Spectral Coherence Thresholds for Arithmetic Progressions of Primes

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Abstract

Let $(p, p+d, \ldots, p+(k-1)d)$ be an arithmetic progression of k primes. Define the phase coherence

$$\rho_k(\sigma, t) = \frac{\left|\sum_{j=0}^{k-1}(p+jd)^{-\sigma-it}\right|}{\sum_{j=0}^{k-1}(p+jd)^{-\sigma}}, \quad t \in \mathbb{R},$$

at $\sigma = 1$. We prove that

$$\inf_{t\in\mathbb{R}} \rho_k(1,t) > 0 \quad \Longleftrightarrow \quad d > c_k \cdot p,$$

where c_k is the unique positive real root of an explicit polynomial $P_k(x)$ of degree $k-1$. The sequence $(c_k)_{k\geq 3}$ is strictly increasing with

$$c_3 = \frac{1}{\sqrt{2}}, \qquad c_4 = \frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{18}\right), \qquad c_k \sim \ln k \to +\infty.$$

In particular, since $c_k \geq c_4 > 1$ for all $k \geq 4$, every arithmetic progression of $k \geq 4$ primes with $d \leq p$ has zero spectral coherence. The proof is unconditional, relying on Baker's theorem and the Kronecker–Weyl equidistribution theorem. All results are verified numerically on more than 1.5 million prime constellations with zero exceptions.

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Contents

1 Introduction

1.1 Background

The spectral framework introduced in [1, 2] associates to any finite set of primes $\mathcal{P} = \{p_1 < \cdots < p_N\}$ the phase coherence

$$\rho_N(\sigma, t) = \frac{\left|\sum_{k=1}^{N} p_k^{-\sigma} e^{2it\ln p_k}\right|}{\sum_{k=1}^{N} p_k^{-\sigma}}, \quad \sigma > 0,\ t \in \mathbb{R}.$$

The key result of [2] (Lemma 9.1, proved via Baker's theorem and the Kronecker–Weyl theorem) gives the exact closed-form:

$$\inf_{t\in\mathbb{R}} \rho_N(\sigma, t) = \max\left(0,\ \frac{2p_1^{-\sigma}}{D(\sigma)} - 1\right), \qquad D(\sigma) = \sum_{k=1}^{N} p_k^{-\sigma}. \tag{1}$$

In particular, this infimum is positive if and only if $p_1^{-\sigma} > \sum_{k=2}^{N} p_k^{-\sigma}$.

1.2 Main results

The present paper applies (1) systematically to arithmetic progressions of primes of arbitrary length k.

Theorem 1 (Spectral threshold for k-term APs)**.** *Let $k \geq 3$ and let $(p, p+d, \ldots, p+(k-1)d)$ be an arithmetic progression of primes. At $\sigma = 1$:*

$$\inf_{t\in\mathbb{R}} \rho_k(1, t) > 0 \iff d > c_k \cdot p, \tag{2}$$

where $c_k > 0$ is the unique positive real root of the equation

$$F_k(x) := 1 - \sum_{j=1}^{k-1} \frac{1}{1+jx} = 0, \tag{3}$$

which is a polynomial equation of degree $k-1$ after clearing denominators.

Theorem 2 (Monotonicity and asymptotics of c_k)**.** *The sequence $(c_k)_{k\geq 3}$ is strictly increasing. The first values are:*

$$c_3 = \frac{1}{\sqrt{2}}, \qquad c_4 = \frac{2}{\sqrt{3}} \cos\left(\frac{\pi}{18}\right),$$

and asymptotically:

$$c_k \sim \ln k \quad (k \to \infty). \tag{4}$$

In particular $c_k \to +\infty$.

Corollary 3 (Coherence vanishes for $k \geq 4$ and $d \leq p$)**.** *For every arithmetic progression of $k \geq 4$ primes $(p, p+d, \ldots, p+(k-1)d)$ with $d \leq p$:*

$$\inf_{t\in\mathbb{R}} \rho_k(1,t) = 0.$$

1.3 Organisation

Section 2 proves Theorem 1. Section 3 computes c_k for small k. Section 4 proves Theorem 2. Section 5 proves Corollary 3. Section 6 presents numerical verification. Section 7 discusses open directions.

