A Machine-Checked Volume-Uniform Wilson-Loop Area Law via a Formalized Cluster Expansion

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July 2, 2026

Abstract

We report a complete formalization, in Lean 4 over Mathlib, of volume-uniform Wilson-loop area laws for $SU(N_c)$ lattice gauge theory in an explicit strong-coupling window — including the case of the *exact* Wilson Boltzmann factor $\prod_p e^{z_p}$, not a linearized surrogate. The headline theorem bounds the normalized Wilson-loop expectation by $N_c\, e^{P\cdot 4d\cdot K}\, \sigma^{\mathrm{A}(C)}\, e^{P\cdot 4d\cdot S(\sigma)}$, where $\mathrm{A}(C)$ is an intrinsic combinatorial filling area of the loop, P is its edge-support size, and *every constant is volume-free*: the bound holds uniformly over all finite lattice sizes. The partition function is cancelled through a fully formalized volume-restricted cluster expansion (loop-tagged factorization, restricted Mayer inversion, Z-ratio bounds, and a pinned-gas resummation built on a Kotecký–Preiss layer with Penrose-style spanning-tree counting). A reusable repackaging converts the bound into manifest exponential area decay with a strictly positive string tension, and — reflecting a methodological lesson from the project's own history — the *non-vacuity* of every hypothesis window is itself machine-checked — both the cluster smallness window and the decay-repackaging window, the latter with an explicit witness of tension $\log 2 - \frac{1}{2}$. For every exported theorem in this chain the Lean kernel's axiom oracle reports exactly `[propext, Classical.choice, Quot.sound]`; there is no `sorry` and no project axiom in the dependency cone. To our knowledge this is the first machine-checked cluster-expansion proof in lattice quantum field theory. All artifacts are public, with per-theorem oracle records in a verification ledger and continuous-integration builds.

1 Introduction

Wilson's confinement criterion asks whether the expectation of a Wilson loop decays with the *area* it spans rather than with its perimeter [1]. At strong coupling this is a classical theorem of lattice gauge theory: the character/cluster expansions of Osterwalder and Seiler establish area-law decay for the Wilson action in a small-β regime [2, 3]. The underlying machinery — polymer representations, Mayer expansions, and tree-graph bounds in the tradition of Penrose [4] and Kotecký–Preiss [5] — is standard, but it is also notoriously bookkeeping-heavy: the mathematical content of "volume-uniform" is precisely that *every* constant produced by the expansion can be tracked through the thermodynamic limit without silent volume dependence. This is the kind of proof in which human errors hide in constants rather than in ideas, and hence a natural target for machine verification.

This paper reports such a verification. Within the author's larger audit-first programme on lattice Yang–Mills theory [8, 9], the following family of results has been formalized end-to-end in Lean 4 [6] over the Mathlib library [7]:

1. **Area laws in four variants** — finite-volume and volume-uniform, each for linearized activities and for the *exact* Wilson Boltzmann factor $\prod_p e^{z_p}$ — with explicit constants and explicit smallness windows (Section 3).
2. **A formalized cluster expansion** strong enough to cancel the partition function with volume-independent losses: loop-tagged factorization, restricted Mayer inversion, Z-ratio bounds, and a pinned-gas resummation over a Kotecký–Preiss layer with Penrose-style breadth-first-search spanning-tree counting (Section 4).
3. **A methodological discipline** in which integrability side conditions are theorems rather than hypotheses, per-theorem axiom oracles are recorded in a public ledger, and — after an early episode in which the repository itself had advertised a vacuous headline statement, since disclaimed in its README — the non-vacuity of every hypothesis window is itself machine-checked (Section 5).

The mathematics is classical in spirit; the contribution is the machine checking, the explicitness, and the reusable formal toolbox. We are not aware of any prior machine-checked cluster or polymer expansion for a lattice quantum field theory in any proof assistant.

2 Setting

Fix a dimension $d \geq 1$ and a linear lattice size $N \geq 1$ (both positive; the theorems are stated for all such d, N). The repository's lattice layer provides a concrete complex of oriented edges and plaquettes over the finite hypercubic lattice of linear size N; a gauge configuration is a map A from oriented edges to the gauge group $G = SU(N_c)$ (as Mathlib's `Matrix.specialUnitaryGroup`) satisfying $A(e^{-1}) = A(e)^{-1}$. The reference measure μ_N is the product of normalized Haar measures over one orientation class of edges, transported to the configuration space (`gaugeMeasureFrom`, `sunHaarProb`).

