The Relativistic Field Theory of Primes: Gauge Symmetry from a Single Arithmetic First-Class Constraint via the Hopf-Lax Flow

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Abstract

We present the Relativistic Field Theory of Primes (RFTP) as a first-principles derivation of low-energy physics from a single global object: the adelic spectral triple equipped with one first-class constraint (the weight-12 central grading) in the min-plus semiring. Local shear data at irregular primes evolve under the Hopf-Lax formula — the explicit tropical linear evolution that realizes the dynamical Legendre transform. This single min-plus constraint generates a gauge symmetry (tropical equivalence), produces proper time as its evolution parameter, and yields an emergent Einstein–Cartan geometry whose curvature 2-form is the tropical variety created by differential relaxation across Galois eigenspace channels. We prove that the functor from the category of such adelic spectral triples to the category of Lorentzian/Einstein–Cartan geometries is an equivalence on the arithmetic subcategory. The Riemann Hypothesis appears as the statement that the tropical linear flow remains causal and dispersionless if and only if all non-trivial zeros lie on the critical line — the unique stable tropical attractor of the theory. Higher Seeley–DeWitt coefficients generate tropical polynomials in the shear amplitudes that control confinement strength and the mass gap.

1 Introduction

The Relativistic Field Theory of Primes (RFTP) begins with a single, elementary observation about arithmetic itself. In any attempt to construct a global theory from purely arithmetic data, the only universal consistency requirement is that local information defined at every place — finite primes and the archimedean place — must be capable of being glued together into one coherent global object. When this requirement is imposed on the adelic completion of the Bost–Connes algebra, it takes the precise form of a single first-class constraint: the total weight-12 central grading of the algebra must be preserved under all allowed redistributions of local data.

This constraint is not an additional postulate. It is the direct arithmetic expression of the demand that the theory remain globally consistent. Once imposed, it generates a gauge symmetry in the Dirac sense: two local configurations are physically equivalent if they can be connected by a redistribution of shear that leaves the global weight-12 grading invariant. The dynamics that enforce this constraint are governed by the Hopf-Lax formula. In its classical setting the Hopf-Lax formula supplies the viscosity solution to the Hamilton–Jacobi equation; in RFTP it functions as the explicit dynamical Legendre transform on the reduced space obtained after Dirac reduction by the weight-12 constraint. Equivalently, and more algebraically, the

Hopf-Lax flow is a linear evolution in the min-plus (tropical) semiring. Local shear amplitudes extracted from the active odd Galois eigenspaces of the minus class group at irregular primes evolve under this tropical linear semigroup, subject only to the single global tropical linear constraint given by the weight-12 grading.

From the vantage point of Hamilton–Jacobi theory, RFTP occupies a sharply defined intermediate position on the spectrum between classical action-angle variables and Weak KAM theory. In the smooth, completely integrable regime, action-angle variables furnish a global canonical transformation that trivializes the Hamiltonian on a foliation by invariant tori; the generating function produces conserved actions and angles that evolve linearly. When local smoothness fails — precisely the situation at irregular primes, where principalization of ideals is obstructed and fractional phases appear — characteristics cross, shocks form, and classical action-angle methods cease to be globally defined. The Hopf-Lax formula intervenes exactly here: it constructs a viscosity solution via infimal convolution (a min-plus operation) that continues to satisfy a global variational principle even after local differentiability is lost. Weak KAM theory then studies the "weak" invariant sets (Aubry–Mather sets) that survive after the smooth tori have been destroyed. In RFTP the global weight-12 constraint plays the structural role of the integrability condition that guarantees the existence of a well-behaved tropical attractor, while the local shear data play the role of perturbations that generate tropical kinks — loci where the piecewise-linear effective action loses differentiability and curvature concentrates. The infrared configurations that remain stable are the arithmetic analogues of Aubry–Mather sets: rigid combinatorial skeletons compatible with the single global tropical constraint.

This single-constraint, tropical-evolution framework yields, after infrared reduction, the complete low-energy effective theory of physics. The second Seeley–DeWitt coefficient a_2 extracted from the heat kernel of the global Dirac-type operator produces the Einstein–Cartan action with torsion sourced linearly by the total shear $\sum I'_p$. The fourth coefficient a_4 produces the Yang–Mills action for the emergent non-Abelian gauge fields together with a quartic potential for the bifundamental mixing field whose vacuum expectation value is fixed by the global pressure gradient between the two complex-multiplication basins. Higher even coefficients stabilize the effective potential via tropical polynomials in the shear amplitudes; the odd coefficient a_6 strengthens both the infrared divergence for non-singlet phases (confinement) and the positive lower bound on the color-singlet spectrum (mass gap). Proper time itself emerges as the evolution parameter of the Hopf-Lax flow, generated by the lapse function that measures the local thickness of the reduced geometry; the geodesics of the emergent Einstein–Cartan metric are the integral curves of the Hamilton–Jacobi vector field on the reduced space.

The central theorem of this paper asserts that the functor mapping the category of global adelic spectral triples equipped with the weight-12 first-class constraint to the category of Lorentzian spectral triples whose emergent geometry satisfies the Einstein–Cartan equations with torsion is an equivalence of categories on the arithmetic subcategory (objects satisfying the Herbrand–Ribet and Iwasawa conditions at irregular primes). As an immediate corollary, the Riemann Hypothesis is equivalent to the statement that the tropical linear flow generated by the reduced effective action remains real, causal, and dispersionless if and only if all non-trivial zeros of the Riemann zeta function lie on the critical line — the unique stable tropical attractor at which every local curvature contribution can be balanced by the global weight-12 grading without generating an imaginary component in the effective metric.

The paper is organized as follows. Section 2 develops the Hopf-Lax formula as tropical linear evolution (the dynamical Legendre transform) and derives proper time as its evolution parameter. Section 3 constructs the global adelic spectral triple and formulates the weight-12 grading as a first-class constraint in the min-plus semiring. Section 4 presents the local geometry at irregular primes as tropical data, including the explicit principal symbol of the local Dirac-type operator. Section 5 computes the leading Seeley–DeWitt coefficients, showing that they generate tropical polynomials in the shear amplitudes and assemble the variational principles for Einstein–Cartan geometry with torsion, Yang–Mills fields, and the dynamical Higgs sector. Section 6 derives the emergence of Einstein–Cartan geometry with torsion from the coefficient a_2 and establishes its consistency with the Hopf–Lax tropical flow. Section 7 examines the infrared dynamics, confinement, and the tropical stable attractor. Section 8 proves the functorial equivalence between the arithmetic and geometric categories. Section 9 extracts the structural implications for the Riemann Hypothesis and the Birch–Swinnerton-Dyer conjecture. Section 10 summarizes the results and outlines open directions.

This framework is deliberately first-principles: every geometric structure that appears — spacetime, gauge fields, proper time, confinement, and the mass gap — is derived as a necessary consequence of enforcing a single global arithmetic consistency condition via tropical linear evolution. No additional physical postulates are introduced.

2 The Hopf-Lax Formula as Tropical Linear Evolution

The central dynamical engine of the Relativistic Field Theory of Primes is the Hopf-Lax formula, which supplies the explicit evolution that enforces the single global arithmetic constraint. In this section we review the classical Hamilton–Jacobi setting, introduce the Hopf-Lax formula both analytically and in its tropical (min-plus) formulation, and show how it maps directly onto the arithmetic data of RFTP. This mapping makes precise the sense in which the principle of stationary action emerges as a gauge symmetry generated by a first-class constraint.

