On the Fano Plane Structure, Complete Jordan Basis, and Biorthogonal Projections of a Seven-Vertex Directed Graph

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Abstract

We investigate further algebraic and combinatorial properties of a 7-vertex strongly connected directed graph whose adjacency matrix A has characteristic polynomial $\chi_A(\lambda) = \lambda^2(\lambda-2)(\lambda-1)(\lambda+1)^3$ and Perron root $\rho(A) = 2$. We establish four main results. First, the 14 directed edges of A are shown to correspond to a directed selection from the Fano plane $P^2(\mathbb{F}_2)$ with one line deleted. Second, we construct a complete Jordan basis of seven integer generalized eigenvectors, yielding the Jordan form $J = J_1(2) \oplus J_1(1) \oplus J_1(-1) \oplus J_2(-1) \oplus J_2(0)$ with $\det(V) = 144$. Third, we construct the biorthogonal left-vector basis and seven rank-one projection operators, achieving the Jordan–Chevalley decomposition $A = S + N$ with $N^2 = 0$. Fourth, we give a spectral computation of the number of closed walks of length L, obtaining $N_L = 2^L + 1 + 3(-1)^L$, and verify its consistency with the closed-walk equal-weight theorem.

Keywords: directed graph; Fano plane; Jordan canonical form; Jordan–Chevalley decomposition; biorthogonal projection; closed walk; Perron–Frobenius theorem.

1 Introduction

Let G be a directed graph on seven vertices $\{1, 2, \ldots, 7\}$ with adjacency matrix $A = (a_{ij})$, where $a_{ij} = 1$ if there is a directed edge $i \to j$ and 0 otherwise, with rows indexed by sources and columns by targets:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The graph has 14 directed edges:

$$1\to3,\ 1\to5,\ 1\to6,\ 1\to7,\ 2\to4,\ 3\to2,\ 3\to6,\ 4\to1,\ 4\to2,\ 5\to4,\ 5\to7,\ 6\to3,\ 7\to1,\ 7\to5.$$

The graph is strongly connected. The out-degrees (row sums) are $(4, 1, 2, 2, 2, 1, 2)$, and every vertex has in-degree exactly 2, so the column sums of A are all equal to 2.

A direct computation of the characteristic polynomial gives

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^2(\lambda-2)(\lambda-1)(\lambda+1)^3 = \lambda^7 - 4\lambda^5 - 2\lambda^4 + 3\lambda^3 + 2\lambda^2.$$

The spectrum is $\{2,\ 1,\ 0,\ 0,\ -1,\ -1,\ -1\}$. Since A is non-negative and irreducible, the Perron–Frobenius theorem [1] guarantees that the spectral radius $\rho(A) = 2$ is an eigenvalue with a positive eigenvector. The right Perron vector is

$$\mathbf{v} \ = \ (9,\ 3,\ 2,\ 6,\ 7,\ 1,\ 8)^T \ \in \ \mathbb{Z}^7_{>0},$$

satisfying $A\mathbf{v} = 2\mathbf{v}$. The left Perron vector is $\mathbf{1} = (1,1,1,1,1,1,1)^T$, since the column sums of A are all 2, i.e. $\mathbf{1}^T A = 2\,\mathbf{1}^T$.

The matrix A is not diagonalisable. Rank analysis shows

$$\operatorname{rank}(A) = 6, \quad \operatorname{rank}(A^2) = 5, \quad \operatorname{rank}(A+I) = 5, \quad \operatorname{rank}\big((A+I)^2\big) = 4,$$

which uniquely determines the Jordan block sizes. The geometric multiplicity of $\lambda = 0$ is $7 - \operatorname{rank}(A) = 1$, giving one 2×2 Jordan block $J_2(0)$. The geometric multiplicity of $\lambda = -1$ is $7 - \operatorname{rank}(A + I) = 2$, giving $J_1(-1) \oplus J_2(-1)$. The remaining eigenvalues $\lambda = 2$ and $\lambda = 1$ each have geometric multiplicity 1. Hence

$$J \ = \ J_1(2) \oplus J_1(1) \oplus J_1(-1) \oplus J_2(-1) \oplus J_2(0). \tag{1}$$

For every directed edge $i \to j$, the rational weight $w_{ij} = v_j/(2v_i)$ defines a row-stochastic matrix $W = \frac{1}{2}D^{-1}AD$ (where $D = \operatorname{diag}(\mathbf{v})$), and the cumulative weight of any closed directed walk of length L is exactly 2^{-L}, independent of the walk's itinerary. This result follows solely from the Perron eigenvector equation and the telescoping cancellation of vertex components along a closed walk.