For a list of edges es tracing a closed loop C, the Wilson line $W_C(A)$ is the ordered product of the edge variables and $\operatorname{tr} W_C$ its (matrix) trace. Plaquette activities take the conjugate-pair form

$$z_p(A) \;=\; c_p \cdot \operatorname{tr} H_p(A) \;+\; \overline{c_p} \cdot \overline{\operatorname{tr} H_p(A)}, \qquad \|c_p\| \leq \delta,$$

where H_p is the plaquette holonomy; the conjugate pairing makes each z_p real, and the weight $\prod_p e^{z_p}$ is the *exact* Wilson Boltzmann factor. In particular the standard Wilson action weight $\exp\big((\beta/N_c)\operatorname{Re}\operatorname{tr} H_p\big)$ is the case $c_p = \beta/(2N_c)$, so $\delta = \beta/(2N_c)$. The partition function is $Z_N = \int \prod_p e^{z_p}\, d\mu_N$, and the object of interest is the normalized expectation $\langle \operatorname{tr} W_C \rangle_N = Z_N^{-1} \int \operatorname{tr} W_C \cdot \prod_p e^{z_p}\, d\mu_N$.

Two combinatorial quantities of the loop control the bound. The *perimeter charge* $P = P(C)$ is the number of distinct edges supporting the loop (`edgeSupport`). The *area* $\mathrm{A}(C)$ is intrinsic and algebraic: the loop determines a 1-chain with coefficients in $\mathbb{Z}/N_c$, and

$$\mathrm{A}(C) \;=\; \min\big\{\, \#\mathrm{supp}(s) \;:\; s \text{ a plaquette 2-chain over } \mathbb{Z}/N_c,\ \partial_2 s = [C] \,\big\}$$

(`chainAreaA`, `loopChain`) — the minimal number of plaquettes in an algebraic filling of the loop. No area hypothesis is imposed anywhere; the exponent in the theorems below is this defined quantity.

3 Main results

Write $\rho(\delta) = e^{2\delta N_c} - 1$ for the effective exact-activity size, and define the two window functions

$$K(t;\delta) \;=\; \frac{e\cdot\rho(\delta)\,e^{t}}{1-(16d+1)^2\,\rho(\delta)\,e^{t}}, \qquad S(\sigma) \;=\; \frac{\sigma}{1-(16d+1)^2\sigma}.$$

Theorem 3.1 (volume-uniform area law, exact activities; `normalized_exp_wilson_loop_area_law`). *Let $N_c \geq 1$, let $\delta \geq 0$ bound the conjugate-pair activities as above, and let $t, \varepsilon \geq 0$ and $\sigma \in [0,1]$ satisfy the explicit window inequalities*

$$(16d+1)^2\,\rho(\delta)\,e^{t+\varepsilon+1} < 1, \qquad 16d\cdot\frac{\rho(\delta)\,e^{t+\varepsilon+1}}{1-(16d+1)^2\rho(\delta)e^{t+\varepsilon+1}} \leq t,$$

$$(16d+1)^2\sigma < 1, \qquad \rho(\delta)\cdot\exp\bigl(16d\cdot K(t;\delta)\bigr) \leq \sigma^2.$$

Then for every lattice size N and every closed loop C,

$$\bigl\|\langle \operatorname{tr} W_C\rangle_N\bigr\| \;\leq\; N_c\cdot e^{P(C)\cdot 4d\cdot K(t;\delta)}\cdot\sigma^{\mathrm{A}(C)}\cdot e^{P(C)\cdot 4d\cdot S(\sigma)}.$$

Every constant on the right is independent of N.

The hypotheses are pure smallness and geometry inequalities: in an earlier stage of the development the integrability of the relevant families was carried as a hypothesis, and it has since been discharged as a theorem (ledger Addendum 17u; `normalized_wilson_loop_area_law_unconditional`), so no measure-theoretic side condition remains. The linearized companion (activities $1 + f_p$ with $\|f_p\|$-control) is `normalized_wilson_loop_area_law`; the pipeline is activity-agnostic and the exact version is obtained from the linearized proof by the single substitution $2\delta N_c \rightsquigarrow e^{2\delta N_c} - 1$. Together with the finite-volume pair the programme comprises four variants:

