The Gravitational Rapidity Potential: A Lorentz-Rotor Formulation of Gravity from Galactic Dynamics to Black-Hole Geodetic Precession

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Abstract

We identify the gravitational rapidity potential chi-g as the primary scalar field of a Lorentz-rotor formulation of gravity. Unlike conventional approaches, which take either the Newtonian potential or the spacetime metric as fundamental, the present framework constructs the local Lorentz rotor directly from chi-g. The gravitational four-velocity, the Newtonian potential, the Painlevé–Gullstrand (PG) coframe, and the corresponding spacetime metric are recovered as successive projections of this underlying Lorentz geometry. The resulting projection hierarchy preserves the exact hyperbolic structure of local Lorentz boosts until the final metric representation. The Schwarzschild and cosmological horizons appear as the two asymptotic limits of a single rapidity field, while the physical velocity remains bounded by construction through v=c tanh(chi-g). Applying the previously derived constant-Lagrangian (CL) condition D/Dt(Gamma)=0 to the gravitational four-velocity yields a geometric interpretation of stationary galactic flows. The conserved quantity Gamma=cosh(chi-r)cosh(chi-phi) defines an isorapidity circle on the Poincaré disk, from which the flat galactic rotation curve, the universal sigma-channel residual, and a weak galactic geodetic precession emerge as complementary projections of the same hyperbolic geometry. The same Lorentz construction extends continuously into the strong-field regime of Schwarzschild accretion disks. Exact composition of the radial vacuum inflow and azimuthal orbital rapidity yields a fully relativistic expression for geodetic precession, recovering the weak-field Gravity Probe B limit while predicting a universal ISCO frequency ratio of 0.374, together with a set of observational tests that distinguish this Lorentz-rotor formulation of geodetic precession from conventional spin-dependent frame-dragging models.

Keywords: gravitational rapidity potential; Lorentz-rotor gravity; Painlevé–Gullstrand geometry; hyperbolic geometry; constant-Lagrangian condition; isorapidity geometry; geodetic precession; galactic rotation curves; Schwarzschild accretion disks; quasi-periodic oscillations

Contents

1 Introduction

Gravity is traditionally formulated in terms of a scalar potential in Newtonian mechanics and by the spacetime metric in General Relativity (GR). Although these two theories differ profoundly in mathematical structure, they share a common viewpoint: the local gravitational field is derived from a more fundamental object. In Newtonian gravity this object is the potential, whereas in GR it is the spacetime metric, represented here in its Painlevé–Gullstrand (PG) form together with its river model of space (Painlevé 1921; Gullstrand 1922; Hamilton and Lisle 2008). The present paper follows the same philosophy but identifies a different primary object: the gravitational rapidity potential χ_g. From this field the local Lorentz rotor is constructed exactly, while the Newtonian potential, the PG coframe, and the corresponding spacetime metric emerge as successive projections of the underlying rapidity geometry.

Rapidity occupies a distinguished position within Special Relativity. Unlike physical velocity, it is additive under successive Lorentz boosts, remains unbounded as the projected velocity approaches the speed of light, and measures geodesic distance on hyperbolic velocity space. Since the work of Varičak (1912); Borel (1913), together with the discoveries of Thomas precession (Thomas 1926, 1927) and its group-theoretical interpretation by Wigner (1939), hyperbolic geometry has been recognised as the natural geometry of Lorentz transformations. Modern geometric formulations of relativity (Hestenes 1966; Doran and Lasenby 2003) make extensive use of this structure, yet rapidity itself rarely serves as the primary gravitational field variable.

The formulation developed here is based on the BQ algebra introduced and developed in de Haas (2018, 2020, 2026a). Within this framework the fundamental gravitational object is the local Lorentz rotor

$$G_g(\mathbf{r}, t) \in \mathrm{Spin}(1, 3),$$

generated by the gravitational rapidity potential $\chi_g(\mathbf{r}, t)$. The gravitational rapidity potential is a scalar field on spacetime whose value specifies the local Lorentz boost relative to the asymptotic cosmological frame. Through exponentiation it generates the local Lorentz rotor G_g, from which the remaining gravitational quantities arise by successive projections. It is important to distinguish the present construction from the conventional Lorentz transformation relating two inertial reference frames. In Special Relativity a Lorentz transformation is normally characterised by a single constant rapidity χ (or Lorentz factor $\gamma(v)$) describing the relative motion of two observers. Here, by contrast, the rapidity is promoted to a spacetime-dependent field, $\chi_g = \chi_g(\boldsymbol{r}, t)$, so that the corresponding Lorentz rotor $G_g = G_g(\boldsymbol{r}, t)$ becomes a local geometric object. The theory therefore describes not a transformation between reference frames, but a field of local Lorentz transformations generated by the gravitational rapidity potential. The familiar global Lorentz transformation is recovered as the special case in which χ_g is constant throughout spacetime. The rotor G_g acts directly on BQ four-vectors through the exact double-sided adjoint action, providing a first-order description of the local Lorentz geometry. Successive projections of this rotor generate the familiar gravitational quantities,

$$\chi_g \longrightarrow G_g \longrightarrow U_g \longrightarrow g_{\mu\nu},$$

where χ_g denotes the gravitational rapidity, G_g the local Lorentz rotor, U_g the gravitational four-velocity field, and $g_{\mu\nu}$ the spacetime metric. Within this hierarchy the metric is not postulated but recovered as a derived object, while the rapidity field and Lorentz rotor retain the complete hyperbolic structure of the local gravitational flow.

