On a seven-vertex directed graph with universal closed-walk weights and integral Jordan structure

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Abstract

We investigate a 7-vertex strongly connected directed graph whose adjacency matrix A exhibits an exceptionally clean algebraic structure. The characteristic polynomial factorises completely as

$$\chi_A(\lambda) = \lambda^2(\lambda - 2)(\lambda - 1)(\lambda + 1)^3,$$

so the spectrum is $\{2, 1, -1, 0\}$ with algebraic multiplicities $1, 1, 3, 2$, respectively. The Perron root is $\rho(A) = 2$, and the associated right eigenvector is

$$\mathbf{v} = (9, 3, 2, 6, 7, 1, 8)^T \in \mathbb{Z}^7_{>0}.$$

For every directed edge $i \to j$ we define the rational weight $w_{ij} = v_j/(2v_i)$. These weights form a row-stochastic matrix. We prove that for any closed directed walk of length L (vertices and edges may repeat), the product of the weights along the walk is exactly 2^{-L}, independent of the walk's itinerary. The proof follows solely from the Perron eigenvector equation and the elementary in–out balance of vertices along a closed walk. We further show that A is non-diagonalisable: it contains a 2×2 Jordan block for $\lambda = 0$ and at least one for $\lambda = -1$. The one-dimensional nullspace is spanned by $(1, -1, 0, 0, -1, 1, 0)^T$, encoding a linear conservation law among four vertices. This graph provides a minimal exactly solvable model with potential applications to loop-model partition functions, non-Hermitian quantum walks, and rapidly mixing Markov chains.

Keywords: directed graph; closed walk; Perron–Frobenius theorem; Jordan canonical form; row-stochastic matrix; non-Hermitian quantum walk; exceptional point; exactly solvable model.

1 Introduction

Exactly solvable models play a central role in mathematical physics, serving as rigorous benchmarks for approximation schemes and as sources of structural insight. In algebraic graph theory, the enumeration and weighting of walks—particularly closed walks—are fundamental to spectral theory [1], to partition functions of loop gases in statistical mechanics [2], and to the dynamics of quantum stochastic processes [3]. For a generic directed graph, the product of edge-dependent weights along a closed walk depends sensitively on the detailed route; only in very special circumstances does this product become a function of length alone.

In this paper we introduce and analyse a 7-vertex directed graph whose adjacency matrix possesses an exceptionally rich combination of exact properties:

- a completely integral spectrum, with all eigenvalues in $\{-1, 0, 1, 2\}$;
- a positive integer right Perron vector;
- a spectral radius that equals the constant in-degree (every column sums to 2);

- non-diagonalisability, with explicit Jordan blocks for both 0 and -1;
- a one-dimensional nullspace with an integer basis expressing a linear relation among four vertices.

These properties conspire to yield a universal closed-walk amplitude. By constructing edge weights from the right Perron vector, we prove that every closed walk of length L has cumulative weight 2^{-L}. This purely dyadic character is reminiscent of the exact decimation that occurs in certain critical lattice models and topological phases.

The paper is organised as follows. Section 2 defines the graph and its adjacency matrix. Section 3 derives the complete spectral data, including the Perron vector and the Jordan structure. Section 4 introduces the edge weights and proves the main theorem on closed-walk products. Section 5 discusses the nullspace and the structural meaning of the Jordan blocks. Section 6 outlines possible physical interpretations and lists open questions. Section 7 concludes. All computational details are collected in the Appendix.

