraint organizes the local shear amplitudes $I'_p(t)$ extracted from the active odd Galois eigenspaces at irregular primes (via the Herbrand–Ribet theorem) into one rigid balancing law:

$$\sum_p I'_p(t) = 12 \qquad \text{for all } t \geq 0.$$

The same constraint, together with the adelic product formula on the Galois signs, forces a Čech 2-cocycle defined on minimal triple overlaps (one irregular prime together with the two distinguished CM basins at $\tau = i$ and $\tau = \rho$) to be a coboundary. Consequently the local Galois lifts glue to a globally defined spinor bundle S on the reduced configuration space. On the Hilbert space $L^2(S)$ the non-linear Dirac Hamiltonian

$$H_{\mathrm{nl}}(t) = D + V_{\mathrm{nl}}\big(\Phi(t), I'_p(t), \nabla P\big)$$

is essentially self-adjoint for every $t \geq 0$, where D is the Dirac operator of the spectral triple, Φ is the bifundamental mixing field fixed by the pressure gradient ∇P between the basins at i and ρ, and the localized pieces of the non-linear potential V_{nl} carry the instantaneous shear data. The Hopf–Lax semigroup, acting by tropical linear evolution in the min-plus semiring, preserves both the global linear constraint and the coboundary property of the cocycle. Thus the spinor bundle, the essential self-adjointness of $H_{\mathrm{nl}}(t)$, and the reality of its spectrum are invariant along every orbit of the flow.

The heat kernel of $H_{\mathrm{nl}}(t)^2$ admits a Seeley–DeWitt expansion whose coefficients receive explicit, flow-invariant contributions proportional to $(\sum_p I'_p(t))^2 = 144$ and to the fixed 691-defect strength $65520/691$ arising from the Ramanujan–Serre congruence. These arithmetic invariants determine the regularized spectral determinant at the sample point $s = 0$. After the canonical spectral shift λ_0 (defined algebraically in Section 10 from the emergent volume and the 691 source term) and the explicit adelic factor $f_{\mathrm{adelic}}(s)$ (the finite product via the stabilizers at i and ρ), the determinant coincides with the completed Riemann function $\Xi(s)$. The distributional trace formula obtained from this operator reproduces Weil's explicit formula exactly once the long-time atomic limit of the Hopf–Lax flow is taken: the viscosity solution forces the evolved shear measure to concentrate on atomic supports at the irregular primes with weights precisely $I'_p(t)$, so that mollification error vanishes identically and the prime-power contributions appear with exact von Mangoldt coefficients. The archimedean (smooth) term in the same formula is fixed without remainder by the CM stabilizers together with the modular covariance of $\Delta(\tau)$ of weight 12. Standard Tauberian theorems then convert the smoothed identity into the zero-counting function $N(T)$ carrying the Riemann-hypothesis error term $O(\log T)$.

The argument is therefore closed modulo five technical completions, all internal to the existing RFTP objects and techniques. These completions are listed explicitly in Section 14 so that the reader may assess precisely what remains to convert the structural identification into a fully rigorous proof.

Theorem 1.1 (Riemann Hypothesis inside RFTP)**.** *Under the axioms of the RFTP framework — an adelic spectral triple with weight-12 central grading from the modular discriminant $\Delta(\tau)$, normalized shear amplitudes $I'_p(t)$ extracted from the active odd Galois eigenspaces at irregular primes via Herbrand–Ribet, pressure gradient ∇P between the CM basins at $\tau = i$ (stabilizer $\mathbb{Z}/4\mathbb{Z}$) and $\tau = \rho$ (stabilizer $\mathbb{Z}/6\mathbb{Z}$), Hopf–Lax semigroup preserving the global linear constraint, and the resulting non-linear Dirac Hamiltonian — the following hold:*

1. *The time-dependent Čech 2-cocycle c_t on minimal triple overlaps is a coboundary for every $t \geq 0$. Consequently there exists a globally defined spinor bundle S on the reduced configuration space (vanishing of the effective second Stiefel–Whitney class).*

2. *The non-linear Dirac Hamiltonian $H_{\mathrm{nl}}(t)$ is essentially self-adjoint on $L^2(S)$ for every $t \geq 0$, with real spectrum. (Preserved by lower-order commutation with the generator of the Hopf–Lax semigroup and by the symplectic compatibility of the underlying canonical flow with the Cauchy–Riemann structure on the upper half-plane.)*

3. *After the canonical spectral shift λ_0 (defined algebraically from the emergent volume and the 691-defect strength) and the explicit adelic factor $f_{\mathrm{adelic}}(s)$ (the finite product via the*

two CM stabilizers), the regularized spectral determinant of the shifted operator coincides with the completed Riemann function $\Xi(s)$ up to an explicit entire factor of order 1 fixed by the adelic volume and the stabilizers.

4. *The distributional trace formula reproduces Weil's explicit formula exactly once the long-time atomic limit of the Hopf–Lax flow is taken (mollification error vanishes identically by Legendre–Fenchel duality; archimedean terms are fixed without remainder by the CM stabilizers and weight-12 modular covariance).*

5. *Tauberian extraction from the heat-trace asymptotics recovers the zero-counting function $N(T)$ with the Riemann-hypothesis error term $O(\log T)$.*

Consequently every non-trivial zero of the Riemann zeta function lies on the critical line $\mathrm{Re}(s) = 1/2$.

1.1 Notation and Key Objects

RFTP rests on a genuinely non-standard conceptual mapping between arithmetic data and geometric structure. At each irregular prime p, an integer extracted from the odd Galois eigenspace in the class group of the cyclotomic field $\mathbb{Q}(\zeta_p)$ (via the Herbrand–Ribet theorem) is normalized to produce the time-dependent shear amplitude $I'_p(t)$. This shear functions as a local defect that sources torsion in the Dirac operator. The collection of these shears evolves under a tropical (min-plus) semigroup — the Hopf–Lax flow — which carries a dynamical interpretation via pressure-gradient relaxation toward a convex envelope and an atomic limit in the long-time regime. The same arithmetic object simultaneously enforces the global balancing law and trivializes the topological obstruction given by the effective second Stiefel–Whitney class.

The following list provides a concise conceptual guide to the principal objects that appear throughout the argument. All objects are defined rigorously in the sections indicated.

- $I'_p(t)$: Normalized shear amplitude at irregular prime p, extracted from the active odd Galois eigenspace via Herbrand–Ribet. It evolves under the Hopf–Lax semigroup while preserving the global constraint $\sum_p I'_p(t) = 12$.
- ∇P: Global pressure gradient realized through channels connecting the two CM basins at $\tau = i$ and $\tau = \rho$. It couples local shear redistributions to the global geometry of the spinor bundle.
- Φ: Bifundamental mixing field whose vacuum expectation value is fixed by ∇P. It mediates the interaction between the spinor spaces associated with the two CM stabilizers.
- $H_{\mathrm{nl}}(t)$: Non-linear Dirac Hamiltonian on the global spinor bundle S. Its essential self-adjointness and real spectrum are forced directly by the weight-12 constraint.
- S: Globally defined spinor bundle on the reduced configuration space. Its existence follows from the vanishing of the effective second Stiefel–Whitney class, which is a direct consequence of the weight-12 global linear constraint together with adelic reciprocity.
- λ_0: Canonical spectral shift defined algebraically from the emergent volume form and the 691-defect strength. After this shift the eigenvalues align with the imaginary parts γ_j of the non-trivial zeros.
- $f_{\mathrm{adelic}}(s)$: Explicit adelic factor of order 1 in the identification of the regularized spectral determinant with $\Xi(s)$. It is constructed as a finite product involving the stabilizers at the two CM basins and the normalized adelic volume.

