

Square Roots of Spacetime: A Dirac–Penrose Primer in Geometric Algebra

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Abstract

This primer presents Dirac’s and Penrose’s “square-root” methods in one language: real geometric (Clifford) algebra. Dirac’s square root of the wave operator forces the Clifford relation and leads to rotors, spinors, and the Dirac equation; Penrose’s square root of a null vector leads to 2-spinors, flags, twistors, and the spin-network idea that geometry may be recovered from quantum angular momentum. The exposition is written for graduate students who know special relativity and elementary quantum mechanics. It keeps the historical claims modest, separates convention from invariant content, and works every major idea through a concrete computation: the belt trick, the Hestenes form of the Dirac equation, the null flagpole, the Dirac–Penrose conformal bridge, the spin-geometry theorem, and a complete 2+1-dimensional miniature in which bivectors, Lorentz orbits, unitary representations of $SU(1,1)$, and Clifford-signature superselection all fit into one diagram. Sixteen exercises are included so that the reader can verify the algebra rather than merely admire it.

1 Two habits of thought

Open Dirac’s 1928 paper [1] and Penrose’s 1967 paper [3] side by side and the contrast is immediate. Dirac starts from a dynamical requirement: a first-order relativistic wave equation whose square gives the Klein–Gordon equation. The requirement forces an algebra. Penrose starts from a geometrical requirement: null directions and conformal relations should be more primitive than coordinates on a manifold. The requirement forces a new arena. Both moves are square roots: the Dirac operator is a square root of the wave operator; a 2-spinor is a square root of a future null vector; twistor space is the spinorial, hence square-root, representation of conformally compactified spacetime; and spin networks ask whether ordinary spatial angles can emerge as the large-spin shadow of pure recoupling combinatorics.

The purpose of this primer is to teach these two habits in one notation. Geometric algebra — Clifford algebra used directly as a geometric calculus, in the tradition of Hestenes [7, 8] and Doran–Lasenby [9] — is the right notation because it keeps three facts visible at once: vectors square to intervals, bivectors generate rotations and boosts, and spinors are the even objects that transform before vectors do. The payoff is not a new formalism for its own sake. It is that a small number of computations recur in many disguises: Dirac matrices, Lorentz rotors, Penrose flags, twistors, spin networks, and low-dimensional representation theory all become instances of the same square-root pattern.

Prerequisites and conventions. Linear algebra, special relativity, and a first quantum mechanics course. We use metric signature $(+, -, -, -)$ throughout (Penrose’s convention [6]), units $\hbar = c = 1$,

and we state every further convention where it is introduced. The Dirac–Penrose literature harbors at least three mutually inconsistent sign conventions for spinors and helicity; the single most useful habit this primer can teach is: *never compute across two sources without re-deriving the convention-sensitive signs once yourself*.

Plan. Section 2 derives the Clifford relation from Dirac’s factorization and introduces rotors through the belt trick. Section 3 rewrites the Dirac equation in Hestenes’ real spacetime algebra. Section 4 constructs 2-spinors as square roots of null vectors and explains the zig-zag electron. Section 5 is the meeting point: Dirac’s projective light cone and Penrose’s twistor space are the vector and spinor linearizations of the conformal group. Section 6 states what the original spin-network program really recovers, and what it does not. Section 7 gives a complete 2+1-dimensional miniature. Section 8 points to continuations.

Square-root object	Geometry recovered	Group seen first
Dirac operator	wave operator / mass shell	$\text{Spin}^+(1, 3)$
2-spinor	future null flagpole	$\text{SL}(2, \mathbb{C})$
Twistor	conformal incidence geometry	$\text{SU}(2, 2) \simeq \text{Spin}(2, 4)$
Spin network	large-spin Euclidean angles	$\text{SU}(2)$ recoupling
2+1 bivector	Lorentz orbits and representation families	$\text{SU}(1, 1)$

2 Dirac’s square root and the algebra it forces

2.1 The factorization

Dirac sought a first-order operator $D = \gamma^\mu \partial_\mu$ whose square is the wave operator $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$. Expanding,

$$D^2 = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu, \quad (1)$$

since partial derivatives commute. The demand $D^2 = \square$ is therefore *exactly* the statement