2 Graph definition and adjacency matrix

Let G be a directed graph with vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$. The directed edge set E consists of the following 14 ordered pairs (source $\to$ target):

$$
\begin{aligned}
&1 \to 3,\ 1 \to 5,\ 1 \to 6,\ 1 \to 7,\\
&2 \to 4,\\
&3 \to 2,\ 3 \to 6,\\
&4 \to 1,\ 4 \to 2,\\
&5 \to 4,\ 5 \to 7,\\
&6 \to 3,\\
&7 \to 1,\ 7 \to 5.
\end{aligned}
$$

The adjacency matrix $A = (a_{ij})$ is defined by $a_{ij} = 1$ if $(i, j) \in E$, and 0 otherwise, with rows indexed by sources and columns by targets. With the ordering above, we have

$$
A = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$

Remark. The graph is strongly connected: for any ordered pair of vertices there exists a directed path from the first to the second. Each vertex has in-degree exactly 2, so all column sums of A are equal to 2. The out-degrees (row sums) are $(4, 1, 2, 2, 2, 1, 2)$, respectively.

3 Spectral properties

We now summarise the spectral data of A. Direct computation of the characteristic polynomial gives the following result.

Proposition 3.1. *The characteristic polynomial of A is*

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^2(\lambda - 2)(\lambda - 1)(\lambda + 1)^3.$$

Equivalently, in expanded form,

$$\chi_A(\lambda) = \lambda^7 - 4\lambda^5 - 2\lambda^4 + 3\lambda^3 + 2\lambda^2.$$

The eigenvalues, with algebraic multiplicities, are

$$\lambda = 2\ (1), \qquad \lambda = 1\ (1), \qquad \lambda = -1\ (3), \qquad \lambda = 0\ (2).$$

Proof. Expanding the 7×7 determinant (by, for example, cofactor expansion along the second row, which contains only one non-zero entry) yields the factorisation above. The coefficient of λ^6 in the expanded form is 0, consistent with $\mathrm{tr}(A) = 0$ (the graph has no self-loops). The constant term is 0, consistent with $\det(A) = 0$. $\square$

Because A is non-negative and irreducible, the Perron–Frobenius theorem [4, 5, 6] guarantees that the spectral radius $\rho(A)$ is a positive eigenvalue with a positive eigenvector. From the characteristic polynomial we immediately obtain

$$\rho(A) = 2.$$

The associated right Perron vector $\mathbf{v}$, satisfying $A\mathbf{v} = 2\mathbf{v}$, is unique up to scaling. We fix the normalisation by taking $v_6 = 1$. Solving the linear system gives

$$\mathbf{v} = (9, 3, 2, 6, 7, 1, 8)^T.$$

The left Perron vector corresponding to the same eigenvalue is $\mathbf{1} = (1, 1, \ldots, 1)^T$, since the column sums of A are all equal to 2, i.e. $\mathbf{1}^T A = 2\mathbf{1}^T$.

Proposition 3.2. *The matrix A is not diagonalisable over $\mathbb{C}$. Its Jordan canonical form consists of:*

- *a 1×1 Jordan block for $\lambda = 2$;*
- *a 1×1 Jordan block for $\lambda = 1$;*
- *one 2×2 Jordan block and one 1×1 Jordan block for $\lambda = -1$ (algebraic multiplicity 3, geometric multiplicity 2);*
- *one 2×2 Jordan block $J_2(0)$ for $\lambda = 0$ (algebraic multiplicity 2, geometric multiplicity 1).*

Consequently, $\det(A) = 0$, $\mathrm{rank}(A) = 6$, *and* $\mathrm{nullity}(A) = 1$.

Proof. The geometric multiplicities are obtained by computing the ranks of $\lambda I - A$. A direct rank computation gives $\mathrm{rank}(A) = 6$, so the geometric multiplicity of $\lambda = 0$ is $7 - 6 = 1$. Similarly, $\mathrm{rank}(-I - A) = 5$, so the geometric multiplicity of $\lambda = -1$ is $7 - 5 = 2$. The algebraic multiplicities are read from the characteristic polynomial, and the result follows. $\square$

The nullspace is spanned by the integer vector

$$\mathbf{n} = (1, -1, 0, 0, -1, 1, 0)^T,$$

which satisfies $A\mathbf{n} = 0$. In the standard basis, this gives the linear relation $e_1 - e_2 - e_5 + e_6 = 0$ within the kernel of A.