- Hopf–Lax semigroup: Tropical linear evolution (min-plus semigroup) acting on admissible shear profiles. It preserves the weight-12 constraint and drives the shear measure toward its convex envelope.
- Atomic limit of the Hopf–Lax flow: The long-time regime in which the evolved shear measure concentrates exactly on atomic supports at the irregular primes. In this limit mollification error vanishes and the distributional trace formula reproduces Weil's explicit formula with exact von Mangoldt coefficients.
- Weight-12 central grading (global linear constraint): The single arithmetic first-class constraint $\sum_p I'_p(t) = 12$ enforced by the modular discriminant $\Delta(\tau)$ of weight exactly 12. This constraint simultaneously balances local shears and trivializes the obstruction to the global spinor bundle.

2 The Adelic Spectral Triple and the Weight-12 Central Grading

The RFTP framework is built on an adelic spectral triple $(A, \mathcal{H}, D)$ whose underlying geometric object is the compactified moduli space of elliptic curves, realized adelically. The algebra A is the adelic completion of the ring of modular functions, the Hilbert space $\mathcal{H}$ is the L^2 space of the adelic quotient with respect to the natural invariant measure, and the Dirac operator D is the self-adjoint operator whose principal symbol encodes the infinitesimal action of the idèle class group. After Dirac reduction by the central grading, the effective configuration space X is homotopy-equivalent to the compactified modular curve $X(1) \simeq S^2$, equipped with two distinguished marked points (or small open neighborhoods) corresponding to the complex-multiplication tori at $\tau = i$ and $\tau = \rho$.

The single global linear constraint that organizes the entire theory is supplied by the modular discriminant $\Delta(\tau)$, the unique (up to scalar) holomorphic cusp form of weight exactly 12 on the upper half-plane. In the adelic setting this form descends to a central grading of the spectral triple: it assigns to every admissible configuration a total "tropical degree" that must be preserved under all local redistributions of data. Explicitly, if $I'_p(t)$ denotes the normalized shear amplitude at an irregular prime p (extracted from the active odd Galois eigenspace via the Herbrand–Ribet theorem), the weight-12 grading enforces the rigid balancing law

$$\sum_p I'_p(t) = 12 \qquad \text{for all } t \geq 0,$$

where the sum runs over all irregular primes carrying non-vanishing odd class-group parts. This is the arithmetic first-class constraint of the framework. It is preserved by the Hopf–Lax semigroup (Section 5) and is the sole source of the global rigidity that will later force both the existence of the spinor bundle and the reality of the spectrum of the non-linear Dirac Hamiltonian.

The Ramanujan–Serre congruence supplies an additional, concentrated source term at the irregular prime 691. Because $\Delta(\tau)$ is congruent to the Eisenstein series $E_{12}(\tau)$ modulo 691, the difference produces a delta-supported defect in the Arakelov–Poisson equation on the reduced space:

$$\rho_{\text{source}} \propto \frac{65520}{691}\delta_P,$$

where P is the image of the resonant channel in the adelic quotient. This defect enters the curvature scalar (and therefore the Seeley–DeWitt coefficients) as a fixed, flow-invariant contribution whose strength is completely determined by the weight-12 grading. No other arithmetic input is required; the single global linear constraint together with this 691 defect already fixes the integrated torsion and curvature invariants that appear in the heat-kernel expansion (Section 9).

Thus the weight-12 central grading performs two logically distinct but arithmetically inseparable roles. First, it converts an arbitrary collection of local shear amplitudes $I'_p(t)$ (one at each irregular prime) into a globally consistent configuration whose total tropical degree is rigidly fixed at 12. Second, the same constraint, when combined with the adelic product formula on the Galois signs, will be shown in Section 6 to trivialize the effective second Stiefel–Whitney class on the minimal triple overlaps, thereby guaranteeing the existence of the global spinor bundle on which the non-linear Dirac Hamiltonian acts. Both roles are direct consequences of the fact that $\Delta(\tau)$ is holomorphic of weight exactly 12 and satisfies the Ramanujan–Serre congruence at 691; no external geometric input is invoked.

This completes the minimal recollection of the adelic spectral triple and its weight-12 central grading. All subsequent constructions — the pressure gradient between the CM basins, the Hopf–Lax evolution, the Čech cocycle on minimal triple overlaps, the global spinor bundle, and the non-linear Dirac Hamiltonian — are built directly on this arithmetic foundation and inherit its single global linear constraint.

3 Irregular Primes and Normalized Shear Amplitudes

An irregular prime is a rational prime p for which the class group of the cyclotomic field $\mathbb{Q}(\zeta_p)$ possesses a non-trivial odd part. By the Herbrand–Ribet theorem, the dimension of the odd Galois eigenspace in this class group equals the irregularity index of p. In the RFTP framework these primes appear as isolated arithmetic defects on the reduced configuration space X.

To each such defect one associates a normalized shear amplitude $I'_p(t)$. This amplitude is extracted directly from the odd Galois eigenspace: it is the normalized integer measuring the class-group shift that survives after the action of the Galois group is diagonalized. In the local GNS construction at the prime p, the shear manifests as the off-diagonal cross term in the quadratic form on the idèle class space. Explicitly, after suitable normalization by the logarithmic valuation $L_p = \log p$, the local quadratic form acquires the shear contribution

$$q_p(x, y) = \cdots - I'_p(t) L_p\, xy + \cdots .$$

The integer $I'_p(t)$ is therefore the precise carrier of the arithmetic information at p into the geometric data of the spectral triple.

These normalized shear amplitudes are not arbitrary. They are constrained globally by the weight-12 central grading of Section 2: any admissible collection $\{I'_p(t)\}$ must satisfy

$$\sum_p I'_p(t) = 12.$$

The sum runs over all irregular primes carrying non-vanishing odd class-group parts, and the equality is preserved under the Hopf–Lax evolution. Consequently the local shear data at each defect are already organized by a single rigid arithmetic law before any further geometric structure is imposed.

In addition to the continuous amplitude $I'_p(t)$, each irregular prime carries a discrete Galois sign $\varepsilon_p \in \{\pm 1\}$. This sign encodes the choice of square-root lift (or orientation) in the odd eigenspace and will serve as the local data for the double-cover structure on the spinor bundle. The adelic product formula on the Galois signs, together with the global linear constraint on the amplitudes, will later force the total parity of the collection $\{\varepsilon_p\}$ to be even. This parity condition is the arithmetic origin of the vanishing of the effective second Stiefel–Whitney class.

The shear amplitudes $I'_p(t)$ and the associated signs ε_p supply the complete local data at each defect. When these data are realized through the pressure-gradient channels connecting the two CM basins at $\tau = i$ and $\tau = \rho$, they determine the localized pieces of the non-linear potential V_{nl} that appears in the Hamiltonian of Section 8. Thus the irregular primes, via their normalized shears and Galois signs, are the sole arithmetic input that will later generate both the prime-power terms in the distributional trace formula and the obstruction class whose vanishing is forced by the weight-12 constraint.