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \quad (2)$$

Dirac realized (2) with 4×4 complex matrices. The geometric-algebra reading, due in its mature form to Hestenes [7], drops the matrices: relation (2) is the definition of an *orthonormal basis of spacetime itself*, provided one allows vectors to be multiplied. Define the *geometric product* of vectors to be associative, distributive, and to satisfy $v^2 = \eta(v, v)$ for every vector v ; then (2) follows by polarization, and the γ_μ are the basis vectors of Minkowski space. The algebra they generate is the real Clifford algebra $\text{Cl}(1, 3)$, the *spacetime algebra* (STA): a 16-dimensional real space spanned by

$$\underbrace{1}_{\text{scalar}} \quad \underbrace{\gamma_\mu}_{4 \text{ vectors}} \quad \underbrace{\gamma_\mu \gamma_\nu \ (\mu < \nu)}_{6 \text{ bivectors}} \quad \underbrace{\gamma_\mu \gamma_\nu \gamma_\rho}_{4 \text{ trivectors}} \quad \underbrace{I \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3}_{\text{pseudoscalar}}. \quad (3)$$

For any two vectors the geometric product splits into symmetric and antisymmetric parts,

$$ab = a \cdot b + a \wedge b, \quad (4)$$

a scalar plus a bivector: the product remembers both the metric and the oriented plane. This single identity is the engine of everything below.

Exercise 1. Derive (2) from $v^2 = \eta(v, v)$ by expanding $(a + b)^2$. Show $I^2 = -1$ and that I anticommutes with every vector in $\text{Cl}(1, 3)$.

Exercise 2. Show that for a *simple* bivector $B = a \wedge b$,

$$B^2 = (a \cdot b)^2 - a^2 b^2 \quad (5)$$

is a scalar. Conclude that in Minkowski signature simple bivectors come in three kinds: *elliptic* ($B^2 < 0$: spacelike planes, e.g. $\gamma_1 \gamma_2$), *hyperbolic* ($B^2 > 0$: timelike planes, e.g. $\gamma_1 \gamma_0$), and *parabolic* ($B^2 = 0$: null planes). This trichotomy is the recurring character of the primer; it will return as the classification of Lorentz transformations, of faces of a tetrahedron, and of unitary representation families.

2.2 Rotors and the belt trick

Let B be a bivector with $B^2 = -1$ and set

$$R(\theta) = e^{-B\theta/2} = \cos \frac{\theta}{2} - B \sin \frac{\theta}{2}. \quad (6)$$

The map $v \mapsto Rv\tilde{R}$ (where $\tilde{R}$ reverses the order of vector factors, $\tilde{ab} = ba$) is a rotation by angle θ in the plane of B , as a two-line computation with $v_{\parallel}, v_{\perp}$ shows. Because the group element acts *two-sidedly with half angles*, we get the fact that makes spinors spinors:

$$R(2\pi) = e^{-B\pi} = -1 \neq +1, \quad R(4\pi) = +1. \quad (7)$$

A full turn of the plane returns every *vector* to itself ($(-1)v(-1) = v$) but not the *rotor*; only after two full turns does the rotor return. This is the algebraic content of the belt trick, and of the statement that the rotor group $\text{Spin}^+(1, 3) (\cong \text{SL}(2, \mathbb{C}))$ is the double cover of the restricted Lorentz group $\text{SO}^+(1, 3)$. Objects that transform one-sidedly under rotors — $\psi \mapsto R\psi$ — are *spinors*, and they are the objects for which $2\pi \neq 4\pi$. Hyperbolic bivectors exponentiate the same way to boosts, $e^{-B\phi/2}$ with $B^2 = +1$ giving $\cosh$ – $\sinh$; parabolic ones give null rotations, $e^{-B\lambda/2} = 1 - B\lambda/2$ exactly, since $B^2 = 0$.

Exercise 3. Verify $Rv_{\parallel}\tilde{R}$ rotates $v_{\parallel}$ by θ and $Rv_{\perp}\tilde{R} = v_{\perp}$, for $v_{\parallel}$ in the plane of B and $v_{\perp}$ orthogonal to it. (Hint: $v_{\parallel}B = -Bv_{\parallel}$ and $v_{\perp}B = Bv_{\perp}$.)