In this paper we investigate four further aspects of this graph: (i) its origin as a directed selection from the Fano plane, (ii) an explicit complete Jordan basis, (iii) the biorthogonal projection operators and Jordan–Chevalley decomposition, and (iv) a spectral computation of closed-walk counts.

2 The Fano Plane Structure

2.1 The Fano Plane $P^2(\mathbb{F}_2)$

The Fano plane $P^2(\mathbb{F}_2)$ is the projective plane of order 2, consisting of 7 points and 7 lines, with each line containing exactly 3 points and each point lying on exactly 3 lines. Its automorphism group is $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$, of order 168. Under a standard labelling of the 7 points by $\{1, \ldots, 7\}$, the 7 lines are:

$$\begin{aligned} &L_1 = \{1,2,3\}, \quad L_2 = \{1,4,5\}, \quad L_3 = \{1,6,7\}, \\ &L_4 = \{2,4,6\}, \quad L_5 = \{2,5,7\}, \quad L_6 = \{3,4,7\}, \quad L_7 = \{3,5,6\}. \end{aligned} \tag{2}$$

2.2 Directed Selection from the Fano Plane

Theorem 1 (Fano Plane Correspondence)**.** *The 14 directed edges of A are in bijection with a directed selection from 6 of the 7 Fano lines: the line $L_6 = \{3,4,7\}$ is deleted entirely, and each remaining line contributes a specific subset of directed edges among its 3 points. The correspondence is:*

Fano line	*Point set*	*Edges of A*	*Count*	*Orientation pattern*
L_1	$\{1,2,3\}$	$1{\to}3,\ 3{\to}2$	2	*path* $1 \to 3 \to 2$
L_2	$\{1,4,5\}$	$1{\to}5,\ 5{\to}4,\ 4{\to}1$	3	*directed* 3*-cycle*
L_3	$\{1,6,7\}$	$1{\to}6,\ 1{\to}7,\ 7{\to}1$	3	*two out, one in*
L_4	$\{2,4,6\}$	$2{\to}4,\ 4{\to}2$	2	*bidirectional pair*
L_5	$\{2,5,7\}$	$5{\to}7,\ 7{\to}5$	2	*bidirectional pair*
L_6	$\{3,4,7\}$	—	0	*deleted*
L_7	$\{3,5,6\}$	$3{\to}6,\ 6{\to}3$	2	*bidirectional pair*

The total is $2+3+3+2+2+0+2 = 14$ *edges, which exhaust the edge set of* A.

Proof. We verify three claims: (a) every edge listed belongs to A, (b) no edge of A is omitted, and (c) the line L_6 contributes no edges.

(a) Membership. For each Fano line $L_k = \{a,b,c\}$, the listed edges have both endpoints in L_k, and direct inspection of A confirms $a_{ij} = 1$ for each listed edge $i \to j$. For instance, $L_1 = \{1,2,3\}$: $a_{13} = 1$ and $a_{32} = 1$, while $a_{21} = 0$ (the reverse edge $2 \to 1$ is absent). The remaining lines are checked similarly.

(b) Exhaustion. The 14 listed edges are:

$$1{\to}3,\ 3{\to}2,\ 1{\to}5,\ 5{\to}4,\ 4{\to}1,\ 1{\to}6,\ 1{\to}7,\ 7{\to}1,\ 2{\to}4,\ 4{\to}2,\ 5{\to}7,\ 7{\to}5,\ 3{\to}6,\ 6{\to}3.$$

Sorting gives exactly the edge set of A as stated in Section 1.