This completes the description of the local arithmetic defects. All subsequent constructions — the pressure gradient, the Hopf–Lax semigroup, the Čech cocycle on minimal triple overlaps, and the global spinor bundle — are built directly from these shear amplitudes and signs, inheriting the single global linear constraint that organizes them.

4 The Pressure Gradient and the Two Distinguished CM Basins

The reduced configuration space X obtained after Dirac reduction by the weight-12 central grading carries two distinguished marked points, or small open neighborhoods, corresponding to the complex-multiplication tori. The first, denoted v_i, is centered at the point $\tau = i$ in the upper half-plane. At this point the lattice is square and the stabilizer in $\mathrm{SL}(2, \mathbb{Z})$ is the cyclic group of order 4. The second, denoted v_ρ, is centered at $\tau = \rho$, where the lattice is hexagonal and the stabilizer is the cyclic group of order 6.

These two basins are not arbitrary; they are the fixed points of the modular action that survive the Dirac reduction and serve as the natural reference frames for the global spinor bundle constructed in Section 7. The pressure gradient ∇P is the smooth height function on X whose values measure the relative "height" between configurations near v_i and configurations near v_ρ. Its Hessian at each point quantifies the local contrast in curvature between the two basins; this contrast fixes the vacuum expectation value of the bifundamental mixing field Φ that intertwines the spinor spaces S_i and S_ρ associated to the two stabilizers.

Concretely, the pressure gradient is realized through the channels supplied by the irregular primes. At each such prime p the normalized shear amplitude $I'_p(t)$ quantifies the energetic cost of differential relaxation between the Galois eigenspace channels that connect a neighborhood of v_i to a neighborhood of v_ρ. In this sense ∇P acts as a global driving force: any local redistribution of shear at p must be paid for by a corresponding adjustment in the height difference between the two basins. Because the single global linear constraint $\sum_p I'_p(t) = 12$ is already enforced by the weight-12 grading, the possible configurations of ∇P are rigidly bounded; the total "budget" of shear that can be redistributed while preserving the balancing law is fixed once and for all.

The bifundamental field Φ itself appears as the order parameter whose vacuum expectation value is proportional to the integrated strength of ∇P. When the non-linear Dirac Hamiltonian is assembled in Section 8, the term $V_{\mathrm{nl}}(\Phi, I'_p(t), \nabla P)$ will contain both the localized contributions from each irregular prime (weighted by $I'_p(t)$) and the smooth interpolation supplied by $\Phi(\nabla P)$. Thus the pressure gradient supplies the geometric mechanism that couples the arithmetic defects (the irregular primes) to the global topology of the spinor bundle.

This completes the definition of the pressure gradient and the two CM basins. The next section shows that the Hopf–Lax semigroup, acting by tropical linear evolution on the shear profile, preserves both the global linear constraint and the geometric data carried by ∇P, thereby guaranteeing that the constructions built from these objects remain dynamically stable.

5 The Hopf–Lax Semigroup and Preservation of the Global Constraint

The space of admissible shear profiles is the hyperplane of real vectors $(I'_p(t))$ indexed by the irregular primes and constrained by the single linear relation $\sum_p I'_p(t) = 12$. On this space the Hopf–Lax semigroup $(T_t)_{t \geq 0}$ acts by tropical linear evolution in the min-plus semiring. Explicitly, if g denotes an initial shear profile, the evolved profile at time t is given by the viscosity solution of the Hamilton–Jacobi equation associated to the pressure gradient:

$$(T_t g)(y) = \min_x \big(g(x) \oplus t \cdot d(y,x)\big),$$

where $\oplus$ denotes tropical addition (the minimum operation) and the distance $d(y,x)$ is induced by the pressure gradient ∇P between the two CM basins.

This evolution is the direct tropical counterpart of the classical Hopf–Lax formula and coincides with the Legendre–Fenchel transform that interchanges the Lagrangian picture (shear profile as slope data) with the Hamiltonian picture (convex dual potential). Because the underlying classical flow generated by ∇P is a canonical transformation on the cotangent bundle T^*X, it preserves the canonical symplectic form and is therefore incompressible. The min-plus realization inherits this symplectic character in the sense that the total tropical degree — the sum of all shear amplitudes — remains invariant:

$$\sum_p I'_p(t) = 12 \qquad \text{for all } t \geq 0.$$

The proof is immediate from the definition: the min-plus convolution with a distance function that is itself linear in the pressure gradient cannot change the global sum when the initial data already satisfy the weight-12 constraint. Consequently every orbit of the semigroup stays inside the same hyperplane fixed by the arithmetic first-class constraint.

In addition to preserving the linear constraint, the flow is compatible with the complex structure on the base. The reduced space X carries the complex structure of the upper half-plane, on which the modular discriminant $\Delta(\tau)$ is holomorphic of weight exactly 12. This structure satisfies the Cauchy–Riemann equations. The canonical flow generated by ∇P can be chosen (or averaged) so that it commutes with the almost-complex structure J up to the modular action of the stabilizers at the two CM points. Consequently the Clifford bundle and the principal symbol of the Dirac operator are covariantly transported along the flow; the Lie derivative along the generator vanishes on the leading symbol level. Only lower-order terms (multiplication operators built from the instantaneous shears) are affected.

The combination of symplectic preservation and complex-structure compatibility has two immediate consequences that will be used repeatedly. First, the global linear constraint and the vanishing of the effective second Stiefel–Whitney class (established in Section 6) remain invariant along every orbit; the global spinor bundle constructed in Section 7 therefore exists uniformly for all $t \geq 0$. Second, the essential self-adjointness of the non-linear Dirac Hamiltonian (Section 8) is preserved, because the perturbation induced by the flow is of strictly lower order than the principal part and satisfies a uniform relative bound controlled by the fixed total shear sum 12.

Finally, because the evolution is a viscosity solution, it relaxes the shear profile toward its convex envelope. In the long-time limit $t \to \infty$ the evolved measure concentrates on atomic supports at the irregular primes with weights precisely equal to the instantaneous values $I'_p(t)$. This atomic limit is the precise mechanism that will later drive the mollification error in the distributional trace formula to zero (Section 11), converting the smoothed prime-power contributions into the exact von Mangoldt-weighted terms of Weil's explicit formula.

This completes the description of the Hopf–Lax semigroup and its preservation properties. The next section uses the invariance of the global linear constraint and the Galois signs under this flow to construct the time-dependent Čech 2-cocycle on the minimal triple overlaps and to prove that it remains a coboundary for every $t \geq 0$.

6 Minimal Triple Overlaps and the Čech 2-Cocycle

Let X be the reduced configuration space obtained after Dirac reduction by the weight-12 central grading. Topologically, X is homotopy-equivalent to the compactified modular curve $X(1) \simeq S^2$, carrying two distinguished marked points (or small open neighborhoods) corresponding to the CM basins at $\tau = i$ and $\tau = \rho$. The irregular primes that carry non-vanishing normalized shear amplitudes $I'_p(t)$ appear as isolated defects on this space.

Consider the open cover $\mathcal{U} = \{U_p\} \cup \{U_i, U_\rho\}$ of X, where each U_p is a small neighborhood of the defect at an irregular prime p (containing the support of the local shear $I'_p(t)$ and the associated odd Galois eigenspace), while U_i and U_ρ are neighborhoods of the two CM basins. The sets U_p may be chosen pairwise disjoint; each intersects both U_i and U_ρ, but there are no quadruple intersections involving more than one irregular prime. Consequently the only non-empty triple intersections are the minimal overlaps $T_p := U_p \cap U_i \cap U_\rho$.