Exercise 4. Show that every rotor satisfies $R\tilde{R} = 1$, and that the composition of rotations is the geometric product of rotors — no matrices, no Euler angles, no gimbal lock.

3 The Dirac equation in the real spacetime algebra

Warning. Everything in this section is a *retrospective* reading. Dirac worked with complex matrices and column spinors; the reformulation below is due to Hestenes [8], decades later. It is presented because it is illuminating, not because it is history.

In the matrix theory a Dirac spinor has four complex components — eight real numbers. The even subalgebra $\text{Cl}^+(1, 3)$ (scalars, bivectors, pseudoscalar) is an eight-dimensional real space. Hestenes' observation is that the two are isomorphic *as representation spaces*, and that under the isomorphism the Dirac equation becomes an equation among real geometric objects:

$$\nabla \psi I\sigma_3 = m \psi \gamma_0, \quad \nabla \equiv \gamma^{\mu} \partial_{\mu}, \quad \sigma_3 \equiv \gamma_3 \gamma_0, \quad (8)$$

with ψ an even multivector field. (Equivalent forms with the spatial bivector $\gamma_2 \gamma_1$ or $\gamma_1 \gamma_2$ in place of $I\sigma_3$ circulate in the literature; the choice is a convention for which laboratory plane carries the role of the ordinary imaginary unit.) Three structural lessons, each independent of the convention:

1. **The imaginary unit is a plane.** In (8) the i of quantum mechanics has been replaced by right multiplication by a *real bivector* of square -1 . The “complex phase” of the electron is a rotation angle in a definite spin plane. Nothing intrinsically complex remains.
2. **The spinor is an instruction to rotate and dilate.** Where $\psi\tilde{\psi} \neq 0$ one can factor $\psi = \rho^{1/2}e^{I\beta/2}R$ with $\rho \geq 0$, β an angle, and R a rotor. The observables of Dirac theory are then manifestly geometric: the probability current is the vector field $J = \psi\gamma_0\tilde{\psi}$ (the rotor R turning the fixed γ_0 into the local flow direction, scaled by ρ), and the spin plane is $R\sigma_3\tilde{R}$ scaled likewise.
3. **Zitterbewegung becomes a picture.** The phase rotation of ψ is a literal rotation in the local spin plane at frequency $2m$; Schrödinger’s zitterbewegung, and Penrose’s zig-zag of the next section, are two bookkeepings of the same internal rotation.

Exercise 5. Show that $\psi\tilde{\psi}$ for even ψ is always of the form $\rho e^{I\beta}$ (scalar plus pseudoscalar), so the polar factorization above exists wherever it is nonzero. Show $J = \psi\gamma_0\tilde{\psi}$ is a vector (odd grades cancel) and compute it for $\psi = R$ a pure rotor.

Exercise 6. Substitute plane waves $\psi(x) = \psi_0 e^{-I\sigma_3 p \cdot x}$ into (8) and derive the momentum-space condition; show $p^2 = m^2$ follows by squaring, i.e. re-run Dirac’s square root backwards.

4 Penrose’s square root: the flagpole and the zig-zag

Dirac took the square root of an *operator*. Penrose’s tradition begins by taking the square root of a *vector*. The vector must be null, because only a rank-one spinor square has zero determinant.

4.1 Two-component spinors and null flags

Encode a real 4-vector v^μ as the Hermitian 2×2 matrix

$$V = v^\mu \sigma_\mu = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}, \quad \sigma_\mu = (\mathbb{I}, \vec{\sigma}), \quad \det V = v \cdot v. \quad (9)$$

(Penrose–Rindler [6] insert a factor $1/\sqrt{2}$ in (9), which moves factors of 2 around in what follows — convention warning number one.) The action $V \mapsto AVA^\dagger$ with $A \in \text{SL}(2, \mathbb{C})$ preserves Hermiticity and $\det$, and realizes exactly the restricted Lorentz group, two-to-one: this is the matrix face of the rotor group of Section 2, since $\text{Cl}^+(1, 3) \cong \text{Mat}_2(\mathbb{C})$.