Variant	Activities	Lean declaration
finite volume	linearized	`finite_volume_area_law`
finite volume	exact $\prod_p e^{z_p}$	`finite_volume_area_law_exp`
volume-uniform	linearized	`normalized_wilson_loop_area_law`
volume-uniform	exact $\prod_p e^{z_p}$	`normalized_exp_wilson_loop_area_law`

Theorem 3.2 (manifest exponential decay; `area_law_to_exp_area_decay`). *On any loop family whose perimeter charge is area-subdominant — i.e. $P\cdot 4d\,(K+S) \leq \lambda\cdot\mathrm{A}(C)$ for some $\lambda < -\log\sigma$ — the bound of Theorem 3.1 repackages as*

$$\bigl\|\langle \operatorname{tr} W_C\rangle_N\bigr\| \;\leq\; N_c\cdot e^{-\tau\,\mathrm{A}(C)}, \qquad \tau = (-\log\sigma) - \lambda > 0,$$

a manifest area law with strictly positive string tension τ.

Proposition 3.3 (machine-checked non-vacuity). *Two distinct hypothesis windows are involved, and each carries its own machine-checked witness.* (i) The repackaging window of Theorem 3.2. *Its quantities K, S, σ, λ enter only through the inequalities displayed there — they are free parameters of the repackaging lemma, not constrained by the cluster expansion — and the witness $\sigma = \frac{1}{2}$, $\lambda = \frac{1}{2}$ yields the strictly positive tension $\tau = \log 2 - \frac{1}{2} > 0$ (`area_law_to_exp_area_decay_window_nonempty`, proved via Mathlib's certified numeric bound*

Real.log_two_gt_d9). We emphasize that $\sigma = \frac{1}{2}$ is not *claimed to lie in the cluster window of Theorem 3.1, which requires the much smaller $(16d+1)^2\sigma < 1$.* (ii) The cluster-expansion smallness window of Theorem 3.1 itself. *For every d and N_c there exists $\delta_0 > 0$ such that all activities with $\delta \le \delta_0$ satisfy the window (lemmas clustering_window_nonempty and sun_clustering_window_nonempty, instantiated at $t = \varepsilon = 1$; ledger Addendum 17t).*

Remark 3.4. Making non-vacuity a theorem is a deliberate methodological choice; see Section 5.

4 Proof architecture

The proof is a cluster expansion engineered so that every loss is charged either to the loop's perimeter or to its area, never to the volume.

Loop-tagged expansion and the pinned dichotomy. Writing the exact weight as $\prod_p(1+f_p)$ with $f_p = e^{z_p} - 1$ and expanding, the numerator becomes a sum over *pinned* plaquette sets S_0 of terms $\int \operatorname{tr} W_C \cdot \prod_{p\in S_0} f_p \cdot$ (far factor). Haar selection rules on $SU(N_c)$ kill every pinned term whose plaquette set cannot fill the loop: the formalized dichotomy (`norm_integral_exp_pinned_term_le`) bounds each term by $\mathbb{1}[\mathrm{A}(C) \le \#S_0] \cdot N_c \cdot \rho(\delta)^{\#S_0}$. This is where the intrinsic $\mathbb{Z}/N_c$-chain area of Section 2 enters: a surviving monomial exhibits an algebraic filling of the loop chain inside S_0.

Exact activities via shifted Cauchy products. For the exact factor the expansion of $\prod_{p\in S_0}(e^{z_p} - 1)$ is organized as a shifted multi-variable Cauchy product $\sum'_m \prod_p z_p^{m_p+1}/(m_p+1)!$ — every exponent is at least one, so the occupied plaquette set of each monomial is exactly S_0 and the Haar kill fires uniformly. The interchange of integral and series is justified by dominated summability (`summable_prod_pow_succ_div_factorial`); this is the point at which the linearized proof generalizes by the substitution $2\delta N_c \rightsquigarrow e^{2\delta N_c} - 1$.

Cancelling the partition function. The far factor of a pinned term is a restricted partition function $Z_{\mathrm{far}}(S_0)$. A volume-restricted Mayer inversion and a Z-ratio bound (`norm_normalized_exp_wilson_loop_le_pinned_sum`) cancel Z_N against $Z_{\mathrm{far}}(S_0)$ at a cost exponential in $\#S_0$ and in the loop's perimeter charge only — the decisive volume-uniformity step.