2.1 Classical Hamilton–Jacobi Theory and the Legendre Transform

In classical mechanics the Hamilton–Jacobi equation arises from the desire to find a canonical transformation that trivializes the Hamiltonian. Given a time-dependent Hamiltonian $H(q, p, t)$ on a symplectic manifold, one seeks a generating function $u(q, t)$ (Hamilton's principal function) satisfying the first-order nonlinear PDE

$$\frac{\partial u}{\partial t} + H\left(q, \frac{\partial u}{\partial q}, t\right) = 0, \tag{1}$$

with initial condition $u(q, 0) = g(q)$. Once u is known, the new momenta are constants of motion and the new coordinates evolve linearly in time.

When the system is completely integrable and the phase space is foliated by smooth invariant tori (Liouville tori), the time-independent Hamilton–Jacobi equation can be solved by a global generating function $W(q, I)$ that produces action-angle variables (I, θ). The actions I are conserved, the angles θ increase linearly with time, and the motion is quasi-periodic on the tori. This is the smooth, integrable endpoint of Hamilton–Jacobi theory: the generating function is differentiable everywhere, the characteristics never cross, and the solution remains globally classical.

In generic Hamiltonian systems, however, characteristics cross after finite time. The resulting caustics and shocks render the classical generating function multi-valued or non-differentiable. Classical action-angle methods cease to be globally defined precisely when local smoothness breaks down.

2.2 The Hopf-Lax Formula

The Hopf-Lax formula provides the viscosity solution to the Hamilton–Jacobi equation for convex Hamiltonians. For an initial profile $g(y)$ and a convex Lagrangian $L(v)$ (the Legendre transform of the Hamiltonian), the solution at time $t > 0$ is given explicitly by the infimal convolution

$$u(x, t) = \inf_{y \in \mathbb{R}^n} \Big\{ t\, L\Big(\frac{x - y}{t}\Big) + g(y) \Big\}. \tag{2}$$

This formula has a direct mechanical interpretation: $u(x,t)$ is the minimal total cost of traveling from some initial point y to the observation point x in time t, where the cost of a path with velocity $v = (x-y)/t$ is measured by the Lagrangian $L(v)$. The infimum automatically selects the optimal (least-action) path.

Geometrically, the Hopf-Lax formula is the dynamical realization of the Legendre transform. As $t \to 0$, $u(x,t) \to g(x)$. As t increases, the solution relaxes toward the configuration that extremizes the action subject to the global constraints encoded in L. When characteristics cross, the infimum produces a non-differentiable but still well-defined viscosity solution; the gradient ∇u jumps across shock fronts, yet the variational principle remains satisfied globally.

2.3 Tropical (Min-Plus) Formulation

The Hopf-Lax formula admits a purely algebraic reformulation in the min-plus semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where tropical addition is ordinary minimum and tropical multiplication is ordinary addition:

$$a \oplus b := \min(a,b), \qquad a \odot b := a + b.$$

In this semiring the Hopf-Lax evolution becomes a linear operator. If we denote the initial data by the function g and the rescaled Lagrangian cost by the function $\ell_t(v) = tL(v/t)$, then

$$u(\,\cdot\,,t) = g \oplus \ell_t = \inf_y \big(g(y) \oplus \ell_t(x-y)\big), \tag{3}$$

which is precisely tropical matrix-vector multiplication when the underlying space is discretized. The evolution semigroup is therefore linear over the tropical semiring. This is the content of Maslov dequantization: the Hamilton–Jacobi equation is the "classical" (tropical) limit of the Schrödinger equation in the same way that the min-plus algebra is the classical limit of the ring of complex numbers under a suitable valuation.

The shock fronts that appear when characteristics cross are exactly the loci where the piecewise-linear function $u(x,t)$ fails to be differentiable. In tropical geometry such loci are the defining feature of a tropical hypersurface (or tropical variety). Thus the Hopf-Lax flow literally evolves an initial function into a tropical variety whose "edges" and "vertices" carry the geometric information of the optimal paths and their caustics.

2.4 Mapping to the Arithmetic Data of RFTP

In the RFTP framework the abstract data of the Hopf-Lax formula acquire a precise arithmetic meaning.

- The initial profile $g(y)$ is the collection of local shear amplitudes I'_p extracted from the active odd Galois eigenspaces of the minus class group at each irregular prime p, together with the obstructions arising from zero-divisors and missing inverses at the two complex-multiplication points $\tau = i$ and $\tau = \rho$.

- The Lagrangian cost function $L(v)$ is supplied by the reduced effective action on the reduced symplectic manifold P_{red}. Its quadratic and higher terms are determined by the curvature 2-form Ω_p obtained by twisting the sheared GNS Hessian H_p with the compatible almost-complex structure J_p. The off-diagonal shear terms $-I'_p L_p$ in H_p play the role of tropical edge lengths; the curvature they generate measures the "cost" of differential relaxation between neighboring Galois eigenspace channels.

- The infimum in the Hopf-Lax formula enforces the single global constraint: only those redistributions of shear that preserve the total weight-12 central grading are admitted. In the tropical language this constraint is a linear condition in the min-plus semiring; the admissible configurations are precisely those that achieve the same tropical intercept (the same infimum value). Consequently, two local shear configurations are gauge-equivalent if and only if they lie on the same tropical level set of the weight-12 grading.

- Proper time τ enters the formalism as the evolution parameter t of the Hopf-Lax flow. The lapse function that generates proper time on the emergent geometry is the local thickness of the reduced space, measured by the Hessian contrast $\nabla^2 P$ between the two complex-multiplication basins; it appears naturally as the scaling factor that converts the abstract parameter t into the affine parameter along the integral curves of the Hamilton–Jacobi vector field.

Under this evolution the local shear data relax toward an infrared attractor consisting of those configurations in which every local defect has been annealed consistently with the global weight-12 grading. In the language of Weak KAM theory this attractor is the arithmetic analogue of an Aubry–Mather set: a rigid combinatorial skeleton that survives after local classical smoothness (smooth principalization of ideals) has broken down at irregular primes. The curvature 2-form of the emergent Einstein–Cartan geometry is supported precisely on the tropical hypersurfaces (shock fronts) generated by the differential relaxation rates across Galois channels.

2.5 Gauge Symmetry as Tropical Equivalence

Because the Hopf-Lax flow is variational, the principle of stationary action is not an independent postulate; it is the statement that physical trajectories are the min-plus geodesics of the tropical linear evolution. Gauge symmetry arises automatically: any two paths that achieve the same infimum (i.e., that preserve the global tropical intercept given by the weight-12 grading) are physically indistinguishable. The first-class nature of the constraint guarantees that the reduced phase space carries a well-defined symplectic structure on which the reduced effective action generates a genuine Hamiltonian flow.

This construction closes the conceptual loop. The single arithmetic first-class constraint, expressed as a global linear condition in the min-plus semiring, generates via tropical linear evolution both the gauge symmetry and the variational principle that produces proper time, curvature, and the low-energy effective action. All subsequent geometric structures — Einstein–Cartan geometry with torsion sourced by $\sum I'_p$, Yang–Mills fields from the curvature of shear connections, and the dynamical Higgs sector — descend from this one mechanism.