Now let $\kappa = \begin{pmatrix} \kappa^0 \\ \kappa^1 \end{pmatrix} \in \mathbb{C}^2$ be a 2-spinor and form the outer product

$$N = \kappa \kappa^\dagger. \quad (10)$$

N is Hermitian, $\det N = 0$, and $\text{tr } N = |\kappa^0|^2 + |\kappa^1|^2 \geq 0$: by (9) it encodes a *future-pointing null vector*, the *flagpole* of κ . Every future null vector arises this way, and κ is determined by its flagpole up to phase, $\kappa \mapsto e^{i\varphi}\kappa$. The phase is not junk: it rotates a half-plane attached to the flagpole (the *flag*), through *twice* φ — the 2:1 cover appearing yet again, now attached to a single null direction. In index notation this is the celebrated $v^{AA'} = \kappa^A \bar{\kappa}^{A'}$ of the 2-spinor calculus, where unprimed indices transform under $\text{SL}(2, \mathbb{C})$ and primed ones under the conjugate representation, raised and lowered with the antisymmetric $\epsilon_{AB}, \epsilon_{A'B'}$.

The geometric-algebra translation makes the “square root of a null vector” literal. Null vectors of the form $n = \rho R(\gamma_0 + \gamma_3)\tilde{R}$ satisfy $n^2 = 0$ and stand in the same relation to even elements ψ as N does to κ : the null vector is obtained by sandwiching a fixed null direction between spinorial data. Square roots of geometry live one grade *below* the geometry, on objects that feel 2π .

Exercise 7. Verify from (9) and (10) that the flagpole is null and future-pointing, that it is insensitive to the phase of κ , and that $\kappa \rightarrow A\kappa$ induces the Lorentz action on it. For $\kappa = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ compute the flagpole explicitly.

Exercise 8. Show that two flagpoles determine the relativistic invariant $|\langle \kappa, \lambda \rangle|^2 = |\kappa^0 \lambda^1 - \kappa^1 \lambda^0|^2 = \frac{1}{2} u \cdot v$, where u, v are the flagpoles (in convention (9)): angles between null directions are squared spinor brackets. This identity is the seed of the modern spinor-helicity formalism for scattering amplitudes.

4.2 The zig-zag electron

In the Weyl (chiral) basis the Dirac equation splits into two coupled 2-spinor equations,

$$i \bar{\sigma}^\mu \partial_\mu \psi_L = m \psi_R, \quad i \sigma^\mu \partial_\mu \psi_R = m \psi_L, \quad \sigma^\mu = (\not{x}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\not{x}, -\vec{\sigma}). \quad (11)$$

Each equation alone (set $m = 0$) is a massless Weyl equation, whose solutions propagate along null flagpoles. Penrose’s zig-zag picture [6] reads (11) dynamically: the electron is a massless *zig* (ψ_L) that converts, at rate set by m , into a massless *zag* (ψ_R) and back, each leg lightlike, the mass being the amplitude for the flip. The mean progress of the zig-zag is the timelike worldline; the internal flip frequency is the zitterbewegung of Section 3, met from the other side. Dirac’s square root and Penrose’s square root are the two halves of one equation.

Exercise 9. Derive (11) from the matrix Dirac equation in the Weyl basis, and check that each of $\psi_{L,R}$ separately satisfies Klein–Gordon. Where exactly does the convention choice in σ^μ vs $\bar{\sigma}^\mu$ enter?

5 The conformal meeting point: Dirac 1936 and Penrose 1967

The two traditions converge on the conformal group, from opposite sides, and the convergence is the best single advertisement for the “square root” viewpoint.