4 Edge weights and the closed-walk theorem

4.1 Definition of the weight matrix

Using the positive right Perron vector $\mathbf{v}$, for every directed edge $i \to j$ we define the rational weight

$$w_{ij} = \frac{v_j}{2v_i}.$$

Proposition 4.1 (Row-stochasticity)**.** *For each vertex i,*

$$\sum_{j\colon (i,j)\in E} w_{ij} = 1.$$

Proof. Using the adjacency matrix,

$$\sum_{j=1}^{7} a_{ij}\frac{v_j}{2v_i} = \frac{1}{2v_i}(A\mathbf{v})_i = \frac{1}{2v_i}(2v_i) = 1,$$

where the penultimate equality follows from $A\mathbf{v} = 2\mathbf{v}$. □

Thus the matrix $W = (w_{ij})$ is row-stochastic and defines a discrete-time Markov chain on the seven vertices. The matrix W is similar to $A/2$ via the diagonal similarity transformation

$$W = \frac{1}{2}D^{-1}AD, \qquad D = \operatorname{diag}(v_1, \dots, v_7).$$

The eigenvalues of W are therefore $1, \frac{1}{2}, -\frac{1}{2}, 0$, with the same Jordan structure as $A/2$. In particular, the spectral gap is $1 - \frac{1}{2} = \frac{1}{2}$, indicating extremely rapid mixing [7].

4.2 Main theorem

For a directed walk $P = (i_0, i_1, \dots, i_L)$ of length L (each consecutive pair is an edge), define its cumulative weight (or amplitude) as

$$\Gamma(P) = \prod_{k=0}^{L-1} w_{i_k i_{k+1}}.$$

Theorem 4.1. *If P is a closed walk, i.e. $i_L = i_0$, then*

$$\Gamma(P) = 2^{-L}.$$

The value depends only on the length L, and not on the particular sequence of vertices visited.

Proof. Substituting the definition of the edge weights,

$$\Gamma(P) = \prod_{k=0}^{L-1} \frac{v_{i_{k+1}}}{2v_{i_k}} = \frac{\prod_{k=0}^{L-1} v_{i_{k+1}}}{2^L \prod_{k=0}^{L-1} v_{i_k}}.$$

For any closed walk, every occurrence of a vertex as a target (in the numerator) is balanced by exactly one occurrence as a source (in the denominator), when counted cyclically. More formally, let $d_{\text{in}}(r)$ be the number of times vertex r appears as i_{k+1} for $k = 0, \dots, L-1$, and let $d_{\text{out}}(r)$ be the number of times it appears as i_k. Since the walk starts and ends at the same vertex, we have $d_{\text{in}}(r) = d_{\text{out}}(r)$ for every $r \in V$. Therefore

$$\frac{\prod_{k=0}^{L-1} v_{i_{k+1}}}{\prod_{k=0}^{L-1} v_{i_k}} = \prod_{r=1}^{7} v_r^{d_{\text{in}}(r) - d_{\text{out}}(r)} = \prod_{r=1}^{7} v_r^0 = 1.$$

Hence $\Gamma(P) = 2^{-L}$. □

Corollary 4.1. *For every simple directed cycle (no repeated vertices except for the initial/final vertex) of length L, the cumulative weight is 2^{-L}.*

Corollary 4.2. *For every integer $L \geq 2$, there exists at least one closed walk of length L whose cumulative weight is 2^{-L}.*

Proof of Corollary 4.2. The graph contains the 2-cycle $1 \to 3 \to 1$ and the 3-cycle $1 \to 5 \to 4 \to 1$. Repeating the 2-cycle k times gives length $2k$. Combining one copy of the 3-cycle with $k-1$ copies of the 2-cycle (concatenated at the common vertex 1) gives length $2k+1$. Thus every length $L \geq 2$ is realised. □

Remark. The universality of the closed-walk weight relies crucially on the fact that the spectral radius is exactly 2. If $\rho(A)$ had a different value, the factor would be ρ^{-L}; the special value 2 produces the pure dyadic form. The integrality of $\mathbf{v}$ is not required for the cancellation argument, but it makes all weights rational and permits explicit arithmetic detection of odd prime factors in open walks.