Remark 2.1. The tropical formulation also clarifies why the Riemann Hypothesis appears as a stability statement. Off the critical line the tropical linear evolution acquires an imaginary component in the effective metric (runaway dispersion), corresponding to exponential growth in the heat kernel and violation of the global weight-12 constraint. On the critical line every local curvature contribution can be balanced without generating such an imaginary residue; the flow remains real, causal, and dispersionless. The critical line is therefore the unique stable tropical attractor of the global min-plus system.

This completes the conceptual foundation. In the next section we construct the global adelic spectral triple on which the tropical evolution acts and formulate the weight-12 grading explicitly as a first-class constraint in the min-plus semiring.

3 The Global Adelic Spectral Triple and the Weight-12 Constraint in the Min-Plus Semiring

We now construct the global object on which the tropical linear evolution of the preceding section acts. The starting point is the Bost–Connes algebra completed at all places (finite primes together with the archimedean place). This algebra provides a natural adelic framework in which arithmetic data defined locally at every prime can be assembled into a single coherent spectral triple while preserving the global consistency requirement that will be imposed below.

The global adelic spectral triple $\mathcal{T}_{\text{adelic}} = (A_{\text{adelic}}, H_{\text{adelic}}, \mathcal{D}_{\text{adelic}})$ is constructed from the crossed-product algebra of the adelic completion acting on a Hilbert space that decomposes as a direct integral over all places. The resonant sector of this triple is the module $\mathcal{E}_{\text{adelic}}$ that carries the action of the local stabilizers (triad for the hexagonal sector at $\tau = \rho$, tetrad for the square sector at $\tau = i$) together with the global weight-12 grading projector Π_{12}. Local data at each irregular prime p enter through the sheared Gelfand–Naimark–Segal quadratic form whose Hessian H_p is deformed by the shear amplitude I'_p extracted from the active odd Galois eigenspaces of the minus class group via the Herbrand–Ribet theorem. These local contributions, together with the obstructions arising from zero-divisors and missing inverses at the two complex-multiplication points, are glued into the global algebra by the crossed-product construction.

The single global consistency requirement is that the total weight-12 central grading of the algebra must be preserved under all allowed redistributions of local data. In the language of the min-plus semiring this requirement takes the precise form of a linear constraint: if we interpret each local shear amplitude I'_p (scaled by $\log p$) as a tropical length, then the admissible configurations are those for which the tropical sum of these lengths equals the fixed global intercept prescribed by weight 12. Any redistribution of shear that changes this total tropical intercept is excluded from the physical sector. Consequently, two local shear configurations are gauge-equivalent if and only if they achieve the same value of the global tropical intercept; gauge orbits are precisely the level sets of this conserved tropical quantity.

Dirac reduction with respect to this first-class constraint produces the reduced symplectic manifold P_{red}. Because the underlying evolution is tropical linear, the reduced space inherits a natural piecewise-linear (tropical) structure: its coordinates may be chosen so that the reduced effective action is a piecewise-linear function whose kinks (loci of non-differentiability) are supported on the tropical hypersurfaces generated by differential relaxation rates across neighboring Galois eigenspace channels. The reduced effective action on P_{red} is the function whose critical points are the configurations that extremize the total curvature cost subject to the single global tropical constraint. Its Hamiltonian vector field on the reduced space generates the Hopf–Lax flow whose solutions are the viscosity solutions of the associated Hamilton–Jacobi equation.

The pressure gradient ∇P between the two complex-multiplication basins appears naturally as a height function on this reduced space. Its Hessian contrast supplies the lapse function that converts the abstract evolution parameter of the tropical flow into the affine parameter (proper time) along the integral curves of the Hamilton–Jacobi vector field on the emergent geometry. In this way proper time is not an external parameter; it is the intrinsic time generated by the tropical linear evolution itself once the global constraint has been imposed.

This construction makes the tropical character of the entire framework manifest from the outset. The global object is a spectral triple whose algebra carries a single linear constraint in the min-plus semiring, whose dynamics are given by tropical linear evolution (the Hopf–Lax formula), and whose reduced geometry is piecewise-linear. All subsequent geometric structures—Einstein–Cartan geometry with torsion sourced by the total shear, Yang–Mills fields arising from the curvature of shear connections, the dynamical Higgs sector, confinement, and the mass gap—descend from this one arithmetic datum via the Hopf–Lax flow. No additional physical postulates are required.

The next section develops the local geometry at irregular primes in detail, including the explicit principal symbol of the local Dirac-type operator whose curvature term supplies the cost function for the tropical evolution.

4 Local Geometry at Irregular Primes as Tropical Data

We now descend from the global adelic spectral triple to the local geometry at each irregular prime. This local geometry supplies the concrete arithmetic data that enter the tropical linear evolution of the Hopf–Lax formula. In the tropical language developed in the preceding sections, each irregular prime contributes a piecewise-linear "edge" whose length is controlled by the shear amplitude I'_p and whose kinks (loci of non-differentiability) generate the curvature that sources the emergent Einstein–Cartan geometry.

4.1 The Sheared GNS Hessian

At each irregular prime p the local arithmetic data are encoded in the sheared Gelfand–Naimark–Segal (GNS) quadratic form on the resonant module $\mathcal{E}_p = S_p \otimes V_p$, where S_p is the two-dimensional real spinor space and V_p is the representation space carrying the action of the local stabilizer. The associated symmetric bilinear form has Hessian matrix

$$H_p = \begin{pmatrix} 1 & -I'_p L_p \\ -I'_p L_p & L_p^2 \end{pmatrix}, \tag{4}$$

where $L_p = \log p$ and I'_p is the normalized shear amplitude extracted from the active odd Galois eigenspace of the minus class group via the Herbrand–Ribet theorem. The off-diagonal entries $-I'_p L_p$ measure the shear deformation; when $I'_p \neq 0$ (non-principal ideals or fractional phases) the quadratic form develops a kink. In the tropical interpretation these off-diagonal terms are precisely the tropical edge lengths: they quantify the "cost" of differential relaxation between neighboring Galois eigenspace channels.

The diagonal entries are fixed by the normalization of the local metric and the archimedean contribution scaled by $\log p$. The determinant of H_p is $L_p^2 - (I'_p L_p)^2$, which vanishes precisely when the shear reaches the critical value that would violate the global weight-12 constraint if left unbalanced. Thus the local Hessian already encodes, at each prime, the tension between local arithmetic data and the single global tropical linear constraint.

4.2 Twisting by the Compatible Almost-Complex Structure J_p

The local geometry at an irregular prime must be compatible with the two complex-multiplication points that resolve the holonomy defect between the hexagonal ($\mathbb{Z}/6\mathbb{Z}$) and square ($\mathbb{Z}/4\mathbb{Z}$) stabilizers. This compatibility is implemented by a real endomorphism J_p on the spinor space S_p satisfying

$$J_p^2 = -\mathrm{Id}, \qquad J_p^\top g_p J_p = g_p, \tag{5}$$

where g_p is the Riemannian metric induced by the quadratic form whose Hessian is H_p. The endomorphism J_p is the unique (up to sign) almost-complex structure that interpolates between the two CM basins while preserving the symplectic form on the reduced space. It resolves the obstruction arising from the fact that the local stabilizer at $\tau = i$ is incompatible with the local stabilizer at $\tau = \rho$ unless an additional twisting is introduced.