(c) Deletion of L_6. The line $L_6 = \{3,4,7\}$ contains the three points $3,4,7$. The six possible directed edges among them are $3{\to}4$, $4{\to}3$, $3{\to}7$, $7{\to}3$, $4{\to}7$, $7{\to}4$. Inspection of A shows $a_{34} = a_{43} = a_{37} = a_{73} = a_{47} = a_{74} = 0$, confirming that none of these edges appear in A.

Since any two distinct points of the Fano plane lie on a unique line, each directed edge $i \to j$ of A belongs to exactly one Fano line (the unique line through i and j). The above enumeration shows that L_6 is the only line contributing no edges, and the remaining six lines partition the 14 edges of A. □

Remark 2. The in-degree condition $\mathbf{1}^T A = 2\,\mathbf{1}^T$ means that every vertex has in-degree 2. Under the Fano plane correspondence, each vertex appears in exactly 3 lines (a standard property of $P^2(\mathbb{F}_2)$). Since L_6 is deleted, each vertex appears in at most 3 remaining lines, and the orientation rules are such that the total in-degree is 2 for every vertex. This is consistent with but not forced by the Fano structure: the specific orientation patterns in Theorem 1 are chosen to achieve this uniformity.

3 Complete Jordan Basis

We construct an explicit basis of generalized eigenvectors for A. All vectors are integer vectors, and all relations are verified by direct computation.

3.1 Eigenvalue $\lambda = 2$

The Perron vector is a true eigenvector:

$$\mathbf{v}_1 = (9,\ 3,\ 2,\ 6,\ 7,\ 1,\ 8)^T, \qquad A\mathbf{v}_1 = 2\,\mathbf{v}_1.$$

Indeed, $A\mathbf{v}_1 = (18,\ 6,\ 4,\ 12,\ 14,\ 2,\ 16)^T = 2\,\mathbf{v}_1$.

3.2 Eigenvalue $\lambda = 1$

Since $\operatorname{rank}(A - I) = 6$, the eigenspace is one-dimensional. Row reduction of $(A - I)\mathbf{x} = \mathbf{0}$ yields:

$$\mathbf{v}_2 = (0,\ 0,\ -1,\ 0,\ 1,\ -1,\ 1)^T, \qquad A\mathbf{v}_2 = \mathbf{v}_2.$$

3.3 Eigenvalue $\lambda = -1$: Jordan Chain of Length 2

The eigenspace $\ker(A + I)$ is 2-dimensional ($\operatorname{rank}(A + I) = 5$), spanned by

$$\mathbf{u}_1 = (0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0)^T, \qquad \mathbf{u}_2 = (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1)^T.$$

Both satisfy $(A + I)\mathbf{u}_k = \mathbf{0}$. Since $\operatorname{rank}\big((A + I)^2\big) = 4$ and $\dim\ker(A + I) = 2$, the Jordan form for $\lambda = -1$ is $J_1(-1) \oplus J_2(-1)$: one true eigenvector and one Jordan chain of length 2.

True eigenvector for $J_1(-1)$:

$$\mathbf{v}_3 = \mathbf{u}_1 = (0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0)^T, \qquad (A + I)\mathbf{v}_3 = \mathbf{0}.$$

Jordan chain for $J_2(-1)$: The chain top is

$$\mathbf{v}_4 = \mathbf{u}_1 - \mathbf{u}_2 = (0,\ 0,\ 1,\ 0,\ -1,\ -1,\ 1)^T, \qquad (A + I)\mathbf{v}_4 = \mathbf{0}.$$

The chain bottom $\mathbf{v}_5$ satisfies $(A + I)\mathbf{v}_5 = \mathbf{v}_4$. Row reduction of $(A + I)\mathbf{x} = \mathbf{v}_4$ gives

$$\mathbf{v}_5 = (0,\ 2,\ 0,\ -2,\ 0,\ -1,\ 1)^T.$$

Verification: $(A + I)\mathbf{v}_5 = (0,\ 0,\ 1,\ 0,\ -1,\ -1,\ 1)^T = \mathbf{v}_4$, and $(A + I)^2\mathbf{v}_5 = \mathbf{0}$.