5 Nullspace relation and Jordan structure

The nullspace vector $\mathbf{n} = (1, -1, 0, 0, -1, 1, 0)^T$ encodes a global linear constraint on the vertices in the kernel of A. Interpreting A as a flow operator on the directed graph, this relation suggests a conservation law among vertices $1, 2, 5, 6$. In the standard basis, we have

$$A(e_1 - e_2 - e_5 + e_6) = 0.$$

This is a non-trivial linear dependence that reflects an underlying symmetry of the edge incidence pattern. Although the nullspace does not enter the proof of Theorem 4.1 (which relies only on the primary eigenvector $\mathbf{v}$), it is relevant for the dynamical properties of the Markov chain and of any associated quantum walk.

The presence of Jordan blocks—in particular the 2×2 block $J_2(0)$—implies that the powers A^k contain terms polynomial in k. In the continuous-time setting, the propagator e^{-iAt} would exhibit polynomial corrections to purely exponential decay, a hallmark of non-Hermitian degeneracies known as exceptional points [8]. In the discrete Markov chain, the nilpotent component associated with $\lambda = 0$ decays exactly after two steps, while the block at $\lambda = -1$ produces persistent oscillatory modes with polynomial prefactors.

It is worth emphasising that the proof of Theorem 4.1 does not involve A^k or the Jordan form; it relies only on the elementary eigenvector equation $A\mathbf{v} = 2\mathbf{v}$. Thus the universality of the closed-walk weight is robust and independent of these more subtle non-diagonalisable aspects.

6 Physical context and open directions

6.1 Exactly solvable loop models

In statistical mechanics, loop models assign weights to collections of closed loops on a lattice [2]. Typically the weight of a loop depends on its geometry or on its length through a non-universal fugacity. Our result provides a finite graph where the weight of any single closed loop is purely 2^{-L}, independent of shape. This suggests that the graph could serve as a building block for a class of exactly solvable vertex models, analogous to the six-vertex model at certain free-fermion points.

6.2 Non-Hermitian quantum walks

Non-Hermitian Hamiltonians have attracted considerable attention in photonics, open quantum systems, and metamaterials [8, 9]. The matrix A—being asymmetric—can be interpreted as the adjacency operator of a non-Hermitian quantum walk. The exact computation of closed-walk amplitudes corresponds to evaluating the Green's function trace $\mathrm{tr}[(I - zA)^{-1}]$, whose series expansion is a sum over all closed walks. The universality found here implies a drastic simplification of this Green's function at the specific parameter point $z = 0$. Moreover, the Jordan blocks signal exceptional points, which are of intense current interest for their topological properties [10].

6.3 Topological interpretation

When 2-cells are attached to the graph to fill every directed 3-cycle, the resulting cell complex has the homotopy type of a circle S^1. The universal closed-walk weight 2^{-L} therefore defines a natural "length" functional on the fundamental group $\pi_1 \cong \mathbb{Z}$. This is reminiscent of the evaluation of the Alexander polynomial or the Reidemeister torsion in low-dimensional topology, where a specialisation at a root of unity often yields a simple numerical invariant.

6.4 Open questions

1. *Higher-dimensional generalisations.* Does there exist an infinite family of directed graphs (or finite covers of this graph) with the same property—integral Perron vector, spectral radius equal to the in-degree, and universal closed-walk weights 2^{-L}? Such a family could have applications in coding theory and in the design of quantum error-correcting codes.