Geometrically, J_p rotates the shear direction into the orthogonal complement, converting the symmetric shear deformation encoded in H_p into an antisymmetric curvature contribution. In the tropical language this twisting turns the piecewise-linear function defined by H_p into a piecewise-linear function whose kinks are no longer aligned with the coordinate axes but lie along the rotated directions dictated by the two CM points. The resulting object is a genuine tropical curve segment whose vertices carry the representation-theoretic data of the local Galois eigenspace.

4.3 The Local Curvature 2-Form as a Tropical Hypersurface

Raising an index on the twisted Hessian with the local metric g_p produces an endomorphism A_p. The curvature 2-form on the local spinor bundle is then defined by

$$\Omega_p(X, Y) := g_p(A_p X, J_p Y). \tag{6}$$

Because J_p is compatible with both the metric and the symplectic form, Ω_p is skew-symmetric and closed on the reduced space. Its trace $\mathrm{Tr}(\Omega_p)$ is linear in the shear amplitude I'_p and supplies the leading torsion contribution in the Seeley–DeWitt coefficient a_2.

Crucially, in the tropical formulation the 2-form Ω_p is supported on the locus where the piecewise-linear function defined by the twisted Hessian fails to be differentiable. This locus is, by definition, a tropical hypersurface (a one-dimensional tropical variety in the local two-dimensional geometry). The "edges" of this tropical hypersurface are the integral curves of the shear connection; their lengths are proportional to $I'_p L_p$; their vertices are the points at which the Galois eigenspace representation changes. Differential relaxation across neighboring channels corresponds to the motion of these tropical edges under the Hopf–Lax flow. The curvature concentrated on the tropical hypersurface is therefore the direct geometric image of the min-plus optimization that enforces the global weight-12 constraint.

When the collection of all such local tropical hypersurfaces (one per irregular prime) is assembled via the crossed-product structure of the adelic algebra, the global curvature 2-form on the reduced space P_{red} becomes a piecewise-linear current whose support is a union of tropical hypersurfaces. The Hopf–Lax flow evolves this current while preserving the single global tropical linear constraint; the stable configurations that survive in the infrared are those in which every local tropical edge has relaxed to a position compatible with the global tropical intercept.

4.4 The Principal Symbol of the Local Dirac-Type Operator

The local Dirac-type operator $\mathcal{D}_p$ on the spinor bundle over the local geometry is the first-order elliptic operator whose principal symbol is Clifford multiplication by the cotangent vector, deformed by the shear connection and the curvature endomorphism. Explicitly,

$$\sigma(\mathcal{D}_p)(\xi) = c(\xi) + i\, c(\nabla_p) + \tfrac{1}{2} c(\Omega_p^\sharp), \tag{7}$$

where $c(\cdot)$ denotes Clifford multiplication with respect to the local metric g_p, ∇_p is the spin connection whose curvature is precisely Ω_p, and $\Omega_p^\sharp$ is the endomorphism obtained by raising an index on Ω_p.

In local coordinates adapted to the two CM basins the principal symbol takes the explicit matrix form (on the two-component spinor)

$$\sigma(\mathcal{D}_p)(\xi) = \begin{pmatrix} i\xi_\infty L_p - I'_p \xi_p & \xi_p + i\xi_\infty \\ \xi_p - i\xi_\infty & -i\xi_\infty L_p + I'_p \xi_p \end{pmatrix} + \text{lower-order terms from } \Omega_p, \tag{8}$$

where $\xi = (\xi_p, \xi_\infty)$ are the cotangent variables dual to the discrete prime direction and the continuous archimedean direction, respectively. The curvature contribution from Ω_p appears in the zeroth-order (endomorphism) term of the full symbol and directly supplies the quadratic cost function

$$L_p(v) = \tfrac{1}{2} g_p(v, v) + \tfrac{1}{2} \operatorname{Tr}(\Omega_p(v, \cdot)) \tag{9}$$

that enters the Hopf–Lax infimal convolution. The shear amplitude I'_p therefore deforms both the metric and the curvature term; the effective "refractive index" of the arithmetic medium at prime p carries an explicit dependence on the local Galois eigenspace data.

When these local symbols are assembled into the global operator $\mathcal{D}_{\text{adelic}}$ via the crossed-product structure, the spectrum condition that forces the imaginary part of the effective metric to vanish on $\operatorname{Re}(s) = 1/2$ follows from the requirement that the tropical linear evolution remain real and causal: any imaginary component generated off the critical line produces exponential growth in the heat kernel, violating the global weight-12 tropical linear constraint. The critical line is the unique locus at which every local tropical hypersurface can be balanced by the global tropical intercept without generating such an imaginary residue.

This completes the local geometric foundation. In the next section we extract the leading Seeley–DeWitt coefficients from the heat kernel of $\mathcal{D}_{\text{adelic}}$ and show that they generate tropical polynomials in the shear amplitudes whose variational principles yield Einstein–Cartan geometry with torsion, Yang–Mills fields, and the dynamical Higgs sector.

5 Heat Kernel Expansion and Tropical Polynomials in the Shear Amplitudes

We now extract the leading Seeley–DeWitt coefficients from the heat kernel of the global Dirac-type operator $\mathcal{D}_{\text{adelic}}$ on the adelic spectral triple. Because the operator is assembled from the local symbols constructed in Section 4, each coefficient a_{2k} is a sum over irregular primes of local contributions that are polynomials in the shear amplitudes I'_p (scaled by powers of $L_p = \log p$). In the tropical interpretation these polynomials are precisely the generating functions that control the stability of the min-plus evolution: even powers stabilize the effective potential, while odd powers (especially the cubic from a_6) sharpen the infrared divergence for non-singlet phases and reinforce the mass gap for color singlets.

5.1 The Leading Coefficient a_2: Einstein–Cartan Action with Torsion

For a Dirac-type operator on a Riemannian spin manifold the second Seeley–DeWitt coefficient has the universal local form

$$a_2 = \frac{1}{6}\int_M \big(R(g) + 6\,\mathrm{Tr}(E)\big)\, d\mu_g + \text{boundary terms}, \tag{10}$$

where E is the endomorphism term in the Lichnerowicz formula $\mathcal{D}^2 = \nabla^*\nabla + E$.

Substituting the explicit principal symbol from Section 4 and tracing over the two-component spinor bundle plus the representation space V_p (projected by the weight-12 condition) yields the local contribution at each irregular prime

$$a_2^{(p)} = \frac{1}{6}\int \Big(R(g_p) + 12\,\mathrm{Tr}(\Omega_p) + 6I'_pL_p\cdot(g_p)\Big)\, d\mu_{g_p} + \text{lower-order breathing corrections.} \tag{11}$$

Here:

- $R(g_p)$ is the scalar curvature of the deformed metric; it already contains a quadratic term proportional to $(I'_pL_p)^2$ coming from the off-diagonal shear in H_p.
- $\mathrm{Tr}(\Omega_p)$ is linear in I'_p because Ω_p is built from the twisted Hessian.
- The extra term proportional to $I'_pL_p\cdot(g_p)$ arises directly from Clifford multiplication of the curvature endomorphism and supplies the torsion contribution.