5.1 Dirac’s projective light cone

In *Wave Equations in Conformal Space* [2], Dirac embeds compactified Minkowski space in six dimensions. Take coordinates X^A on $\mathbb{R}^{4,2}$ with metric $\eta_{AB} = \text{diag}(-, +, +, +, +, -)$ (the overall sign is conventional here), impose the cone condition $X \cdot X = 0$, and identify $X \sim \lambda X$, $\lambda \neq 0$. The resulting space of null *rays* is conformally compactified Minkowski space $M^\#$. A convenient section: writing $X^\pm = X^5 \pm X^4$ (say), points with $X^+ \neq 0$ can be scaled to $X^+ = 1$, and the remaining four coordinates $x^\mu = X^\mu / X^+$ are ordinary Minkowski coordinates; the flat metric is recovered on the section, and different sections differ by conformal factors. The payoff is *linearization*: the conformal group of Minkowski space, an awkward nonlinear action involving inversions, becomes the linear group $\text{SO}(2, 4)$ acting on the ambient $\mathbb{R}^{4,2}$; and conformally covariant wave equations become homogeneous equations on the cone. This is a direct ancestor of the embedding-space method in conformal field theory and is closely related in spirit to the tractor calculus of conformal geometry.

Readers who know conformal geometric algebra (CGA) should feel at home: CGA performs the analogous trick one signature over, embedding Euclidean $\mathbb{R}^n$ in the null cone of $\mathbb{R}^{n+1,1}$ so that the conformal group of $\mathbb{R}^n$ acts by rotors of $\text{Cl}(n+1, 1)$, and spheres and planes become blades. Dirac’s 1936 construction is the Lorentzian ancestor of that method: the same null-cone idea applied to spacetime rather than to Euclidean space.

Exercise 10. On the section $X^+ = 1$, show that a linear $\text{SO}(2, 4)$ transformation that mixes X^μ with $X^\pm$ induces on x^μ precisely the special conformal transformations and dilations, and locate the inversion. Where do the points with $X^+ = 0$ go? (They are the conformal boundary — the germ of Penrose’s $\mathcal{I}$.)

5.2 Penrose’s twistors

Penrose’s 1967 linearization [3] of the same group is spinorial. Twistor space is $\mathbb{T} = \mathbb{C}^4$ with elements

$$Z^\alpha = (\omega^A, \pi_{A'}), \quad (12)$$

a pair of 2-spinors, equipped with the Hermitian form $Z \cdot \bar{Z} = \omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}$ of signature $(2, 2)$. The invariance group is $\text{SU}(2, 2)$, and the isomorphism

$$\text{SU}(2, 2) \cong \text{Spin}(2, 4) \quad (13)$$

says that *twistor space is the spinor representation of Dirac’s conformal group*: Penrose’s arena is the square root of Dirac’s. The two constructions of $M^\#$ — null rays in $\mathbb{R}^{4,2}$, and holomorphic lines in projective twistor space — are the vector and spinor descriptions of one geometry.

Spacetime re-enters through the *incidence relation*

$$\omega^A = i x^{AA'} \pi_{A'}. \quad (14)$$

Fix a real point x : equation (14) is linear in Z , and its solutions form a complex 2-plane in $\mathbb{T}$, i.e. a projective line $\mathbb{CP}^1$ in $P\mathbb{T}$. Fix instead a twistor with $Z \cdot \bar{Z} = 0$ (a *null* twistor): the real points incident with it form a null geodesic — a light ray. So the dictionary is deliberately non-local: *point* $\leftrightarrow$ *sphere of light rays through it*; *light ray* $\leftrightarrow$ *point (of PT)*. Causal structure becomes intersection theory: two points are null separated iff their $\mathbb{CP}^1$ ’s meet. For non-null twistors, $\frac{1}{2} Z \cdot \bar{Z}$ measures the helicity of an associated massless particle (up to normalization conventions that differ between sources — convention warning number two).

Warning. A twistor is *not* a Dirac spinor, although both are pairs of 2-spinors with four complex components. A Dirac spinor (ψ_L, ψ_R) carries a representation of the Lorentz group $\text{SL}(2, \mathbb{C})$ at a point; a twistor carries a representation of the conformal group $\text{SU}(2, 2)$ and, through (14), is an intrinsically non-local object. Penrose himself warned against the identification. The components match; the geometry does not.

Twistor theory is at its most natural for massless fields, conformally invariant equations, and self-dual sectors. Extending it to full Lorentzian Einstein gravity and to massive matter — through the googly problem, ambitwistors, and two-twistor constructions — is subtler, and a primer should not oversell it. What is not oversold is the kinematics above — it is exact, and it is the conformal bridge on which the modern scattering-amplitude and tractor literatures stand.