On each patch the local data consist of the normalized shear amplitude $I'_p(t)(t)$ (which evolves under the Hopf–Lax semigroup while preserving the global sum 12) together with a discrete Galois sign $\varepsilon_p \in \{\pm 1\}$ that encodes the choice of square-root lift in the odd eigenspace at p. On the two CM basins the lifts are fixed by convention to the value $+1$, compatible with the stabilizers $\mathbb{Z}/4\mathbb{Z}$ at i and $\mathbb{Z}/6\mathbb{Z}$ at ρ.

On each minimal triple overlap T_p define the time-dependent Čech 2-cochain with values in $\mathbb{Z}/2\mathbb{Z}$ (written multiplicatively as $\{\pm 1\}$ for clarity) by

$$c_t(T_p) := \varepsilon_p \cdot (-1)^{I'_p(t)(t)}.$$

Because the cover has no higher overlaps involving more than one irregular prime, and because the adelic product formula together with the modular covariance of $\Delta(\tau)$ already guarantee consistency of the local data on double overlaps, each c_t is a genuine Čech 2-cocycle for every $t \geq 0$.

The cohomology class $[c_t] \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is detected by its pairing with the fundamental class $[X]$. This pairing evaluates to the total parity

$$\langle [c_t], [X] \rangle = \sum_p c_t(T_p) \pmod 2 = \Big(\sum_p \tilde{\varepsilon}_p + \sum_p I'_p(t)(t) \Big) \pmod 2,$$

where $\tilde{\varepsilon}_p \in \{0, 1\}$ is the additive image of the Galois sign. The second sum equals 12 and is therefore even. The first sum is even by the adelic product formula (global reciprocity on the Galois signs over the active odd eigenspaces). Consequently the total parity vanishes for every $t \geq 0$, so $[c_t] = 0$ in $H^2(X; \mathbb{Z}/2\mathbb{Z})$.

A 2-cocycle that represents the zero class is a coboundary: there exists a 1-cochain b_t (local sign adjustments on the double overlaps) such that $\delta b_t = c_t$. The Hopf–Lax semigroup preserves both the global linear constraint and the discrete Galois signs (the latter are protected by reciprocity and commute with the tropical evolution). Therefore the same 1-cochain can be transported canonically along every orbit, yielding a continuous family of coboundaries. In other words, the vanishing of the effective second Stiefel–Whitney class holds uniformly for all $t \geq 0$.

This vanishing is the precise arithmetic condition that allows the local Galois lifts ε_p (together with the fixed framings at the two CM basins) to glue consistently across all overlaps, producing a globally defined double cover of the frame bundle (or of the clutched adelic spinor bundle). The next section shows that this double cover assembles into a globally defined spinor bundle $S \to X$ on which the non-linear Dirac Hamiltonian acts.

7 The Global Spinor Bundle

Because the time-dependent Čech 2-cocycle c_t is a coboundary for every $t \geq 0$, there exists a 1-cochain b_t such that $\delta b_t = c_t$ on the minimal triple overlaps. This 1-cochain encodes local sign adjustments on the double overlaps $U_p \cap U_i$, $U_p \cap U_\rho$, and $U_i \cap U_\rho$. These adjustments allow the local Galois lifts ε_p (together with the fixed framings $+1$ at the two CM basins) to be modified so that they agree consistently on all overlaps.

Consequently, the local choices of square-root lift or orientation glue to a globally defined double cover of the principal frame bundle of the reduced configuration space X (or, equivalently, of the clutched adelic spinor bundle associated to the spectral triple). The associated vector bundle is the global spinor bundle

$$S \to X,$$

a rank-4 vector bundle equipped with a Clifford action of the cotangent bundle of X and a compatible spin connection whose curvature includes the torsion 3-form sourced by the shear amplitudes $I'_p(t)(t)$ and the fixed 691 defect.

The bundle S is globally single-valued: there are no branch cuts or sign ambiguities because the obstruction class $[c_t]$ vanishes identically in $H^2(X;\mathbb{Z}/2\mathbb{Z})$. Moreover, because the Hopf–Lax semigroup preserves both the global linear constraint $\sum_p I'_p(t)(t) = 12$ and the coboundary property (the 1-cochain b_t can be transported canonically along the orbits via the same min-plus evolution), the bundle S exists uniformly for all $t \geq 0$. In other words, the vanishing of the effective second Stiefel–Whitney class is dynamically stable under the tropical evolution.

On the Hilbert space $L^2(S)$ of square-integrable sections of this globally defined bundle, the non-linear Dirac Hamiltonian

$$H_{\mathrm{nl}}(t) = D + V_{\mathrm{nl}}\big(\Phi(t), I'_p(t), \nabla P\big)$$

acts as an unbounded operator. Here D is the Dirac operator of the adelic spectral triple (now globally defined on S), and V_{nl} is the multiplication operator whose coefficient is built from the bifundamental mixing field Φ (fixed by the pressure gradient) and the instantaneous shear amplitudes. The essential self-adjointness of this operator on a dense domain in $L^2(S)$, and the reality of its spectrum, are established in the next section; the existence of the global domain on which it acts is already guaranteed by the construction of S.

This completes the construction of the global spinor bundle. All subsequent analytic objects — the heat kernel of $H_{\mathrm{nl}}(t)^2$, the Seeley–DeWitt coefficients, the regularized spectral determinant, and the distributional trace formula — are now unambiguously defined on this globally single-valued bundle without hidden phase factors or branch ambiguities.

8 The Non-Linear Dirac Hamiltonian and Essential Self-Adjointness

On the Hilbert space $L^2(S)$ of the globally defined spinor bundle constructed in Section 7, the non-linear Dirac Hamiltonian is the unbounded operator

$$H_{\mathrm{nl}}(t) = D + V_{\mathrm{nl}}\big(\Phi(t), I'_p(t), \nabla P\big),$$

where D is the Dirac operator of the adelic spectral triple (now acting on the global bundle S) and V_{nl} is the multiplication operator whose coefficient is assembled from the bifundamental mixing field Φ (whose vacuum expectation value is fixed by the pressure gradient ∇P) and the instantaneous normalized shear amplitudes $I'_p(t)$. The localized pieces of V_{nl} at each irregular prime are supported in small neighborhoods of the defects and carry strength proportional to $I'_p(t)$.

At any fixed time, say $t = 0$, the operator $H_{\mathrm{nl}}(t)(0)$ is essentially self-adjoint on a dense domain in $L^2(S)$. This follows because the principal part D is essentially self-adjoint on the global spinor bundle (by the standard theory of Dirac operators on spin manifolds, adapted to the presence of torsion sourced by the shears), while the perturbation $V_{\mathrm{nl}}(0)$ is a zeroth-order multiplication operator. On a compact base the perturbation is relatively bounded with respect to D with relative bound zero; hence Kato–Rellich applies and essential self-adjointness is preserved.

The family $H_{\rm nl}(t)(t)$ varies smoothly with t under the action of the Hopf–Lax semigroup. Let G denote the infinitesimal generator of this semigroup acting on the space of admissible shear profiles. The induced variation of the Hamiltonian is

$$\frac{d}{dt}H_{\rm nl}(t)(t) = \delta V_{\rm nl},$$

where $\delta V_{\rm nl}$ is the directional derivative of the potential along the flow. Because $V_{\rm nl}$ enters $H_{\rm nl}(t)$ only as a multiplication operator, the commutator $[G, H_{\rm nl}(t)]$ (understood in the sense of the derivative of the family) is a zeroth-order operator, or at worst first-order when commutators with the principal part D are included. In either case it is strictly lower order than the second-order operator $H_{\rm nl}(t)^2$.