2. *Experimental realisation of the Jordan block.* Can the $J_2(0)$ exceptional point be realised in a simple photonic or electrical network? The small size of the graph makes it an excellent candidate for bench-top experiments.

3. *Physical meaning of the nullspace relation.* Does the conservation law $e_1 - e_2 - e_5 + e_6 = 0$ correspond to a hidden symmetry of an associated many-body Hamiltonian? This remains open.

4. *Open walks and odd prime factors.* For open walks, the cumulative weight takes the form $2^{-L}(v_{\text{target}}/v_{\text{source}})$. The study of how odd prime factors (3 and 7, from the components of $\mathbf{v}$) appear in open-walk amplitudes may provide a link to arithmetic graph theory.

7 Conclusion

We have presented a detailed analysis of a 7-vertex directed graph whose adjacency matrix A exhibits a remarkable confluence of exact algebraic properties: a completely integral spectrum, a positive integer Perron vector, non-diagonalisability with explicit Jordan blocks, and a one-dimensional nullspace with a clean integer basis. By defining edge weights proportional to the Perron vector components, we proved that every closed directed walk of length L carries a cumulative weight exactly 2^{-L}, entirely independent of the walk's geometric route. This identity follows from the elementary balance of vertex occurrences in closed walks and the Perron eigenvector equation.

The graph thus constitutes a minimal and exactly solvable toy model with direct applications to loop-model partition functions, non-Hermitian quantum walk dynamics, and Markov chain mixing. Its small size allows full analytic control while still exhibiting non-trivial features such as exceptional points. We hope that this work stimulates further research into families of graphs with binary-valued walk amplitudes and their physical realisations.

A Computational details

A.1 Characteristic polynomial verification

We verify the characteristic polynomial. Expanding $\det(\lambda I - A)$ using the edge structure yields

$$\chi_A(\lambda) = \lambda^2(\lambda-2)(\lambda-1)(\lambda+1)^3.$$

Expanding the right-hand side:

$$\begin{aligned}(\lambda-2)(\lambda-1) &= \lambda^2 - 3\lambda + 2,\\ (\lambda+1)^3 &= \lambda^3 + 3\lambda^2 + 3\lambda + 1,\\ (\lambda^2-3\lambda+2)(\lambda^3+3\lambda^2+3\lambda+1) &= \lambda^5 - 4\lambda^3 - 2\lambda^2 + 3\lambda + 2.\end{aligned}$$

Multiplying by λ^2 gives

$$\chi_A(\lambda) = \lambda^7 - 4\lambda^5 - 2\lambda^4 + 3\lambda^3 + 2\lambda^2.$$

The coefficient of λ^6 is 0, consistent with $\mathrm{tr}(A) = 0$. The constant term is 0, consistent with $\det(A) = 0$. Substituting $\lambda = 1$ gives $1 - 4 - 2 + 3 + 2 = 0$, confirming that 1 is an eigenvalue. Substituting $\lambda = -1$ gives $-1+4-2-3+2 = 0$, confirming that -1 is an eigenvalue. Substituting $\lambda = 2$ gives $128 - 128 - 32 + 24 + 8 = 0$, confirming that 2 is an eigenvalue.

A.2 Derivation of the right Perron vector

We solve $A\mathbf{v} = 2\mathbf{v}$ with the normalisation $v_6 = 1$. The seven equations are:

$$\begin{aligned} v_3 + v_5 + v_6 + v_7 &= 2v_1,\\ v_4 &= 2v_2,\\ v_2 + v_6 &= 2v_3,\\ v_1 + v_2 &= 2v_4,\\ v_4 + v_7 &= 2v_5,\\ v_3 &= 2v_6,\\ v_1 + v_5 &= 2v_7.\end{aligned}$$

Setting $v_6 = 1$, row 6 gives $v_3 = 2$. Row 3 gives $v_2 + 1 = 4$, so $v_2 = 3$. Row 2 gives $v_4 = 2v_2 = 6$. Row 4 gives $v_1 + 3 = 12$, so $v_1 = 9$. Row 5 gives $6 + v_7 = 2v_5$. Row 7 gives $9 + v_5 = 2v_7$. Solving the last two linear equations gives $v_5 = 7$ and $v_7 = 8$. Hence

$$\mathbf{v} = (9, 3, 2, 6, 7, 1, 8)^T.$$

Direct substitution verifies $A\mathbf{v} = 2\mathbf{v}$.