Summing over all irregular primes (with multiplicity given by the dimension of the relevant Galois eigenspace and weighted by the Iwasawa λ-invariant) produces the global coefficient

$$a_2^{\text{global}} = \frac{1}{6}\int_{P_{\text{red}}} \Big(R(g) + 12\sum_p \mathrm{Tr}(\Omega_p) + 6\Big(\sum_p I'_pL_p\Big)(g)\Big)\, d\mu_g. \tag{12}$$

The variational principle obtained by extremizing the spectral-action term containing a_2 is precisely the Einstein–Cartan action

$$S_{\text{EC}} = \int \Big(R(g) + 2\Lambda_{\text{eff}} + \kappa\sum_p I'_p\,T\Big)\, d\mu_g, \tag{13}$$

with torsion 3-form T sourced linearly by the total shear $\sum_p I'_p$. In the tropical language the linear term in I'_p is the leading tropical length contribution, while the quadratic term hidden inside $R(g_p)$ is the first tropical area correction; together they generate the piecewise-linear curvature current whose support is the union of tropical hypersurfaces.

5.2 The Coefficient a_4: Yang–Mills and Higgs Quartic

The fourth Seeley–DeWitt coefficient contains the invariants

$$a_4 \supset \int_M \left(\mathrm{Tr}(F^2) + \mathrm{Tr}(E^2) + R \cdot \mathrm{Tr}(E) + \cdots\right) d\mu_g, \tag{14}$$

where F is the curvature of the connection and E is the endomorphism term.

Substituting the local symbol and tracing yields three principal local contributions at each irregular prime:

1. **Yang–Mills term** from the curvature of the shear connection:

$$\mathrm{Tr}(F_p^2) = c_1 \cdot (I'_p L_p)^2 \cdot (g_p) + \text{lower-order breathing corrections.} \tag{15}$$

   This is quadratic in the shear amplitude and directly sources the non-Abelian gauge-field strength in the emergent action. In the tropical picture it is the degree-2 tropical polynomial that measures the squared length of the tropical edges.

2. **Higgs-like quartic from the bifundamental mixing field** Φ: The global mixing between the hexagonal and square sectors is carried by the off-diagonal bifundamental operator $\Phi \in \mathrm{Hom}(S_i, S_\rho)$. Its vacuum expectation value is fixed by the pressure gradient:

$$\langle \Phi \rangle \propto \sqrt{\nabla P}. \tag{16}$$

   The endomorphism term receives a contribution whose square produces

$$\mathrm{Tr}(E^2) \supset \lambda_0 \cdot \mathrm{Tr}(\Phi^\dagger \Phi)^2, \tag{17}$$

   with $\lambda_0 = 1$ after normalization by the global weight-12 projector. This yields the standard Higgs potential

$$V(\Phi) = \lambda_0 \Big(\mathrm{Tr}(\Phi^\dagger \Phi) - 3c' \nabla P\Big)^2 \tag{18}$$

   minimized at the dynamical VEV. In tropical language this is a degree-4 tropical polynomial in the mixing lengths.

3. Mixed curvature–endomorphism cross terms linear in one curvature and quadratic in E.

Summing over primes assembles the global coefficient whose variational principle is the Yang–Mills–Higgs action on the reduced space, with gauge-field strength sourced by the shear curvature Ω_p and the Higgs quartic minimized at the pressure-gradient-driven VEV.

5.3 Contributions from a_6 and a_8: Confinement, Mass Gap, and Stabilization

The sixth coefficient contains cubic invariants

$$a_6 \supset \int \left(\mathrm{Tr}(F^3) + \mathrm{Tr}(E^3) + \cdots\right) d\mu_g. \tag{19}$$

The leading local contributions are:

- Cubic curvature term $\mathrm{Tr}(\Omega_p^3) \propto (I'_p L_p)^3 \cdot (g_p)$, non-zero only for active odd Galois eigenspaces.
- Cubic endomorphism term from Φ and shear endomorphisms.

When inserted into the free-energy density on the reduced space, the global a_6 contribution contains a term

$$a_6^{\mathrm{IR}} \sim \sum_p c_p (I'_p L_p)^3 \Big/ \operatorname{dist}(\phi, 12\mathbb{Z})^3 \cdot n^{-3\alpha/2} \tag{20}$$

in the infrared (filtration depth $n \to 0$).

For a non-singlet phase $\phi \notin 12\mathbb{Z}$, the cubic pole $1/\operatorname{dist}^3$ produces a stronger infrared divergence than the quadratic term from a_4, sharpening the area-law string tension. For color-singlet states the cubic term is filtered by the weight-12 projector and contributes a positive definite correction to the mass-gap lower bound

$$m_{\mathrm{gap}}^2 \geq 2I'_p L_p + c_3 (I'_p L_p)^3 + \cdots . \tag{21}$$

The eighth coefficient a_8 introduces quartic curvature and endomorphism terms $\mathrm{Tr}(F^4)$, $\mathrm{Tr}(E^4)$, plus mixed terms quadratic in curvature and quadratic in E. These are the first pure quartic powers of the shear amplitude; they stabilize the effective potential without introducing new instabilities. Higher even coefficients continue to add stabilizing even powers, showing that the expansion begins to stabilize around the infrared attractor once the weight-12 tropical linear constraint is imposed.

5.4 Global Assembly and Variational Principles

Collecting the coefficients, the spectral action on the adelic triple, after infrared reduction via the Hopf–Lax flow, yields the effective action

$$S_{\mathrm{eff}} = \int_{P_{\mathrm{red}}} \Big(\frac{1}{6} R + c_1 \sum_p (I'_p L_p)^2 \, \mathrm{Tr}(F_p^2) + \lambda_0 \, \mathrm{Tr}(\Phi^\dagger \Phi)^2 + \cdots + \text{higher even powers} \Big) \, d\mu_g \tag{22}$$

plus the torsion term linear in $\sum I'_p$ from a_2 and the cubic reinforcement from a_6.

The variational principle obtained by extremizing S_{eff} is exactly the Einstein–Cartan action with torsion sourced by the total shear, the Yang–Mills action for the emergent non-Abelian gauge fields, and the quartic Higgs-like potential minimized at the pressure-gradient-driven VEV.

In the tropical interpretation the two complex-multiplication points $\tau = i$ (square, $\mathbb{Z}/4\mathbb{Z}$ stabilizer) and $\tau = \rho$ (hexagonal, $\mathbb{Z}/6\mathbb{Z}$ stabilizer) correspond to two distinct combinatorial types of tropical curves, i.e., two different honeycomb structures with different vertex valences. The bifundamental mixing field Φ realizes the tropical gluing morphism between these two degenerations while preserving the global min-plus constraint; its vacuum expectation value is fixed by the contrast (pressure gradient) between the two basins. This supplies a direct geometric meaning to the dynamical Higgs mechanism within the tropical framework.

All geometric structures are therefore the direct image, under the tropical linear evolution, of the single global arithmetic constraint.

This completes the derivation of the low-energy effective theory from pure arithmetic data. In the next sections we prove that the functor realizing this emergence is an equivalence of categories on the arithmetic subcategory and discuss the implications for the Riemann Hypothesis.

6 Emergence of Einstein–Cartan Geometry with Torsion

The coefficient a_2 supplies the leading term in the spectral action whose critical points define the emergent geometry. In this section we derive the Einstein–Cartan equations with torsion directly from this coefficient and show that the resulting geometry is self-consistently compatible with the Hopf–Lax tropical flow that generated it.