Exercise 11. Show that the incidence relation (14) with x Hermitian forces $Z \cdot \bar{Z} = 0$: real points see only null twistors. Show that two points x, y are incident with a common twistor iff $\det(x - y)^{AA'} = 0$, i.e. iff they are null separated.

6 Geometry from spin: Penrose 1971

The most radical square root comes last. In 1971 Penrose asked [4]: could the directional geometry of space be, not a container for quantum systems, but a *statistical consequence* of quantum angular momentum alone?

A *spin network* (in the original sense) is a closed trivalent graph with edges labeled by non-negative half-integers (spins), obeying the triangle rule at each vertex. To each such network the $SU(2)$ recoupling calculus assigns a number (contract a Wigner $3j$ symbol at each vertex and sum over magnetic indices); ratios of these numbers are probabilities for outcomes of combining and splitting angular momenta. There is no manifold anywhere in the definition — only combinatorics.

Theorem 1 (Spin-geometry theorem; Penrose 1971, rigorous form Moussouris 1983 [5]). *Let two large-spin subsystems emanate from a sufficiently complex spin network, and define the “angle” between them operationally, through the probabilities with which one subsystem’s spin projects onto the other’s axis. In the limit of large spins these operational angles satisfy the consistency relations of directions in three-dimensional Euclidean space.*

Two honest qualifications belong to the statement. The theorem recovers *angles only* — directional geometry, not distances; lengths and areas require additional input. And it recovers *three-dimensional Euclidean* structure, not four-dimensional Lorentzian spacetime. Penrose’s 1971 networks are also not identical to the spin networks of loop quantum gravity, which came a quarter century later: the latter allow arbitrary valence with group-theoretic intertwiners at nodes and arise as a basis of a kinematical Hilbert space, rather than as closed diagrams with evaluations. The lineage, however, is direct, and the modern theory rewards the reader of this primer twice over: its Lorentzian vertex amplitudes are built exactly from the $SL(2, \mathbb{C})$ representation theory whose low-dimensional cousin we dissect next, and its phase space admits a parametrization by pairs of twistors (“twisted geometries”). See [14] to continue.

Exercise 12. The simplest nontrivial evaluation: a theta graph (two vertices joined by three edges labeled a, b, c). Write its value as the square of a $3j$ symbol contraction and verify it vanishes unless (a, b, c) satisfy the triangle rule. For edge labels $(2j, 2j, 2)$ and large j , relate the ratio of evaluations to the probability distribution of the angle between two spin- j systems combined to near-maximal total spin.

7 A worked miniature: 2+1 dimensions from algebra to representations

We close with a complete, self-contained case study that exercises every habit of the primer at once, in the smallest interesting arena: 2+1 Minkowski space, algebra $Cl(1, 2)$, generators e_0, e_1, e_2 with $e_0^2 = +1, e_1^2 = e_2^2 = -1$. The method is Dirac’s — let the algebra classify — executed with Penrose’s aim — read the geometry, and then the quantum theory, off the classification.

7.1 Bivectors classify the group

$Cl(1, 2)$ has exactly three basis bivectors, one of each kind of Exercise 2:

$$e_1e_2 \text{ (elliptic, } B^2 = -1), \quad e_0e_1, e_0e_2 \text{ (hyperbolic, } B^2 = +1), \quad (15)$$

and null combinations such as $(e_0 + e_1)e_2$ (parabolic, $B^2 = 0$). Exponentiating gives the three conjugacy classes of the identity-connected group $Spin^+(1, 2)$: rotations, boosts, null rotations. The even subalgebra $Cl^+(1, 2)$ is spanned by $\{1, e_1e_2, e_0e_1, e_0e_2\}$ and is isomorphic to $Mat_2(\mathbb{R})$ (the split quaternions); its unit-norm elements give

$$Spin^+(1, 2) \cong SL(2, \mathbb{R}) \cong SU(1, 1), \quad (16)$$

the last isomorphism realized by the matrices $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 - |\beta|^2 = 1$.