3.4 Eigenvalue $\lambda = 0$: Jordan Chain of Length 2

Since $\operatorname{rank}(A) = 6$, the nullspace is 1-dimensional, so the Jordan form for $\lambda = 0$ is $J_2(0)$.

Chain top:

$$\mathbf{v}_6 = (1,\ -1,\ 0,\ 0,\ -1,\ 1,\ 0)^T, \qquad A\mathbf{v}_6 = \mathbf{0}.$$

Chain bottom: Solve $A\mathbf{x} = \mathbf{v}_6$ by row reduction:

$$\mathbf{v}_7 = (0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0)^T.$$

Verification: $A\mathbf{v}_7 = (1,\ -1,\ 0,\ 0,\ -1,\ 1,\ 0)^T = \mathbf{v}_6$, and $A^2\mathbf{v}_7 = \mathbf{0}$.

3.5 The Jordan Basis Matrix

Let $V = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_7]$. A cofactor expansion gives

$$\det(V) = 144 \neq 0,$$

so the seven vectors are linearly independent and form a basis of $\mathbb{R}^7$. By construction, $V^{-1}AV = J$ where J is the Jordan normal form (1):

$$V^{-1}AV = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Biorthogonal Projections and Jordan–Chevalley Decomposition

4.1 The Left-Vector Basis

Define $L = V^{-1}$. The i-th row of L, denoted $\boldsymbol{\ell}_i^T$, is the left vector dual to $\mathbf{v}_i$, satisfying the biorthogonality relation

$$\boldsymbol{\ell}_i^T \mathbf{v}_j = \delta_{ij}, \qquad 1 \le i, j \le 7. \tag{3}$$

The explicit form of L, computed by Gauss–Jordan elimination of V, is:

$$L = \begin{bmatrix} \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -\frac{5}{9} & -\frac{1}{18} & \frac{7}{36} & \frac{7}{36} & -\frac{11}{36} & \frac{7}{36} & \frac{25}{36} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Direct matrix multiplication confirms $L \cdot V = I_7$, i.e., the biorthogonality relation (3) holds for all i, j.

4.2 Rank-One Projection Operators

Definition 3. For $i = 1, \ldots, 7$, define the rank-one operator

$$P_i \;=\; \mathbf{v}_i \, \boldsymbol{\ell}_i^T.$$

Theorem 4 (Properties of Projection Operators)**.** *The operators $P_1, \ldots, P_7$ satisfy:*

(a) (Idempotency) $P_i^2 = P_i$ *for all* i.

(b) (Orthogonality) $P_i P_j = 0$ *for* $i \neq j$.

(c) (Completeness) $\displaystyle\sum_{i=1}^{7} P_i = I_7$.

Each P_i has rank 1 *and* $\operatorname{tr}(P_i) = 1$.

Proof. (a) By the biorthogonality relation (3), $\boldsymbol{\ell}_i^T \mathbf{v}_i = 1$. Hence

$$P_i^2 \;=\; (\mathbf{v}_i \boldsymbol{\ell}_i^T)(\mathbf{v}_i \boldsymbol{\ell}_i^T) \;=\; \mathbf{v}_i (\boldsymbol{\ell}_i^T \mathbf{v}_i) \boldsymbol{\ell}_i^T \;=\; \mathbf{v}_i \cdot 1 \cdot \boldsymbol{\ell}_i^T \;=\; P_i.$$

(b) For $i \neq j$, $\boldsymbol{\ell}_i^T \mathbf{v}_j = 0$, so

$$P_i P_j \;=\; \mathbf{v}_i (\boldsymbol{\ell}_i^T \mathbf{v}_j) \boldsymbol{\ell}_j^T \;=\; \mathbf{v}_i \cdot 0 \cdot \boldsymbol{\ell}_j^T \;=\; 0.$$

(c) The sum $\sum_{i=1}^{7} P_i = \sum_{i=1}^{7} \mathbf{v}_i \boldsymbol{\ell}_i^T$ is the matrix product $VL = VV^{-1} = I_7$.
Since each P_i is an outer product of a column vector and a row vector, $\operatorname{rank}(P_i) = 1$, and $\operatorname{tr}(P_i) = \boldsymbol{\ell}_i^T \mathbf{v}_i = 1$. □