Resumming the pinned gas. The remaining sum over pinned sets is controlled by an abstract pinned-gas estimate (`sum_pinned_dichotomy_le`) built on the repository's Kotecký–Preiss layer: a 26-module, ≈11,500-line development of polymer counting with Penrose-style breadth-first-search spanning-tree bounds, Ursell coefficients, and the KP convergence criterion [4, 5]. The geometric rate σ absorbs one factor per pinned plaquette beyond the area threshold, yielding the $\sigma^{\mathrm{A}(C)}$ decay with the perimeter-only prefactors $e^{P\cdot 4d\cdot K}$ and $e^{P\cdot 4d\cdot S(\sigma)}$.

5 The formalization

Scale and toolchain. The area-law dependency cone spans the lattice chain complex (L0), the Gibbs-measure layer (L1), a 3,075-line Wilson-loop monomial module, the 26-module Kotecký–

Preiss layer, and the 2,117-line volume-restricted gate module containing the headline theorems. Everything is Lean 4 over Mathlib and is rebuilt by continuous integration.

Oracle discipline. The repository maintains a verification ledger recording, for every exported theorem, the output of Lean's `#print axioms`. For the declarations of Sections 3 the ledger records exactly the canonical classical trio; e.g.

```
'YangMills.normalized_exp_wilson_loop_area_law'  [propext, Classical.choice, Quot.
    sound]
'YangMills.normalized_wilson_loop_area_law'      [propext, Classical.choice, Quot.
    sound]
'YangMills.area_law_to_exp_area_decay'           [propext, Classical.choice, Quot.
    sound]
```

There is no `sorry` and no project-specific axiom anywhere in this dependency cone. For full disclosure: the repository as a whole also contains clearly labeled exploratory lanes: the experimental modules carry explicit axioms (fifteen, all confined to `Experimental/`) and abandoned proof attempts, while the renormalization-group lane carries explicitly named *hypotheses* rather than axioms. None of these is imported by the area-law chain, and the per-theorem oracle is precisely the mechanism that makes this separation checkable rather than asserted.

Non-vacuity as a theorem. The project's README retains a legacy disclaimer: an early advertised headline of the shape $\exists\, m > 0$ was discovered to be vacuously provable and was publicly disclaimed and removed. The lasting correction is procedural: every smallness window now comes with a machine-checked non-emptiness witness (Proposition 3.3 and ledger Addenda 17t/20), and integrability side conditions were converted from hypotheses into theorems (Addendum 17u). We highlight this as a transferable practice for formalized analysis: *a conditional theorem should ship with a formal witness that its conditions are satisfiable.*

Engineering notes for formalizers. Constants are carried symbolically (K, S, ρ) rather than numerically, so that volume-uniformity is syntactically visible in the statement; the $\int \leftrightarrow \sum'$ interchanges are localized in a small number of dominated-convergence lemmas; and the activity-agnostic design of the pinned pipeline is what reduced the exact-activity upgrade to a one-parameter substitution — the entire V4 stage was closed in five oracle-checked increments (ledger Addenda 18–18d).

6 Reproducibility and artifacts

All artifacts are public in the programme repository

`https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`

including the Lean sources (`YangMills/L1_GibbsMeasure/RestrictedGate.lean` for the headline theorems), the verification ledger (`docs/VERIFICATION-LEDGER.md`) with per-theorem oracle lines, the staged plan documents (`docs/AREA-LAW-VU-PLAN.md`) whose bricks map one-to-one onto the lemmas of Section 4, and continuous-integration builds; the pinned sources and ledger support independent replay.

7 Related work and outlook

The strong-coupling area law is classical [2, 3]; the contribution here is its machine-checked, explicitly volume-uniform form for the exact Boltzmann factor. On the formalization side, proof assistants have reached deep results in analysis and probability, but we are not aware of a prior mechanized polymer or cluster expansion for any lattice field theory. A companion result in the same programme — an exact Catalan evaluation of the factorially weighted spanning-tree sums that appear in the Kotecký–Preiss counting layer [10] — has been formalized with the same oracle discipline and is being prepared as generic Mathlib contributions; sharpening the tree-counting constants of Section 4 with it is natural future work. The larger programme — mass gap, renormalization-group inputs — remains open and clearly labeled as such in the repository; nothing in this paper depends on it.

Acknowledgments

The Lean formalization was developed interactively with Anthropic's Claude, in accordance with the ACM authorship policy on AI-assisted work; every theorem stated here was checked by the Lean 4 kernel, with the axiom oracle outputs archived in the repository ledger.

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