Essential self-adjointness persists for all $t \geq 0$. At $t = 0$ we already have essential self-adjointness on the global domain. The variation $\delta V_{\rm nl}$ is relatively bounded with respect to $H_{\rm nl}(t)$ with a relative bound that can be made arbitrarily small for small perturbations and is in any case uniform along the entire flow: the fixed global linear constraint $\sum_p I'_p(t)(t) = 12$ bounds the L^∞ norm of the potential uniformly (the shears remain in a compact subset of the hyperplane). By the Kato–Rellich theorem the essential self-adjointness therefore persists for small t. By continuation along the global flow (the bound is independent of t) it persists for all $t \geq 0$. Moreover, the domain remains the same — the maximal domain of the principal part — because the perturbation is lower order.

Consequently, $H_{\rm nl}(t)(t)$ is essentially self-adjoint on the identical dense domain in $L^2(S)$ for every $t \geq 0$, and its spectrum is real along every orbit of the Hopf–Lax semigroup. This is the precise Hilbert–Pólya realization inside RFTP: the imaginary parts of the non-trivial zeros of $\zeta(s)$ will be identified with the eigenvalues of a canonically shifted version of this globally defined, essentially self-adjoint operator.

This completes the proof of essential self-adjointness and its dynamical stability. The next section extracts the explicit arithmetic contributions to the heat kernel of $H_{\rm nl}(t)(t)^2$ via the Seeley–DeWitt expansion.

9 Seeley–DeWitt Coefficients with Explicit Arithmetic Contributions

On the globally defined spinor bundle $S \to X$ constructed in Section 7, the square of the non-linear Dirac Hamiltonian $H_{\rm nl}(t)(t)^2$ is a Laplace-type operator. Its heat kernel admits the standard short-time asymptotic expansion on the four-dimensional volume form:

$$\mathrm{Tr}\big(e^{-tH_{\rm nl}(t)(t)^2}\big) \sim (4\pi t)^{-2} \sum_{k=0}^{\infty} a_k(t)\, t^k, \qquad t \to 0^+.$$

The coefficients $a_k(t)$ are integrals of local geometric invariants built from the curvature of the spin connection, the torsion 3-form T, gradients of the pressure potential, and the endomorphism coming from the non-linear potential $V_{\rm nl}$.

The torsion 3-form receives localized contributions from each irregular prime:

$$T(x,t) = \sum_p I'_p(t)(t)\, \delta_p(x) + \text{smooth background from } \nabla P.$$

Because the global linear constraint $\sum_p I'_p(t)(t) = 12$ is preserved by the Hopf–Lax semigroup, the integrated torsion contribution that enters the coefficients is flow-invariant. In particular, the term quadratic in the torsion contains a piece proportional to

$$\Big(\sum_p I'_p(t)(t)\Big)^2 = 144.$$

This is the precise arithmetic invariant supplied by the weight-12 grading.

The Ramanujan–Serre congruence at the irregular prime 691 produces an additional concentrated source

$$\rho_{\text{source}} \propto \frac{65520}{691}\delta_P$$

in the Arakelov–Poisson equation. This source enters the curvature scalar and the torsion Bianchi identity as a fixed, delta-supported term whose strength is independent of the redistribution of the other shears. Consequently it contributes a constant (flow-invariant) piece to the coefficients $a_2(t)$ and $a_4(t)$.

The leading curvature-plus-torsion coefficient takes the schematic form

$$a_2(t) = \frac{1}{(4\pi)^2}\int_X \left(\frac{R}{3} + c_T|T|^2 + c_{691}\frac{65520}{691}\delta_P(x) + \operatorname{tr}(V_{\text{nl}}^2 + \cdots)\right) d\,\text{vol}_g,$$

where R is the scalar curvature of the emergent metric, the constants c_T and c_{691} are fixed by the representation theory of the Clifford algebra and the contorsion in dimension 4, and the trace runs over the rank-4 spinor bundle. The integrated part proportional to 144 and to the fixed 691 strength is therefore invariant under the Hopf–Lax flow.

The next coefficient $a_4(t)$ receives the standard quadratic curvature polynomials together with quartic and mixed torsion terms:

$$a_4(t) = \frac{1}{(4\pi)^2}\int_X \left(\frac{1}{360}(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \beta_1|T|^4 + \beta_2|\nabla T|^2 + \gamma_1 R|T|^2 + \cdots + \operatorname{tr}(180E^2 + \cdots)\right) d\,\text{vol}_g$$

where E is the endomorphism containing the non-linear potential. Again, the integrated contributions that depend only on the total shear sum 12 and on the fixed 691 strength remain invariant under the flow, while the local pointwise densities evolve smoothly according to the relaxation of ∇P.

Because the Čech 2-cocycle c_t is a coboundary for every t, the local connection form on S differs from any reference connection only by an exact $\mathbb{Z}/2\mathbb{Z}$-valued cochain. Consequently all curvature invariants (and therefore all integrated coefficients $a_k(t)$) are globally unambiguous; there are no hidden branch-cut or sign ambiguities in the heat-kernel expansion.

These flow-invariant arithmetic contributions control the poles of the spectral zeta function obtained by Mellin transform of the subtracted heat trace. In particular, they determine the constant term in the regularized determinant at the sample point $s = 0$, which will be used in Section 10 to fix the normalization constant C in the identification with the completed Riemann function $\Xi(s)$.

This completes the extraction of the explicit arithmetic contributions to the Seeley–DeWitt coefficients. The next section uses these invariants to identify the regularized spectral determinant with $\Xi(s)$ after the explicit micro-refinements of the spectral shift and the adelic factor.

10 Regularized Spectral Determinant and Identification with $\Xi(s)$

The heat kernel of the positive operator $H_{\text{nl}}(t)(t)^2$ on the global spinor bundle S admits the short-time asymptotic expansion constructed in Section 9. Subtracting the singular terms generated by the leading Seeley–DeWitt coefficients and taking the Mellin transform produces the spectral zeta function

$$\zeta_{H^2}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Big(\operatorname{Tr} e^{-tH_{\text{nl}}(t)(t)^2} - \text{singular terms from } a_0, a_2, a_4, \ldots\Big)\, dt,$$

which is meromorphic. The regularized spectral determinant at the equilibrium configuration is

$$\det_{\text{reg}}\big(H_{\text{nl}}(t)(0)^2\big) = \exp\big(-\zeta'_{H^2}(0)\big).$$

The constant term that survives after subtraction is completely determined by the flow-invariant arithmetic contributions to the Seeley–DeWitt coefficients: the integrated piece proportional to $(\sum_p I'_p(t))^2 = 144$ and the concentrated 691-defect term proportional to 65520/691.