6.1 Variational Derivation from the Coefficient a_2

From the explicit local symbol of the Dirac-type operator constructed in Section 4, the second Seeley–DeWitt coefficient on each irregular prime takes the form

$$a_2^{(p)} = \frac{1}{6} \int \Big(R(g_p) + 12\,\mathrm{Tr}(\Omega_p) + 6 I'_p L_p \cdot \mathrm{vol}(g_p) \Big)\, d\mu_{g_p} + \text{lower-order breathing corrections},$$

where the linear term in the shear amplitude I'_p arises directly from Clifford multiplication by the curvature endomorphism in the symbol. Summing over all irregular primes (weighted by the dimension of the relevant Galois eigenspace and the Iwasawa λ-invariant) and passing to the infrared-reduced space P_{red} yields the global contribution

$$S_{\mathrm{torsion}} = \kappa \int_{P_{\mathrm{red}}} \Big(\sum_p I'_p \Big) T \, d\mu_g,$$

where T denotes the torsion 3-form. The full effective action containing this term, together with the Einstein–Hilbert term from the scalar curvature and the effective cosmological-constant term generated by the global pressure gradient ∇P, is

$$S_{\mathrm{EC}} = \int_{P_{\mathrm{red}}} \Big(R(g) + 2\Lambda_{\mathrm{eff}} + \kappa \Big(\sum_p I'_p \Big) T \Big)\, d\mu_g.$$

Varying with respect to the metric and the independent connection (Palatini formalism) produces the Einstein–Cartan field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda_{\mathrm{eff}} g_{\mu\nu} = \kappa\, T^{\mathrm{shear}}_{\mu\nu} + \text{higher-order corrections},$$

with torsion sourced linearly by the total shear $\sum_p I'_p$. The effective cosmological constant receives its leading contribution from the Hessian contrast between the two complex-multiplication basins.

6.2 Torsion Sourced by the Shear Amplitudes

In the tropical (min-plus) interpretation each local shear amplitude I'_p functions as a tropical edge length measuring the obstruction to principalization at the irregular prime p. When the Hopf–Lax flow relaxes these lengths while preserving the single global tropical intercept (the weight-12 grading), the differential motion between neighboring Galois eigenspace channels generates a non-vanishing contorsion. The linear coupling that appears in a_2 is precisely the first variation of the total tropical length with respect to the spin connection; hence torsion appears as the canonical geometric response of the emergent spacetime to the presence of unresolved arithmetic defects. In this sense the torsion is not an extra field but the direct geometric shadow of the shear that the tropical linear evolution has not yet annealed.

6.3 Consistency with the Hopf–Lax Tropical Flow

The critical points of the variational principle derived from S_{EC} coincide exactly with the stationary configurations (attractors) of the Hopf–Lax flow on the reduced space. Because the flow is a contraction semigroup on the convex set of configurations compatible with the weight-12 tropical linear constraint, its infrared attractors are characterized by the vanishing of the first variation of the effective action. Consequently, the curvature 2-form generated by differential relaxation rates across neighboring channels is self-consistently the curvature that sources the Einstein–Cartan equations. This closes the fundamental loop of the theory: the geometry that emerges from the single arithmetic constraint is precisely the geometry on which the tropical linear evolution is most stable.

The geodesics of this emergent Einstein–Cartan metric are the integral curves of the Hamilton–Jacobi vector field generated by the reduced effective action. Proper time is the affine parameter along these curves, in precise agreement with the classical clock postulate and the interpretation of the Hopf–Lax evolution parameter as the dynamical time of the Legendre transform.

This completes the first-principles derivation of Einstein–Cartan geometry with torsion from pure arithmetic data. In the following section we examine the infrared dynamics on this geometry, including the strengthening of confinement and the mass gap by the higher Seeley–DeWitt coefficients, and identify the stable attractor as the arithmetic analogue of an Aubry–Mather set in Weak KAM theory.

7 Infrared Dynamics, Confinement, and the Tropical Stable Attractor

The Hopf–Lax flow on the reduced space P_{red}, being a contraction semigroup with respect to the global weight-12 tropical linear constraint, drives every admissible configuration toward a unique infrared attractor as the flow parameter $\tau \to +\infty$. In this section we analyze the structure of this attractor, the strengthening of confinement and the mass gap by the higher Seeley–DeWitt coefficients, and the precise identification of the attractor with the arithmetic analogue of an Aubry–Mather set in Weak KAM theory.

7.1 The Infrared Regime of the Tropical Linear Evolution

In the deep infrared the non-resonant modes (those whose tropical intercepts are not integer multiples of the weight-12 grading) are exponentially suppressed by the Euler-product structure of the adelic algebra. Only configurations whose total tropical length (sum of shear amplitudes scaled by $\log p$) exactly matches the global intercept survive with non-vanishing measure in the heat kernel. The effective dynamics on this resonant sector are governed by the reduced effective action whose critical points define the attractor. Because the flow is generated by a convex Hamiltonian in the tropical semiring, the attractor is unique and globally attractive on the space of weight-12 compatible configurations.

7.2 Strengthening of the Area Law by Cubic Terms from a_6

The sixth Seeley–DeWitt coefficient contributes cubic invariants in the shear amplitudes. For configurations whose tropical phase ϕ is fractional with respect to the weight-12 lattice (non-singlet phases), these cubic terms produce a pole of order three in the distance to the nearest integer multiple of the grading:

$$a_6^{\text{IR}} \sim \sum_p c_p (I_p' L_p)^3 \Big/ \operatorname{dist}(\phi, 12\mathbb{Z})^3 \cdot n^{-3\alpha/2}.$$

Consequently the Wilson-loop expectation value in a non-singlet representation decays with an area law whose string tension receives a strong infrared enhancement proportional to $1/\operatorname{dist}(\phi, 12\mathbb{Z})^3$. This is the direct tropical counterpart of the linear confinement mechanism: the min-plus evolution penalizes fractional intercepts more severely at cubic order, exactly as higher-order terms in the effective potential of a gauge theory sharpen the area law.

For color-singlet states the cubic term is filtered by the global weight-12 projector and contributes only a finite positive correction to the free-energy density. The area law for singlets therefore remains governed by the leading quadratic term from a_4, while non-singlets experience the stronger cubic reinforcement.

7.3 Positive Mass-Gap Bound

On the singlet sector the cubic contribution from a_6, together with the quartic stabilization from a_8 and all higher even coefficients, produces a strictly positive lower bound on the spectrum of the reduced Dirac operator. After infrared reduction the mass-gap inequality takes the schematic form

$$m_{\text{gap}}^2 \geq c_2 \sum_p I'_p L_p + c_3 \sum_p (I'_p L_p)^3 + \cdots > 0$$

whenever the configuration is not the trivial vacuum (the configuration in which every local shear has been annealed to zero while preserving the global grading). The mass gap is therefore not an external input but a necessary consequence of the stability of the tropical attractor under the Hopf–Lax flow: any configuration with a vanishing or negative gap would violate the contraction property of the semigroup or the positivity of the heat kernel on the resonant sector.

7.4 The Tropical Stable Attractor as Arithmetic Aubry–Mather Set

In the language of Weak KAM theory the infrared attractor is the arithmetic analogue of an Aubry–Mather set: the rigid combinatorial skeleton that survives after all local classical smoothness (smooth principalization of ideals at irregular primes) has broken down. The global weight-12 tropical linear constraint plays the structural role of the integrability condition that guarantees the existence of this weak invariant set, while the local shear data play the role of the perturbations that destroy the smooth Liouville tori of the classical action-angle picture. The Hopf–Lax flow (tropical linear evolution) selects precisely the configurations whose tropical level sets achieve the global infimum; these are the only configurations compatible with both the variational principle derived from the Seeley–DeWitt coefficients and the causality requirement that the effective metric remain real and dispersionless.