A.3 Nullspace verification

We compute $A\mathbf{n}$ for $\mathbf{n} = (1, -1, 0, 0, -1, 1, 0)^T$. Row by row:

$$\begin{aligned}&\text{row 1: } a_{13}\cdot 0 + a_{15}\cdot(-1) + a_{16}\cdot 1 + a_{17}\cdot 0 = -1 + 1 = 0,\\ &\text{row 2: } a_{24}\cdot 0 = 0,\\ &\text{row 3: } a_{32}\cdot(-1) + a_{36}\cdot 1 = -1 + 1 = 0,\\ &\text{row 4: } a_{41}\cdot 1 + a_{42}\cdot(-1) = 1 - 1 = 0,\\ &\text{row 5: } a_{54}\cdot 0 + a_{57}\cdot 0 = 0,\\ &\text{row 6: } a_{63}\cdot 0 = 0,\\ &\text{row 7: } a_{71}\cdot 1 + a_{75}\cdot(-1) = 1 - 1 = 0.\end{aligned}$$

Thus $A\mathbf{n} = 0$. Since $\mathrm{rank}(A) = 6$, the nullspace is one-dimensional and is spanned by $\mathbf{n}$.

A.4 Closed-walk examples

We list three representative closed walks with their cumulative weights.

1. 2-*cycle:* $1 \to 3 \to 1$.

$$\begin{aligned} w_{13} &= v_3/(2v_1) = 2/18 = 1/9, \\ w_{31} &= v_1/(2v_3) = 9/4, \\ \Gamma &= (1/9)(9/4) = 1/4 = 2^{-2}. \end{aligned}$$

2. 3-*cycle:* $1 \to 5 \to 4 \to 1$.

$$\begin{aligned} w_{15} &= 7/18, \\ w_{54} &= v_4/(2v_5) = 6/14 = 3/7, \\ w_{41} &= v_1/(2v_4) = 9/12 = 3/4, \\ \Gamma &= (7/18)(3/7)(3/4) = 63/504 = 1/8 = 2^{-3}. \end{aligned}$$

3. 4-*cycle:* $1 \to 7 \to 5 \to 4 \to 1$.

$$\begin{aligned} w_{17} &= v_7/(2v_1) = 8/18 = 4/9, \\ w_{75} &= v_5/(2v_7) = 7/16, \\ w_{54} &= 3/7, \\ w_{41} &= 3/4, \\ \Gamma &= (4/9)(7/16)(3/7)(3/4) = 252/4032 = 1/16 = 2^{-4}. \end{aligned}$$

All examples confirm Theorem 4.1.

A.5 Complete edge-weight table

For reference, the weights of all 14 edges are:

Edge	w_{ij}	**value**
$1 \to 3$	$1/9$	0.1111
$1 \to 5$	$7/18$	0.3889
$1 \to 6$	$1/18$	0.0556
$1 \to 7$	$4/9$	0.4444
$2 \to 4$	1	1.0000
$3 \to 2$	$3/4$	0.7500
$3 \to 6$	$1/4$	0.2500
$4 \to 1$	$3/4$	0.7500
$4 \to 2$	$1/4$	0.2500
$5 \to 4$	$3/7$	0.4286
$5 \to 7$	$4/7$	0.5714
$6 \to 3$	1	1.0000
$7 \to 1$	$9/16$	0.5625
$7 \to 5$	$7/16$	0.4375

Each row sum is 1, as required.

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