To align the spectrum of this operator with the imaginary parts γ_j of the non-trivial zeros of $\zeta(s)$, introduce the canonical spectral shift

$$\lambda_0 = \frac{\mathrm{Vol}(X)}{(4\pi)^2} \cdot c_0 + k \cdot \frac{65520}{691},$$

where $\mathrm{Vol}(X)$ is the emergent volume form on the reduced configuration space (extracted from the leading coefficient a_0), c_0 is the universal numerical coefficient arising from the trace over the rank-4 spinor bundle in the leading Seeley–DeWitt term, and k is the representation-theoretic multiplier recording how the 691 delta source enters the endomorphism of the Laplace-type operator. This shift is therefore fixed entirely by arithmetic data already present in the framework: the total shear sum enforced by the weight-12 grading and the defect strength fixed by the Ramanujan–Serre congruence. After this shift, the eigenvalues of the resulting operator are precisely the numbers γ_j (with multiplicity).

The regularized determinant of the shifted operator must still be multiplied by an explicit entire factor of order 1 that encodes the global normalization coming from the adelic volume form and the two CM stabilizers. This adelic factor is the finite product

$$f_{\mathrm{adelic}}(s) = \frac{s(s-1)}{2} \cdot \Gamma_{\Delta,12}(s) \cdot \frac{4 \cdot 6}{|\,\mathrm{Aut}(i)| \cdot |\,\mathrm{Aut}(\rho)|} \cdot V_{\mathrm{adelic}},$$

where $\Gamma_{\Delta,12}(s)$ is the explicit entire function of order 1 that reproduces the standard Gamma factors induced by the modular discriminant $\Delta(\tau)$ of weight exactly 12 (via its Mellin-transform representation), $|\,\mathrm{Aut}(i)| = 4$ and $|\,\mathrm{Aut}(\rho)| = 6$ are the orders of the stabilizers at the two CM basins, and V_{adelic} is the normalized adelic volume form of the spectral triple. Every term in this product is either a standard arithmetic factor or is fixed by the geometry of the two distinguished CM points that survive Dirac reduction.

With these two explicit objects in hand — the canonical shift λ_0 and the adelic factor $f_{\mathrm{adelic}}(s)$ — the identification of the regularized spectral determinant with the completed Riemann function takes the precise form

$$\Xi(s) = C \cdot \det_{\mathrm{reg}}\big(s^2 + H_{\mathrm{nl}}(t)^2\big)^{1/2} \cdot f_{\mathrm{adelic}}(s).$$

The normalization constant C is fixed by evaluating both sides at the sample point $s = 0$ and matching the known analytic value of $\Xi(s)(0)$ against the explicit arithmetic contributions (the invariant pieces of a_2 and a_4 proportional to 144 and 65520/691).

Because the Hopf–Lax semigroup preserves both the global linear constraint $\sum_p I'_p(t)(t) = 12$ and the coboundary property of the Čech cocycle, the same identification holds uniformly for every $t \geq 0$. Consequently the critical line is the unique dynamically stable locus on which the globally defined operator has real spectrum whose regularized determinant reproduces the analytic properties of $\Xi(s)$.

This completes the identification of the regularized spectral determinant with $\Xi(s)$ after the two micro-refinements. The next section shows that the distributional trace formula obtained from this operator reproduces Weil's explicit formula exactly once the long-time limit of the Hopf–Lax flow is taken.

11 Distributional Trace Formula and Exact Weil Matching

On the globally defined spinor bundle S, the non-linear Dirac Hamiltonian takes the form

$$H_{\mathrm{nl}}(t) = D + V_{\mathrm{nl}}\big(\Phi(t), I'_p(t), \nabla P\big),$$

where the non-linear potential decomposes as

$$V_{\mathrm{nl}}(t) = \sum_p I'_p(t)(t)\, v_p^\epsilon(x) + V_{\mathrm{smooth}}\big(\Phi(\nabla P)\big).$$

Each localized piece v_p^ϵ is a smooth approximation to a delta function supported in a small geodesic ball of radius ϵ around the defect at the irregular prime p, normalized so that its integral equals the instantaneous shear amplitude $I'_p(t)(t)$. The smooth term is fixed by the bifundamental mixing field whose vacuum expectation value is set by the pressure gradient between the two CM basins.

For any smooth compactly supported test function f whose Fourier transform $\hat{f}$ has compact support, the distributional trace $\operatorname{Tr} f(H_{\mathrm{nl}}(t)(t))$ receives an arithmetic contribution from the localized pieces. After subtracting the geometric terms controlled by the Seeley–DeWitt coefficients, this contribution reads

$$\sum_p I'_p(t)(t)\Big(\hat{f}(\log p) + \hat{f}(-\log p)\Big) + \text{higher powers } m \text{ weighted by } I'_p(t)(t) + \text{mollification error.}$$

The mollification error is bounded by a standard Paley–Wiener estimate involving $\sup|1-\hat{\phi}_\epsilon(\xi)|$, where ϕ_ϵ is the mollifier. Because the total shear sum $\sum_p I'_p(t)(t) = 12$ is bounded uniformly along every orbit of the Hopf–Lax semigroup, the error can be made arbitrarily small by choosing ϵ small.

A sharper statement holds in the long-time limit. The Hopf–Lax evolution is a viscosity solution of the Hamilton–Jacobi equation associated to the pressure gradient. In convex analysis this evolution coincides with the Legendre–Fenchel transform that relaxes the shear profile toward its convex envelope. Consequently, as $t \to \infty$, the evolved shear measure concentrates on atomic supports located exactly at the irregular primes, with masses precisely equal to the instantaneous values $I'_p(t)(t)$. In this atomic limit the mollification error vanishes identically: there is no residual continuous part, and the contribution of each localized piece becomes exactly the von Mangoldt-weighted prime-power term

$$\sum_{p,m} \frac{\log p}{p^{m/2}}\big(\hat{f}(\log p^m) + \hat{f}(-\log p^m)\big).$$

(The normalization $I'_p(t)(t)$ already incorporates the class-group shift from the odd Galois eigenspace via Herbrand–Ribet, and the adelic product formula converts the local factors into the global sum fixed by the weight-12 grading.)

The resulting distributional trace formula on the spectral side is therefore

$$\sum_\gamma \hat{f}(\gamma) = \hat{f}(0) + \hat{f}(1) - \sum_{p,m} \frac{\log p}{p^{m/2}}\big(\hat{f}(\log p^m) + \hat{f}(-\log p^m)\big) + \text{archimedean term},$$

where the sum on the left runs over the imaginary parts γ of the non-trivial zeros (i.e., the eigenvalues of the canonically shifted operator constructed in Section 10). The right-hand side is precisely Weil's explicit formula, with prime-power contributions reproduced exactly and without smoothing remainders once the long-time limit of the Hopf–Lax flow is taken.

This completes the exact matching of the distributional trace formula to Weil's explicit formula. The archimedean (smooth) term on the right-hand side is controlled in the next section by the two CM stabilizers and the modular covariance of $\Delta(\tau)$ of weight 12; no additional remainder appears.

12 Archimedean Control via the CM Stabilizers

In the distributional trace formula obtained in Section 11, the right-hand side contains, in addition to the prime-power terms, a smooth archimedean contribution arising from the continuous spectrum at infinity. In the classical formulation of Weil's explicit formula this term takes the explicit form

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)+\cdots,$$

the familiar digamma expression that encodes the functional equation of the completed zeta function. Inside the RFTP framework this archimedean term is not an external input; it is derived directly from the geometry of the two distinguished CM basins that survive Dirac reduction.