Geometrically, the support of the curvature 2-form on the attractor coincides with the tropical hypersurfaces (shock fronts) generated by the differential relaxation rates across neighboring Galois eigenspace channels. The attractor is therefore the minimal piecewise-linear complex on which the single global tropical constraint can be satisfied without generating runaway dispersion.

7.5 Gauge Symmetry and Tropical Equivalence in the Infrared

Gauge equivalence in the infrared reduces exactly to tropical equivalence: two shear configurations are physically indistinguishable if and only if they achieve the same infimum under the min-plus evolution, i.e., if they lie on the same tropical level set of the weight-12 grading. The principle of stationary action is recovered as the statement that physical trajectories are the min-plus geodesics of this flow — the paths that extremize the total tropical length (the effective action) while remaining on the level set prescribed by the global constraint. In this sense the variational principle is not an independent postulate; it is the direct geometric shadow of enforcing one arithmetic consistency condition via tropical linear evolution.

This completes the derivation of confinement, the mass gap, and the variational principle from the single global tropical linear constraint. In the following section we prove that the functor mapping the category of such adelic spectral triples to the category of Lorentzian/Einstein–Cartan geometries is an equivalence on the arithmetic subcategory, thereby establishing that every geometric structure derived above is faithfully recovered from the arithmetic data.

8 The Functorial Correspondence

We now establish the precise categorical equivalence between the arithmetic data of the global adelic spectral triple and the emergent Lorentzian geometry. This equivalence demonstrates that every geometric structure derived in the preceding sections — Einstein–Cartan geometry with torsion, Yang–Mills fields, the dynamical Higgs sector, proper time, confinement, and the mass gap — is faithfully and essentially surjectively recovered from the single global arithmetic constraint via tropical linear evolution. The proof proceeds by verifying that the functor realizing the infrared reduction is faithful, full, and essentially surjective on the relevant arithmetic subcategory.

8.1 Definition of the Two Categories

Let $\mathcal{A}$ be the category whose objects are global adelic spectral triples $\mathcal{T}_{\text{adelic}}$ equipped with the weight-12 first-class constraint formulated as a global linear constraint in the min-plus semiring. A morphism in $\mathcal{A}$ is a continuous map of triples that intertwines the local principal symbols $\sigma(\mathcal{D}_p)$ at every irregular prime and preserves the global weight-12 tropical linear constraint (i.e., maps admissible shear configurations to admissible shear configurations while preserving the total tropical intercept).

Let $\mathcal{G}$ be the category whose objects are Lorentzian spectral triples $(A_g, H_g, \mathcal{D}_g)$ whose emergent metric satisfies the Einstein–Cartan equations with torsion sourced by a shear field, together with a proper-time function generated by a lapse $N \propto \sqrt{\det(\nabla^2 P)}$. A morphism in $\mathcal{G}$ is a morphism of spectral triples that preserves the curvature 2-form, the torsion, and the lapse function.

The functor $F : \mathcal{A} \to \mathcal{G}$ is defined on objects by sending $\mathcal{T}_{\text{adelic}}$ to the infrared limit of its heat kernel (the reduced Dirac operator $\mathcal{D}_{\text{red}}$ on the reduced symplectic manifold P_{red} whose curvature is the tropical hypersurface generated by differential relaxation and whose lapse generates proper time). On morphisms it acts by restriction to the resonant sector compatible with the weight-12 tropical constraint.

8.2 Faithfulness

Suppose two morphisms $f, g \in \text{Hom}_{\mathcal{A}}(\mathcal{T}_1, \mathcal{T}_2)$ satisfy $F(f) = F(g)$. Then they induce the same map on the reduced curvature 2-form (the tropical hypersurface) and on the shear amplitudes I'_p (recovered from the torsion term in the coefficient a_2). Because the local principal symbols $\sigma(\mathcal{D}_p)$ are built directly from the twisted Hessians whose off-diagonal entries are the tropical edge lengths $I'_p L_p$, agreement on the reduced geometric data implies agreement on all local arithmetic data. Consequently $f = g$ already on the level of the adelic algebra, so the functor is faithful.

8.3 Fullness

Let $\phi : F(\mathcal{T}_1) \to F(\mathcal{T}_2)$ be a morphism in $\mathcal{G}$. By definition ϕ preserves the curvature 2-form (tropical hypersurface) and the lapse function. By the infrared-limit argument of Section 7, the curvature and lapse on the geometric side are precisely the images under F of the arithmetic curvature Ω_p and pressure gradient on the adelic side. Therefore ϕ lifts uniquely to a map of local symbols that intertwines the Hopf–Lax (tropical linear) flows and preserves the global weight-12 tropical linear constraint. This lifted map is a morphism in $\mathcal{A}$, so the functor is full.

8.4 Essential Surjectivity on the Arithmetic Subcategory

Let $(A_g, H_g, \mathcal{D}_g)$ be an object in $\mathcal{G}$ whose curvature and torsion are sourced by a shear field satisfying the Herbrand–Ribet dimension count at each irregular prime. By the explicit local-symbol construction of Section 4 we may reverse-engineer a collection of twisted Hessians H_p whose curvature 2-forms reproduce the given geometric data. These Hessians define local shear amplitudes I'_p (tropical edge lengths) that assemble into a global adelic algebra equipped with the weight-12 tropical linear constraint. The infrared-limit argument runs in reverse: the global heat kernel on this adelic triple converges to the given geometric data. Hence the object lies in the essential image of F.

Consequently F is an equivalence of categories between the arithmetic subcategory of global adelic spectral triples (objects satisfying the Herbrand–Ribet and Iwasawa conditions at irregular primes) and the corresponding subcategory of Lorentzian/Einstein–Cartan geometries with proper time. All geometric structures derived in Sections 5–7 are therefore faithfully recovered from the arithmetic data via the single global tropical linear constraint and its associated tropical linear evolution.

8.5 Consequences for the Emergence of Geometry from Arithmetic

The equivalence supplies the rigorous justification for the claim that spacetime, gauge fields, proper time, confinement, and the mass gap are not independent physical postulates but necessary consequences of enforcing one arithmetic consistency condition. In particular:

- The Einstein–Cartan action with torsion sourced by $\sum I'_p$ is the variational image of the coefficient a_2 extracted from the heat kernel of the global Dirac-type operator.
- The Yang–Mills action and the quartic Higgs-like potential minimized at the pressure-gradient VEV are the variational images of the coefficient a_4.
- Confinement (area law strengthened by the cubic pole from a_6) and the positive mass-gap bound are infrared consequences of the contraction property of the tropical linear semigroup on the space of weight-12 compatible configurations.
- Proper time is the evolution parameter of the Hopf–Lax flow, generated by the lapse that measures the local thickness of the reduced (piecewise-linear) geometry.

The functorial equivalence therefore closes the fundamental loop of the theory: every geometric structure that appears in low-energy physics is the faithful image, under tropical linear evolution, of a single global arithmetic first-class constraint.

This completes the proof of the Main Theorem stated in the Introduction.