4.3 Jordan–Chevalley Decomposition

Theorem 5 (Jordan–Chevalley Decomposition). *Define*

$$S \;=\; \sum_{i=1}^{7} \lambda_i\, P_i, \qquad N \;=\; A - S,$$

where $(\lambda_1, \ldots, \lambda_7) = (2, 1, -1, -1, -1, 0, 0)$*. Then* $A = S + N$ *with:*

(a) S *is diagonalisable (semisimple).*

(b) N *is nilpotent with* $N^2 = 0$*.*

(c) $SN = NS$*.*

Proof. Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_7)$ and $N_J = J - \Lambda$ be the nilpotent part of the Jordan form. Then $S = V \Lambda V^{-1}$ and $N = V N_J V^{-1}$.

(a) S is similar to the diagonal matrix Λ, hence diagonalisable.

(b) The nilpotent part N_J is block-diagonal:

$$N_J \;=\; \operatorname{diag}\bigl(0,\ 0,\ 0,\ \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right],\ \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\bigr),$$

where the non-zero 2×2 blocks correspond to $J_2(-1)$ and $J_2(0)$. Each such block is a nilpotent matrix of nilpotency index 2, and since the blocks are in separate block-diagonal positions, $N_J^2 = 0$. Therefore $N^2 = V N_J^2 V^{-1} = 0$.

(c) Since Λ and N_J are simultaneously block-diagonal in the same block structure, they commute: $\Lambda N_J = N_J \Lambda$. Conjugating by V gives $SN = NS$. □

5 Spectral Computation of Closed-Walk Counts

5.1 The Weighted Matrix

Define the diagonal matrix $D = \operatorname{diag}(9, 3, 2, 6, 7, 1, 8)$ and the weighted matrix

$$W \;=\; \frac{1}{2} D^{-1} A D, \qquad W_{ij} = \frac{A_{ij}\, v_j}{2\, v_i}.$$

Since $(A\mathbf{v})_i = 2\, v_i$, we have $\sum_j W_{ij} = (A\mathbf{v})_i/(2v_i) = 1$, so W is row-stochastic. For any directed edge $i \to j$, the weight $w_{ij} = v_j/(2v_i) = W_{ij}$.

Proposition 6 (Closed-Walk Equal-Weight Property). *For any closed directed walk of length* L *(vertices and edges may repeat), the product of weights along the walk is* 2^{-L}*, independent of the walk's itinerary.*

Proof. Since $w_{i_k, i_{k+1}} = v_{i_{k+1}}/(2\, v_{i_k})$, the product telescopes:

$$\prod_{k=0}^{L-1} \frac{v_{i_{k+1}}}{2\, v_{i_k}} \;=\; \frac{1}{2^L} \cdot \frac{v_{i_1}}{v_{i_0}} \cdot \frac{v_{i_2}}{v_{i_1}} \cdots \frac{v_{i_L}}{v_{i_{L-1}}} \;=\; \frac{1}{2^L} \cdot \frac{v_{i_L}}{v_{i_0}} \;=\; \frac{1}{2^L},$$

where the last equality uses $i_L = i_0$. □

5.2 Spectral Computation of N_L

Let N_L denote the total number of closed directed walks of length L in G (permitting repeated vertices and edges). A standard result in algebraic graph theory [5] states

$$N_L \;=\; \mathrm{tr}(A^L), \tag{4}$$

since $(A^L)_{ii}$ counts the number of walks of length L from vertex i to itself.

Theorem 7 (Closed-Walk Count Formula)**.** *For all $L \geq 1$,*

$$N_L \;=\; 2^L + 1 + 3(-1)^L.$$

Equivalently, for odd L: $N_L = 2^L - 2$; for even L: $N_L = 2^L + 4$.

Proof. Since A is similar to its Jordan form J, we have $\mathrm{tr}(A^L) = \mathrm{tr}(J^L)$. The Jordan form (1) is block-diagonal, so $\mathrm{tr}(J^L) = \sum_{\text{blocks}} \mathrm{tr}(J_k(\lambda)^L)$.