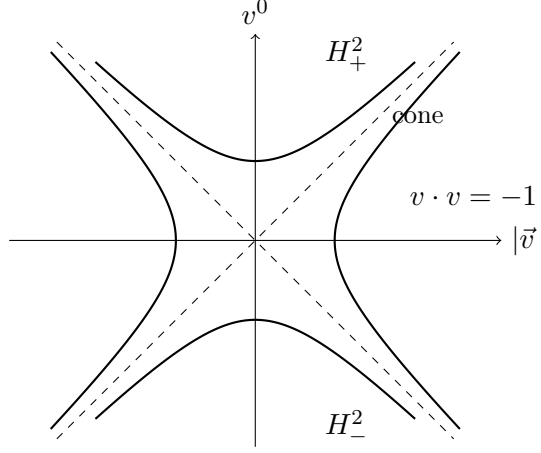


Figure 1: The three orbit families of $\text{Spin}^+(1, 2)$ on unit directions of 2+1 Minkowski space, drawn in the $(|\vec{v}|, v^0)$ half-plane and its mirror. The bound $\Lambda_{00} \geq 1$ of (17) makes H_+^2 and H_-^2 , and the two cone components, disjoint orbits; the one-sheeted hyperboloid is a single orbit containing each antipodal pair.

7.2 The group classifies the orbits

The unit quadrics of the vector space fall into the same trichotomy (Figure 1): the two-sheeted hyperboloid $H_\pm^2 = \{v \cdot v = \pm 1\}$ of timelike directions, the light cone $v \cdot v = 0$, and the one-sheeted hyperboloid $\{v \cdot v = -1\}$ of spacelike directions. Which pairs of points can the group connect? The decisive computation is one line. Encoding $v = v^0 \mathbb{I} + v^1 \sigma_1 + v^2 \sigma_2$ and acting by $v \mapsto g v g^\dagger$, the time-time component of the induced Lorentz matrix is

$$\Lambda_{00} = \frac{1}{2} \text{tr}(g g^\dagger) = |\alpha|^2 + |\beta|^2 = 1 + 2|\beta|^2 \geq 1 : \quad (17)$$

the connected group is *orthochronous*, exactly and for every element. Consequences:

1. the future and past sheets of H^2 are *disjoint orbits*: no group element maps a future timelike direction to a past one, since that would need $\Lambda_{00} \leq -1$;
2. the future and past components of the light cone are likewise disjoint orbits — the parabolic case sides with the elliptic one;
3. on the one-sheeted hyperboloid the antipode *is* reached: conjugating the half-turn $e^{-e_1 e_2 \pi/2}$ by the boost-rotation taking a given spacelike v to $(0, 1, 0)$ gives an explicit g_v with $\Lambda(g_v)v = -v$ (Exercise 13).

So the *orientation* (future/past label) of a direction is a group invariant precisely when the direction is non-spacelike — precisely when the bivector of the orthogonal plane is non-hyperbolic. The trichotomy of Exercise 2 has become a theorem about which geometric data are absolute.

7.3 The orbits classify the representations, and the pseudoscalar has the last word

Bargmann's classification [11] of the unitary irreducible representations of $\text{SU}(1, 1)$ is richer than the orbit slogan, but the main families mirror the picture: the *discrete series* $D_k^\pm$, lowest- and highest-weight families whose two signs correspond to the two sheets $H_\pm^2$; principal continuous families

associated with hyperbolic orbits; and limiting/complementary cases that sit at the boundary of this sketch. The orbit disjointness of the sheets has a sharp quantum image: D_k^+ and D_k^- share the Casimir $k(k-1)$ but have disjoint weight spectra $\{k, k+1, \dots\}$ and $\{-k, -k-1, \dots\}$; a bounded intertwiner must preserve weights, hence vanishes:

$$\text{Hom}_{\text{SU}(1,1)}(D_k^+, D_k^-) = \{0\}. \quad (18)$$

The future/past label is *superselected*: no covariant map connects the two signs. For spacelike data, by contrast, the antipodal flip is implemented by an ordinary connected-group element (item (iii) above) acting *inside* the relevant representation space — no sheet label is protected.