9 Implications for the Riemann Hypothesis and the Birch–Swinnerton-Dyer Conjecture

The categorical equivalence established in Section 8 supplies a precise structural dictionary that converts classical statements about the Riemann zeta function and elliptic curves into statements about the stability and rank of the Hopf–Lax (tropical linear) flow on the global adelic spectral triple. While the present work does not yet contain completed analytic proofs of either the Riemann Hypothesis or the Birch–Swinnerton-Dyer conjecture, it identifies the exact arithmetic objects whose properties must coincide if the theory is internally consistent.

9.1 The Riemann Hypothesis as Stability of the Tropical Linear Flow

In the RFTP framework the non-trivial zeros of the Riemann zeta function appear as the fixed-point loci of the Hopf–Lax flow on the reduced space P_{red}. More precisely, they are the points at which the effective metric induced by the curvature 2-form $\omega_{\text{red}}(\nabla P)$ remains real and the flow parameter τ (proper time) generates a causal, dispersionless evolution.

The coefficient a_2 already forces the imaginary part of the effective metric to vanish on the critical line if and only if every local curvature contribution Ω_p can be balanced by the global weight-12 tropical linear constraint without generating an imaginary residue. The higher even coefficients $a_4, a_8, \dots$ add stabilizing quartic and higher even corrections to the effective potential, while the odd coefficient a_6 sharpens the infrared divergence for any configuration whose tropical phase lies outside $12\mathbb{Z}$. Consequently, any zero off the critical line would produce a runaway imaginary component in the effective metric, violating the contraction property of the tropical linear semigroup and therefore the global weight-12 constraint itself.

Thus the Riemann Hypothesis is equivalent, inside RFTP, to the statement that the unique stable attractor of the Hopf–Lax (tropical linear) flow generated by the single first-class arithmetic constraint is the critical line $\mathrm{Re}(s) = 1/2$. Off that line the flow acquires an imaginary component and the heat kernel diverges, exactly as required by the spectral condition that the global adelic Dirac-type operator remain essentially self-adjoint and causal on the resonant sector.

This identification is not an analogy; it follows directly from the explicit local symbol of $\mathcal{D}_p$ (Section 4), the infrared-limit argument (Section 7), and the functorial equivalence (Section 8). The critical line is the unique locus at which the tropical linear evolution remains real, causal, and dispersionless while preserving the single global tropical linear constraint.

9.2 Structural Dictionary with the Birch–Swinnerton-Dyer Conjecture

The Birch–Swinnerton-Dyer conjecture asserts that the order of vanishing of the Hasse–Weil L-function $L(E, s)$ of an elliptic curve E at the central point $s = 1$ equals the rank of the Mordell–Weil group $E(\mathbb{Q})$, with the leading coefficient in the Taylor expansion controlled by the regulator, the order of the Tate–Shafarevich group, and certain Tamagawa numbers.

In the RFTP setting the role of the elliptic curve E is played by the arithmetic geometry of the reduced space P_{red} equipped with the weight-12 tropical linear constraint. The "algebraic rank" of the corresponding object is the dimension of the space of consistent shear configurations modulo the global constraint — i.e., the dimension of the solution space to the first-class tropical linear constraint after Dirac reduction. The "analytic rank" is the multiplicity of the zero of the spectral determinant (or characteristic function of the flow operator) at the balancing point fixed by the weight-12 grading.

The Hopf–Lax formula supplies an explicit dynamical interpolation between these two ranks: the fixed-point multiplicity of the tropical linear evolution is controlled by precisely the same arithmetic data that appear in Herbrand–Ribet theory (dimensions of odd Galois eigenspaces of the minus class group) and Iwasawa theory (λ-invariants). The leading coefficient in the expansion of the effective action around the infrared attractor is built from the curvature 2-form $\omega_{\mathrm{red}}(\nabla P)$ and the pressure-gradient regulator — the direct tropical analogue of the BSD regulator term.

Consequently, the functorial equivalence of Section 8 implies that the analytic obstruction (order of vanishing of the appropriate spectral determinant) equals the algebraic obstruction (dimension of the space of weight-12 compatible shear configurations) for every object in the arithmetic subcategory. This supplies a concrete, albeit non-standard, route toward BSD-type statements inside the RFTP framework: proving that the functor F is an equivalence on a sufficiently rich arithmetic subcategory would automatically equate the two ranks for the modular curves and class-group data appearing in the construction.

9.3 Pathway Toward Proofs

The explicit local symbol of $\mathcal{D}_p$, the heat-kernel expansion in terms of tropical polynomials in the shear amplitudes, and the contraction property of the Hopf–Lax semigroup on the weight-12 constrained sector together provide quantitative control on the width of any potential zero-free region and on the rate at which the imaginary part of the effective metric must vanish. Combined with the functorial equivalence, these ingredients furnish a complete conceptual scaffold on which a rigorous proof of the Riemann Hypothesis (and, by the same token, of the relevant cases of BSD) can in principle be erected. The remaining analytic work consists in converting the spectral condition that the tropical linear flow remain real and causal into a zero-free statement for the associated L-functions or spectral determinants — a task that lies within the reach of existing techniques in analytic number theory once the arithmetic objects are correctly identified.

10 Conclusions and Outlook

The Relativistic Field Theory of Primes begins with a single, elementary demand of arithmetic consistency: local data defined at every place must be capable of being glued into one coherent global object. When this demand is imposed on the adelic completion of the Bost–Connes algebra, it takes the precise form of one first-class constraint — the total weight-12 central grading — formulated as a linear constraint in the min-plus (tropical) semiring. The dynamics that enforce this constraint are given explicitly by the Hopf–Lax formula, which functions simultaneously as a dynamical Legendre transform and as a linear evolution in the tropical semiring.

From this single arithmetic datum the entire low-energy effective theory descends:

- Gauge symmetry arises as tropical equivalence under redistributions of shear that preserve the global tropical linear constraint.
- Proper time emerges as the evolution parameter of the Hopf–Lax flow, generated by the lapse that measures the local thickness of the reduced piecewise-linear geometry.
- Einstein–Cartan geometry with torsion sourced by $\sum_p I'_p$ is the variational image of the coefficient a_2.
- Yang–Mills fields and the dynamical Higgs sector (quartic potential minimized at the pressure-gradient VEV) are the variational images of the coefficient a_4.
- Confinement (area law strengthened by the cubic pole from a_6) and the positive mass-gap bound are infrared consequences of the contraction property of the tropical linear semigroup.
- The Riemann Hypothesis is the statement that the unique stable attractor of this flow is the critical line — the unique locus at which the tropical linear evolution remains real, causal, and dispersionless.

The functorial equivalence proved in Section 8 supplies the rigorous justification that every geometric structure appearing in low-energy physics is the faithful image, under tropical linear evolution, of one global arithmetic first-class constraint. In this precise sense the principle of stationary action is not an independent physical postulate; it is the geometric shadow of enforcing a single consistency condition on arithmetic data.

Several directions remain open. The explicit computation of higher Seeley–DeWitt coefficients as tropical polynomials in the shear amplitudes will furnish further quantitative control on stability and confinement. The construction of the functor on morphisms and the verification of its naturality properties will complete the categorical picture. Numerical exploration of the infrared attractor for small irregular primes (beginning with $p = 691$) may yield testable predictions for fusion-rate corrections or for the detailed shape of the effective potential. Finally, the conversion of the spectral stability condition into a zero-free theorem for the associated L-functions stands as the principal remaining analytic task on the path to a proof of the Riemann Hypothesis inside the RFTP framework.

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