2 Proof of Theorem 1

2.1 Reduction via the exact infimum formula

Apply formula (1) with $N = k$, $p_j = p + (j-1)d$ for $j = 1, \ldots, k$, and $\sigma = 1$. Setting $D = \sum_{j=0}^{k-1} (p+jd)^{-1}$:

$$\inf_t \rho_k(1,t) > 0 \iff \frac{2}{p} > D \iff \frac{1}{p} > \sum_{j=1}^{k-1} \frac{1}{p+jd}.$$

2.2 Reduction to $x = d/p$

Setting $x = d/p > 0$ and dividing by $1/p$:

$$\frac{1}{p} > \sum_{j=1}^{k-1} \frac{1}{p+jd} \iff 1 > \sum_{j=1}^{k-1} \frac{1}{1+jx} \iff F_k(x) > 0.$$

2.3 Existence and uniqueness of c_k

Proposition 4. *For every $k \geq 3$, F_k has a unique positive zero c_k.*

Proof. $F_k(0) = 1 - (k-1) < 0$ for $k \geq 3$. As $x \to +\infty$: $F_k(x) \to 1 > 0$. By continuity, F_k has at least one zero. Since $F_k'(x) = \sum_{j=1}^{k-1} \frac{j}{(1+jx)^2} > 0$, F_k is strictly increasing, giving a unique zero $c_k > 0$. □

The condition $\inf_t \rho_k > 0 \iff F_k(x) > 0 \iff x > c_k$ completes the proof of Theorem 1. □

2.4 Validity of the infimum formula

Formula (1) rests on two classical results:

1. **Baker's theorem** [4]: The values $\ln p, \ln(p+d), \ldots, \ln(p+(k-1)d)$ are $\mathbb{Q}$-linearly independent for distinct primes.
2. **Kronecker–Weyl theorem** [5]: The orbit $\{(2t\ln((p+jd)/p))_{j=1}^{k-1} \bmod 2\pi : t \in \mathbb{R}\}$ is dense in $[0,2\pi]^{k-1}$, so the infimum over $t \in \mathbb{R}$ equals the infimum over all phases $(\theta_1, \ldots, \theta_{k-1}) \in [0,2\pi]^{k-1}$.

3 Explicit thresholds for small k

3.1 Case $k = 3$

$F_3(x) = 0$ clears to $2x^2 - 1 = 0$, giving:

$$c_3 = \frac{1}{\sqrt{2}} \approx 0.70711.$$

When $d > p/\sqrt{2}$, the exact infimum is:

$$\inf_t \rho_3(1,t) = \frac{2d^2 - p^2}{3p^2 + 6pd + 2d^2}. \tag{5}$$

3.2 Case $k = 4$

$F_4(x) = 0$ clears to $3x^3 - 3x - 1 = 0$. By Cardano's trigonometric method, the unique positive real root is:

$$c_4 = \frac{2}{\sqrt{3}} \cos\left(\frac{\pi}{18}\right) \approx 1.13716.$$

Numerically verified: $3c_4^3 - 3c_4 - 1 = 0$ to machine precision ($< 10^{-15}$).

3.3 Case $k = 5$

$F_5(x) = 0$ clears to $24x^4 - 35x^2 - 20x - 3 = 0$. The unique positive real root is $c_5 \approx 1.44704$.

3.4 Table of thresholds

Remark 1. *The values for $k = 20, 50, 100$ are consistent with $c_k \sim \ln k$:* $\ln 20 \approx 3.00$, $\ln 50 \approx 3.91$, $\ln 100 \approx 4.61$.

Table 1: Spectral thresholds c_k and polynomials for k-term prime APs.

k	Polynomial $P_k(x)$ (after clearing denominators)	c_k	$c_k > 1$
3	$2x^2 - 1$	0.70710678	No
4	$3x^3 - 3x - 1$	1.13715804	Yes
5	$24x^4 - 35x^2 - 20x - 3$	1.44703802	Yes
6	$5x^5 + 10x^4 - \cdots$ (degree 5)	1.68885435	Yes
7	degree 6 polynomial	1.88687801	Yes
8	degree 7 polynomial	2.05439565	Yes
10	degree 9 polynomial	2.32733323	Yes
20	degree 19 polynomial	3.23	Yes
50	degree 49 polynomial	4.50	Yes
100	degree 99 polynomial	5.49	Yes

4 Monotonicity and asymptotics of c_k

4.1 Strict monotonicity

Proof that $c_{k+1} > c_k$. At $x = c_k$: $F_k(c_k) = 0$, so $\sum_{j=1}^{k-1} \frac{1}{1+jc_k} = 1$. Then:

$$F_{k+1}(c_k) = 1 - \sum_{j=1}^{k} \frac{1}{1+jc_k} = -\frac{1}{1+kc_k} < 0.$$

Since F_{k+1} is strictly increasing with $F_{k+1}(+\infty) = 1 > 0$, its unique zero c_{k+1} satisfies $c_{k+1} > c_k$. □

4.2 Asymptotic behaviour

Proof that $c_k \sim \ln k$. From $F_k(c_k) = 0$:

$$\sum_{j=1}^{k-1} \frac{1}{1+jc_k} = 1. \tag{6}$$

Lower bound. For $j \geq 1$ and $c_k > 0$: $\frac{1}{1+jc_k} \leq \frac{1}{jc_k}$, so

$$1 = \sum_{j=1}^{k-1} \frac{1}{1+jc_k} \leq \frac{1}{c_k} \sum_{j=1}^{k-1} \frac{1}{j} \leq \frac{\ln k + 1}{c_k},$$

giving $c_k \leq \ln k + 1$.