Where, algebraically, does the flip live, if not in the connected group? In the pseudoscalar. $I = e_0 e_1 e_2$ is a product of unit vectors — a versor, an element of the Pin group — it is *odd*, it is central (the dimension is odd), and its twisted adjoint action on vectors is

$$\alpha(I) v I^{-1} = -I v I^{-1} = -v : \quad (19)$$

the universal antipode, the exchange of the sheets, implemented one grade up from the rotors and outside the connected group — consistent with, and explaining, (17). And now signature has the last word. In $\text{Cl}(1, 2)$, $I^2 = -1$: the center is $\mathbb{R} \oplus \mathbb{R}I \cong \mathbb{C}$, and the flip's conserved charge is a complex phase. In the opposite convention $\text{Cl}(2, 1)$, $I^2 = +1$: the projectors $P_{\pm} = \frac{1}{2}(1 \pm I)$ are *central idempotents*, the algebra splits into two ideals that no group element connects, and the flip's charge is a real, two-valued, superselected label. Whether “future versus past” is a real charge or a phase is decided by $(p - q) \bmod 8$ — the same periodicity that organizes real spinor modules in every dimension [10]. The reader who has followed the primer this far has just watched, in eight dimensions of algebra, the governing arc: an algebraic trichotomy (Exercise 2) became a geometric orbit theorem (17), the orbit theorem became a representation-theoretic superselection rule, and the deepest invariant of the story — the reality of the orientation charge — turned out to be a property of the Clifford signature itself. That arc, run in four dimensions with $\text{SL}(2, \mathbb{C})$ in place of $\text{SU}(1, 1)$, is the working method of current research on causal structure in quantum gravity.

Exercise 13. Prove (17) from the parametrization of $\text{SU}(1, 1)$, and prove that an orthochronous transformation preserves the sign of v^0 on timelike *and null* vectors but not on spacelike ones (exhibit the failure).

Exercise 14. Construct g_v of item (iii) explicitly for $v = (\sinh \chi, \cosh \chi, 0)$ and verify $\Lambda(g_v)v = -v$ by matrix multiplication.

Exercise 15. Show $I^2 = -1$ in $\text{Cl}(1, 2)$ and $I^2 = +1$ in $\text{Cl}(2, 1)$ by direct computation, and that I is central in both. Solve $(a + bI)^2 = a + bI$ over the reals in each case and conclude that nontrivial central idempotents exist only in $\text{Cl}(2, 1)$.

Exercise 16 (capstone). Repeat Exercise 14 one dimension up: show $I^2 = -1$ in $\text{Cl}(1, 3)$ while in $\text{Cl}(1, 4)$ the pseudoscalar is central with $I^2 = +1$, so that $\text{Cl}(1, 4)$ splits into two ideals. The pattern of the miniature persists: whether a “chirality-like” label is a real superselected charge is always a question about the signature, never about the dynamics.

8 Roads from here

Each section of this primer is the trailhead of a living subject, and the reader should leave with a map rather than a summary.

From Section 3: the spacetime-algebra program continues into a gauge-theoretic formulation of gravity on flat space [12], in which the rotor group is gauged and the Einstein equations reappear as field equations for position and rotation gauge fields — geometric-algebra technique at full strength.

From Section 5: Dirac’s cone became the embedding-space method of modern conformal field theory and the tractor calculus of conformal differential geometry; Penrose’s twistors became a production tool for scattering amplitudes after Witten’s twistor-string observation [13], through spinor-helicity variables the reader met in Exercise 8.

From Section 6: spin networks, generalized and re-derived, became the kinematical basis of loop quantum gravity, whose covariant (spin-foam) dynamics evaluates $SL(2, \mathbb{C})$ integrals over precisely the coherent data built from flagpoles and rotors above; the twistor parametrization of its phase space closes the loop between Sections 5 and 6. The textbook [14] is the natural continuation.

From Section 7: the miniature is not a toy. Its 3+1-dimensional continuation — $SL(2, \mathbb{C})$ data, $SU(1, 1)$ subgroups for timelike faces, discrete/continuous representation sectors, and signature-selected orientation labels — is the representation-theoretic machinery of Lorentzian quantum geometry with timelike boundaries. The habits are Dirac’s and Penrose’s; the frontier is open.

Acknowledgment of method. Every central equation in this primer can be checked by hand; the author encourages the reader to distrust and verify, in that order. That, more than any formula, is the content of the Dirac–Penrose tradition.

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