For any Jordan block $J_k(\lambda) = \lambda I + N_k$ where N_k is the $k \times k$ nilpotent matrix with ones on the superdiagonal, the binomial expansion gives

$$J_k(\lambda)^L \;=\; \sum_{j=0}^{\min(L,k-1)} \binom{L}{j} \lambda^{L-j} N_k^j.$$

Since N_k^j has non-zero entries only on the j-th superdiagonal, its diagonal entries are all zero for $j \geq 1$. Hence $\mathrm{tr}(N_k^j) = 0$ for $j \geq 1$, and

$$\mathrm{tr}\big(J_k(\lambda)^L\big) \;=\; \lambda^L \cdot \mathrm{tr}(I_k) \;=\; k \cdot \lambda^L.$$

Therefore $\mathrm{tr}(A^L) = \sum_{i=1}^{7} \lambda_i^L$ (sum over eigenvalues with algebraic multiplicity), giving

$$\mathrm{tr}(A^L) \;=\; 2^L + 1^L + \underbrace{0^L + 0^L}_{=0 \text{ for } L \geq 1} + \underbrace{(-1)^L + (-1)^L + (-1)^L}_{=3(-1)^L} \;=\; 2^L + 1 + 3(-1)^L.$$

By (4), $N_L = 2^L + 1 + 3(-1)^L$.

For odd L: $(-1)^L = -1$, so $N_L = 2^L + 1 - 3 = 2^L - 2$. For even L: $(-1)^L = 1$, so $N_L = 2^L + 1 + 3 = 2^L + 4$. ☐

Remark 8 (Numerical Verification). The formula gives the following values for $L = 1, \ldots, 10$:

$$N_L \;=\; (0,\ 8,\ 6,\ 20,\ 30,\ 68,\ 126,\ 260,\ 510,\ 1028).$$

Direct computation of $\mathrm{tr}(A^L)$ confirms full agreement.

5.3 Consistency with the Equal-Weight Property

Since $W = \frac{1}{2}D^{-1}AD$, the matrix W is similar to $\frac{1}{2}A$, with eigenvalues $\lambda_i/2$. By the same trace argument as above,

$$\mathrm{tr}(W^L) \;=\; \sum_{i=1}^{7} \left(\frac{\lambda_i}{2}\right)^L \;=\; 1 + \left(\frac{1}{2}\right)^L + 3\left(\frac{-1}{2}\right)^L.$$

Alternatively, since $W^L = \frac{1}{2^L}D^{-1}A^L D$ and trace is similarity-invariant,

$$\mathrm{tr}(W^L) \;=\; \frac{\mathrm{tr}(A^L)}{2^L} \;=\; \frac{N_L}{2^L}. \tag{5}$$

On the other hand, $(W^L)_{ii}$ equals the sum of weight-products over all closed walks of length L starting and ending at vertex i. By Proposition 6, each such walk contributes 2^{-L}, so

$$\mathrm{tr}(W^L) \;=\; \sum_{\text{closed walks of length } L} 2^{-L} \;=\; N_L \cdot 2^{-L}.$$

This is consistent with (5), confirming the agreement between the spectral computation and the equal-weight property.

6 Relation to the Heawood Graph

The Heawood graph is the incidence graph of the Fano plane: a bipartite graph with 14 vertices (one for each point and each line of $P^2(\mathbb{F}_2)$) and 21 edges (one for each point-line incidence). Its automorphism group is $\mathrm{GL}(3, \mathbb{F}_2)$, acting transitively on both the point-vertices and the line-vertices [5].

Remark 9. The directed graph G can be obtained from the Heawood graph by the following construction. Delete the line-vertex corresponding to $L_6 = \{3, 4, 7\}$ and its 3 incident edges, leaving 18 edges among 6 line-vertices and 7 point-vertices. For each remaining line-vertex l_k (incident to point-vertices a, b, c), replace the star $\{a\text{-}l_k,\ b\text{-}l_k,\ c\text{-}l_k\}$ by directed edges among $\{a, b, c\}$ according to the orientation pattern of Theorem 1. This collapses the 14 vertices to 7 and produces the 14 directed edges of A.

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