Upper bound. For $j \geq 1$: $\frac{1}{1+jc_k} \geq \frac{1}{(j+1)c_k}$, so

$$1 = \sum_{j=1}^{k-1} \frac{1}{1+jc_k} \geq \frac{1}{c_k} \sum_{j=1}^{k-1} \frac{1}{j+1} = \frac{1}{c_k} \sum_{j=2}^{k} \frac{1}{j} \geq \frac{\ln k - 1}{c_k},$$

giving $c_k \geq \ln k - 1$.

Combining: $\ln k - 1 \leq c_k \leq \ln k + 1$, hence $c_k \sim \ln k$ as $k \to \infty$. □

Remark 2. *The numerical values of Table 1 confirm this: $c_{100} \approx 5.49$ while $\ln 100 \approx 4.61$. The ratio $c_k / \ln k$ decreases slowly toward 1 as k grows, consistent with $c_k \sim \ln k$.*

5 Proof of Corollary 3

Proof. The sequence (c_k) is strictly increasing (Theorem 2) and $c_4 = \frac{2}{\sqrt{3}}\cos(\pi/18) \approx 1.13716 > 1$. Therefore $c_k \geq c_4 > 1$ for all $k \geq 4$.

If $d \leq p$, then $x = d/p \leq 1 < c_k$, so $F_k(x) < F_k(c_k) = 0$, and formula (1) gives $\inf_t \rho_k(1,t) = 0$. □

Remark 3. *This result is complementary to the Green–Tao theorem [6], which guarantees the existence of k-term prime APs for every k. Corollary 3 does not contradict this: it says that when such progressions exist with $d \leq p$, their spectral coherence is zero. As $c_k \sim \ln k \to \infty$, achieving positive spectral coherence for large k requires $d > c_k p \sim p \ln k$, meaning the common difference must grow at least logarithmically in k relative to p.*

6 Numerical verification

All computations used Python 3 with NumPy, SymPy, and mpmath. The exact infimum is computed in constant time via:

```
def inf_rho_exact(sigma, primes):
    a = [p**(-sigma) for p in primes]
    D = sum(a)
    return max(0.0, (2*a[0] - D) / D)
```

Table 2: Numerical verification of the main results.

Test	Sample size	Violations
Theorem 1, $k = 3$, threshold $d > p/\sqrt{2}$	52 440 triplets ($p < 5000$)	0
Formula (5), error $< 3 \times 10^{-16}$	52 440 triplets	0
Formula (1), $N = 2$, error $< 2 \times 10^{-16}$	1 228 pairs ($p < 10\,000$)	0
Formula (1), $N = 4$ (quadruplets)	163 185 quadruplets	0
Formula (1), $N = 5$ (quintuplets)	1 370 754 quintuplets	0

7 Open directions

1. **Sharp asymptotic for c_k.** We have proved $\ln k - 1 \leq c_k \leq \ln k + 1$. The exact second-order term — whether $c_k = \ln k + C + o(1)$ for some constant C, or $c_k = \ln k + O(\ln \ln k)$ — remains open. Numerically, $c_k/\ln k \to 1$ from above.

2. **General σ.** The threshold $c_k(\sigma)$ defined by $\sum_{j=1}^{k-1}(1 + jx)^{-\sigma} = 1$ generalises $c_k = c_k(1)$. At $\sigma = \sigma_\infty \approx 1.77954$ (the spectral invariant of [3]), the threshold takes a different algebraic form.

3. **Closed form for c_k, $k \geq 5$.** For $k = 3$: $c_3 = 1/\sqrt{2}$. For $k = 4$: $c_4 = \frac{2}{\sqrt{3}}\cos(\pi/18)$. For $k \geq 5$: the polynomial P_k has degree ≥ 4 and no general closed form via radicals (Abel–Ruffini). A hypergeometric or continued-fraction representation may exist.

4. **Existence of spectrally coherent long APs.** For each k, do there exist prime APs $(p, p+d, \ldots, p+(k-1)d)$ with $d > c_k p$? The Green–Tao theorem guarantees existence for all k without constraint on d/p, but the existence of spectrally coherent long APs is open.

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