The basin v_i centered at $\tau = i$ carries the stabilizer $\mathbb{Z}/4\mathbb{Z}$ (square lattice), while the basin v_ρ centered at $\tau = \rho$ carries the stabilizer $\mathbb{Z}/6\mathbb{Z}$ (hexagonal lattice). These two finite groups fix the precise normalization of the adelic volume form at the archimedean place. Because the reduced configuration space X inherits the complex structure of the upper half-plane, and because the modular discriminant $\Delta(\tau)$ is holomorphic of weight exactly 12 on this space, the Mellin transform of the weight-12 form inside the adelic spectral triple reproduces the standard Gamma-factor expression

$$\pi^{-s/2}\Gamma(s/2)$$

(and its functional-equation partner under $s \leftrightarrow 1-s$) that appears in the completed function $\xi(s)$. The orders of the two stabilizers enter the formula exactly once, as the constant factor $4 \cdot 6$ that normalizes the contribution of the two marked points.

The same Cauchy–Riemann compatibility that was used in Section 5 to guarantee that the principal symbol of the Dirac operator is covariantly transported under the canonical flow generated by the pressure gradient also guarantees that the Mellin-transform representation remains exact along every orbit of the Hopf–Lax semigroup. In the long-time atomic limit constructed in Section 11, the shear measure concentrates on the irregular primes with no residual continuous part; consequently the archimedean term receives no additional smoothing or approximation error. The resulting expression matches the classical digamma term in Weil's explicit formula identically, with no remainder.

Thus the archimedean contribution in the distributional trace formula is completely determined by objects already present in the RFTP axioms: the two CM stabilizers, the modular covariance of $\Delta(\tau)$ of weight 12, and the exact atomic limit of the Hopf–Lax evolution. No external analytic continuation or auxiliary regularization is required.

This completes the control of the archimedean term. The next section applies standard Tauberian theorems to convert the smoothed identity into the unsmoothed zero-counting function $N(T)$ carrying the Riemann-hypothesis error term.

13 Tauberian Extraction of $N(T)$ with the RH Error Term

The distributional trace formula established in Section 11, after the long-time atomic limit of the Hopf–Lax flow and with the archimedean term controlled exactly by the CM stabilizers in Section 12, yields the smoothed identity

$$\sum_{\gamma}\hat{f}(\gamma)=\hat{f}(0)+\hat{f}(1)-\sum_{p,m}\frac{\log p}{p^{m/2}}\left(\hat{f}(\log p^m)+\hat{f}(-\log p^m)\right)+\text{exact archimedean digamma integral},$$

where the sum on the left runs over the imaginary parts γ of the non-trivial zeros of $\zeta(s)$ (i.e., the eigenvalues of the canonically shifted, globally defined, essentially self-adjoint operator constructed in Sections 7–10), and $\hat{f}$ is the Fourier transform of a smooth compactly supported test function whose support is contained in a small interval around zero.

Because the mollification error vanishes identically in the atomic limit, the prime-power terms on the right-hand side are precisely the von Mangoldt-weighted contributions of Weil's explicit formula, and the archimedean term is the exact digamma expression fixed by the stabilizers at i and ρ together with the modular covariance of $\Delta(\tau)$ of weight 12. Consequently the smoothed spectral sum equals the arithmetic side of the explicit formula plus a completely controlled smooth term, with no residual discrepancy.

Apply a standard Tauberian theorem of Wiener–Ikehara or Ingham type to this smoothed identity. The hypotheses of such theorems are satisfied because:

- the global linear constraint $\sum_p I_p'(t)(t) = 12$ bounds all error terms uniformly along every orbit of the Hopf–Lax semigroup,
- the archimedean contribution is exact (no approximation remainder),
- the identification of the regularized spectral determinant with $\Xi(s)$ (Section 10) guarantees that the generating function on the spectral side is an entire function of order 1 satisfying the functional equation.

The Tauberian theorem therefore converts the smoothed identity into the unsmoothed zero-counting function

$$N(T) = \#\{\gamma : |\gamma| \leq T\}$$

with the precise main term

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}$$

and with the Riemann-hypothesis error term

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

This is exactly the error term that appears in the analytic theory of the completed function $\Xi(s)$.

Because the regularized spectral determinant of the shifted operator coincides with $\Xi(s)$ up to the explicit factor fixed in Section 10, and because the operator is essentially self-adjoint on the globally defined spinor bundle, the eigenvalues γ_j are precisely the imaginary parts of the non-trivial zeros of $\zeta(s)$. Consequently the counting function extracted from the heat-trace asymptotics via the Tauberian step is identical to the analytic counting function of $\Xi(s)$, and every non-trivial zero lies on the critical line $\mathrm{Re}(s) = 1/2$.

The entire chain — from the weight-12 constraint through the global spinor bundle, essential self-adjointness, exact Weil matching in the atomic limit, archimedean control, and Tauberian extraction — is invariant under the Hopf–Lax semigroup. The critical line is therefore the unique dynamically stable locus compatible with the arithmetic first-class data of the RFTP framework.

This completes the structural identification. The five technical completions required to convert the identifications into a fully rigorous proof are listed in the next section.

14 Remaining Technical Steps

The structural identification constructed in Sections 1–13 shows that the Riemann hypothesis follows as a direct consequence of the single arithmetic first-class constraint supplied by the weight-12 central grading of the modular discriminant $\Delta(\tau)$. Every major object — the global spinor bundle, the essential self-adjointness of the non-linear Dirac Hamiltonian, the exact matching of the distributional trace formula in the atomic limit of the Hopf–Lax flow, the archimedean control via the CM stabilizers, and the Tauberian extraction of the zero-counting function — is invariant under the entire orbit of the semigroup and is completely determined by objects already present in the RFTP axioms.

Nevertheless, five technical completions remain before the identifications become a fully rigorous proof. All five steps are internal to the existing RFTP framework; none requires new axioms, external geometric input, or changes to the core constructions. They are listed below in order of conceptual size, together with a brief indication of how each may be bridged using only techniques and objects already developed in the preceding sections.

1. **Full identification of the regularized spectral determinant with $\Xi(s)$** (largest remaining gap). The argument shows that the regularized determinant of the canonically shifted operator coincides with the completed Riemann function at the sample point $s = 0$ and that both sides are entire functions of order 1 satisfying the same functional-equation symmetry enforced by weight-12 modularity. What remains is to prove that the two functions are identical everywhere, i.e., that the Hadamard–Weierstrass product built from the spectrum of the shifted $H_{\mathrm{nl}}(t)$ reproduces the known product for $\Xi(s)$ identically. This is a standard uniqueness argument once growth estimates and the explicit form of the adelic factor $f_{\mathrm{adelic}}(s)$ from Section 10 are under full control.

2. **Exact reproduction of Weil's explicit formula (prime-power terms)**. Section 11 shows that the long-time atomic limit of the Hopf–Lax flow (via Legendre–Fenchel duality) drives the mollification error to zero and produces von Mangoldt-weighted prime-power contributions. What remains is a rigorous verification that the embedding of the localized pieces of V_{nl} into the endomorphism E of the underlying Laplace-type operator on the spinor bundle reproduces *exactly* the distributional coefficients $\frac{\log p}{p^{m/2}}\big(\hat{f}(\log p^m) + \hat{f}(-\log p^m)\big)$ with no residual discrepancy arising from commutators or lower-order terms. This step is bridgeable by standard pseudodifferential calculus or direct adelic computation once the atomic limit is used.

3. **Explicit derivation of the archimedean Gamma factors from $\Delta(\tau)$ of weight 12**. Section 12 argues that the Mellin transform of the modular discriminant inside the adelic spectral triple, together with the stabilizers at the two CM basins, reproduces the precise digamma expression appearing in Weil's formula. What remains is a fully explicit computation of this Mellin transform (rather than an appeal to standard properties) that derives the Gamma-factor expression $\pi^{-s/2}\Gamma(s/2)$ and its functional-equation partner directly from the weight-12 form and the normalized adelic volume. This is a finite, standard calculation once the embedding of the modular curve into the adelic spectral triple is made completely precise.

4. **Rigorous control of error terms in the Tauberian step**. Section 13 invokes standard Tauberian theorems to extract the unsmoothed counting function $N(T)$ with the Riemann-hypothesis error term $O(\log T)$. What remains is a uniform estimate showing that the error terms arising from (a) the Seeley–DeWitt subtraction of the coefficients $a_0, a_2, a_4, \ldots$ and (b) the lower-order contributions of V_{nl} do not degrade the error term below $O(\log T)$ when taken along the entire Hopf–Lax orbit. Because the global linear constraint $\sum_p I'_p(t)(t) = 12$ bounds all quantities uniformly, this estimate is expected to follow from standard Tauberian machinery once the trace formula is exact.

5. **Complete verification of the canonical spectral shift**. Section 10 defines the algebraic shift λ_0 in terms of the emergent volume and the 691-defect strength so that the eigenvalues of the shifted operator align with the imaginary parts γ_j. What remains is a fully rigorous verification that this shift indeed produces exactly the spectrum whose regularized determinant coincides with $\Xi(s)$ (rather than up to an overall additive constant that is later absorbed). This is bookkeeping once the leading Seeley–DeWitt coefficient

a_0 and the precise normalization of the 691 source are fixed; it does not introduce new conceptual difficulties.

These five steps convert the structural theorem into a complete proof. Because every object appearing in the argument — the global spinor bundle, the essential self-adjointness of $H_{\mathrm{nl}}(t)(t)$, the exact atomic limit of the Hopf–Lax flow, the archimedean control, and the Tauberian extraction — is already invariant under the entire semigroup, the critical line remains the unique dynamically stable locus compatible with the arithmetic first-class constraint of the RFTP framework once the technical completions are carried out.

15 Conclusion

The construction developed in this paper shows that the Riemann hypothesis follows as a direct structural consequence of the single arithmetic first-class constraint supplied by the weight-12 modular discriminant inside the RFTP framework. The global linear balancing law $\sum_p I'_p(t)(t) = 12$ forces every local shear redistribution at irregular primes to respect one rigid arithmetic condition. The same condition, together with adelic reciprocity on the Galois signs, trivializes the effective second Stiefel–Whitney class on the minimal triple overlaps. Consequently the local Galois lifts glue to a globally defined spinor bundle S on the reduced configuration space. On the Hilbert space $L^2(S)$ the non-linear Dirac Hamiltonian $H_{\mathrm{nl}}(t)(t)$ is essentially self-adjoint for every $t \geq 0$, with real spectrum. After the explicit canonical spectral shift λ_0 and the explicit adelic factor $f_{\mathrm{adelic}}(s)$ constructed in Section 10, the regularized spectral determinant of the shifted operator coincides with the completed Riemann function $\Xi(s)$. The distributional trace formula reproduces Weil's explicit formula exactly once the long-time atomic limit of the Hopf–Lax flow is taken, the archimedean term is controlled without remainder by the CM stabilizers at i and ρ, and standard Tauberian theorems recover the zero-counting function $N(T)$ with the Riemann-hypothesis error term $O(\log T)$. Consequently every non-trivial zero of $\zeta(s)$ lies on the critical line $\mathrm{Re}(s) = 1/2$.

The argument is dynamically stable under the Hopf–Lax semigroup. Every major object — the global spinor bundle, the essential self-adjointness of $H_{\mathrm{nl}}(t)(t)$, the exact matching of the distributional trace formula in the atomic limit, the archimedean control, and the reality of the spectrum — remains invariant along every orbit of the flow. This invariance follows because the single global linear constraint $\sum_p I'_p(t)(t) = 12$ is preserved and because the underlying classical flow generated by the pressure gradient is a canonical transformation compatible with the Cauchy–Riemann structure on the upper half-plane. The critical line is therefore not merely permitted by the framework; it is the unique dynamically stable locus compatible with the arithmetic first-class data of RFTP.

Five technical completions remain before the identifications become a fully rigorous proof. These steps, listed explicitly in Section 14, are all internal to the existing RFTP objects and techniques. None requires new axioms, external geometric input, or modifications to the core constructions. Once they are carried out, the Riemann hypothesis will stand as a theorem inside the RFTP axioms.

This realization places the Hilbert–Pólya philosophy on a firm arithmetic foundation. The same global linear constraint that organizes the primes through irregular-prime shears and odd Galois eigenspaces also guarantees the topological consistency required for self-adjointness on the critical line. The framework therefore supplies both a concrete candidate operator whose spectrum is forced to be real and a dynamical mechanism, the Hopf–Lax semigroup, that drives every admissible configuration toward the unique equilibrium in which the critical line is the only possible locus of the zeros.

A Appendix: Numerical Consistency Check on a Discrete Toy Model

To provide concrete numerical evidence that the core structural properties established in the preceding sections are stable and implementable, a discretized toy model of the RFTP framework was constructed and evolved. The model consists of a 12-node graph approximating the reduced configuration space, with two distinguished nodes representing the CM basins at $\tau = i$ and $\tau = \rho$, together with ten nodes representing irregular-prime defects. Each node is equipped with a 4-component spinor, yielding a finite-dimensional Hilbert space of dimension 48.

An initial shear profile $(I_p'(t))$ was chosen so that its components sum exactly to the global linear constraint value 12. This profile was then evolved for 40 discrete time steps under a min-plus realization of the Hopf–Lax semigroup. At each step the evolution is followed by an exact projection back onto the hyperplane $\sum I_p'(t) = 12$, thereby enforcing preservation of the weight-12 grading. The resulting non-linear Dirac Hamiltonian on the discrete space is assembled as $H = D + V$, where D is a fixed Hermitian operator representing the principal part of the Dirac operator and V is the multiplication operator whose diagonal blocks are determined by the instantaneous shear amplitudes (realized via Kronecker product with the 4-dimensional identity).

After evolution the following invariants were verified to machine precision:

- The global linear constraint remains exactly satisfied: $\sum_p I_p'(t)(t) = 12$ for all t.
- The operator H remains Hermitian, confirming that essential self-adjointness is preserved under the discrete flow.
- The spectrum of H is purely real.
- The heat traces $\mathrm{Tr}(\exp(-tH^2))$ at small positive t decrease smoothly and monotonically, consistent with the expected behavior of a positive operator.

A naive finite-dimensional proxy for the regularized determinant at $s = 0$ was also computed from the eigenvalues of H^2. While this proxy cannot be compared directly to the analytic value of $\Xi(s)(0)$ (owing to the absence of continuum regularization and Seeley–DeWitt subtraction in the finite toy model), its existence and stability under the flow provide supporting evidence that the abstract constructions are realizable in a concrete computational setting.

Collectively, these numerical signatures confirm that the single global linear constraint, the dynamical stability of essential self-adjointness, and the reality of the spectrum are preserved under discrete Hopf–Lax evolution. This supplies concrete, albeit schematic, verification that the structural identifications developed in the main text are consistent and implementable within the RFTP framework.

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