This viewpoint is motivated by two complementary developments. The first is the rapidity-space description of geodetic precession presented in de Haas (2014), where the exact composition of two perpendicular Lorentz boosts determines the accumulated Thomas–Wigner rotation through the hyperbolic area enclosed on the Poincaré disk. The second is the Lorentz-rotor formulation of electromagnetism, relativistic fluid dynamics and gravitation developed in de Haas (2026a,c), in which local physical fields are constructed through first-order rotor equations rather than by second-order potential equations. Together these developments motivate that the hyperbolic geometry of local Lorentz boosts may itself provide the natural starting point for a description of gravitation.

The present work develops this idea in three successive steps. First, it constructs the gravitational rapidity field and derives the local Lorentz rotor together with its four-velocity projection. Second, it establishes the approximation hierarchy connecting the rapidity field with the Newtonian potential, the Painlevé–Gullstrand river model (Painlevé 1921; Gullstrand 1922; Hamilton and Lisle 2008), and the metric description of General Relativity. Finally, it applies the same Lorentz-geometric framework to the family of stationary constant-Lagrangian (CL) galactic flows introduced in de Haas (2026c,b), showing that their dynamics are naturally described by conserved isorapidity curves in hyperbolic velocity space. The same geometric construction extends continuously into the strong-field regime of Schwarzschild accretion disks, where the exact composition of radial vacuum inflow and azimuthal orbital motion yields a fully relativistic description of geodetic precession.

The aim of the paper is therefore not to propose an alternative metric theory of gravity, but to identify a more fundamental Lorentz-geometric level from which the familiar descriptions of Newtonian gravity and the Painlevé–Gullstrand representation of General Relativity emerge as successive projections. Within this first-order

Lorentz-rotor hierarchy, the gravitational rapidity potential becomes the primary geometric object, while the Newtonian potential, the gravitational four-velocity, the Painlevé–Gullstrand coframe, and the corresponding spacetime metric appear as successive projections of the same underlying Lorentz-rotor geometry.

The paper is organised as follows. Sections 2–5 introduce the gravitational rapidity field, construct the local Lorentz rotor, and derive the gravitational four-velocity together with its relation to the Newtonian potential and the Painlevé–Gullstrand metric. Section 6 summarises the resulting Lorentz-rotor hierarchy and its connection with the Painlevé–Gullstrand representation of General Relativity. Section 7 develops the geometric interpretation of the constant-Lagrangian condition in terms of isorapidity geometry and extends the same Lorentz construction to the strong-field regime of Schwarzschild accretion disks, leading to a unified description of geodetic precession across gravitational scales.

2 The Combined Newton–Hubble Rapidity Field

In the BQ rotor framework (de Haas 2026c), gravity is encoded through Lorentz boosts generated by rapidity fields rather than by force potentials. For a spherically symmetric mass M embedded in an FRW background with Hubble parameter $H(t)$, two collinear radial rapidities contribute: the Newtonian rapidity $\psi_N(r)$ associated with gravitational infall and the Hubble rapidity $\psi_H(r,t)$ associated with cosmological expansion. The quantities $\psi_N(r)$ and $\psi_H(r,t)$ should not be interpreted as being obtained by transforming existing velocity fields into rapidities. They are introduced as the primary gravitational scalar fields whose low-velocity hyperbolic projections reproduce the Newtonian escape velocity and the Hubble flow, respectively. We define two primary gravitational rapidity potentials:

$$\psi_N(r) \equiv \frac{\sqrt{2GM/r}}{c}, \qquad and \qquad \psi_H(r,t) \equiv \frac{H(t)\,r}{c}. \tag{1}$$

Although the origin of these potential rapidities can obviously be found in the Lorentz rapidities correspondence to existing velocities, the escape velocity and the Hubble velocity, here the potential rapidities are introduced as primary scalar fields whose hyperbolic projections generate the physical velocity field. That is the rapidity approach to gravity: rapidities are the primary objects and everything else is derived from them.

Because the Newtonian and Hubble gravitational rapidity potentials are collinear in rapidity space, visualized by the Poincaré disk, they add linearly and the combined radial gravitational rapidity potentials is

$$\chi_r(r,t) \equiv \psi_N(r) - \psi_H(r,t). \tag{2}$$

so that

$$\boxed{\chi_r(r,t) \equiv \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right).} \tag{3}$$

This is the *gravitational rapidity potential*: a dimensionless scalar field on spacetime whose numerical value is the hyperbolic angle corresponding to the local combined gravitational flow. It is *not* a velocity; it is the gravitational rapidity potentials from which velocities are derived.

3 The Gravitational Four-Velocity as Hyperbolic Projection

Given the gravitational rapidity potential χ_r, the gravitational four-velocity is obtained by the standard Lorentz projection. The rotor

$$G_g = \exp\!\left(\tfrac{1}{2}\chi_r\,\hat{\sigma}_r\right) \tag{4}$$

acts on the timelike reference vector $c\hat{T}$ through the double-sided adjoint action, producing

$$U_g(r,t) = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\chi_r\,\hat{T} + c\sinh\chi_r\,\hat{I}_r. \tag{5}$$

The components are identified as

$$u_0 \equiv c\cosh\chi_r, \qquad u_r \equiv c\sinh\chi_r, \tag{6}$$

so the Lorentz factor and the physical radial flow velocity are

$$\gamma_r^{\mathrm{FR}} = \cosh\chi_r, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r. \tag{7}$$

These are the *exact* flow-relativistic expressions. The observable velocity v_r^{FR} is bounded: $|v_r^{\mathrm{FR}}| < c$ for all finite χ_r, since $|\tanh x| < 1$ identically. As $|\chi_r| \to \infty$ the physical velocity approaches c asymptotically without ever reaching it. The Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$, by contrast, is unbounded and grows without limit as $|\chi_r| \to \infty$: it is the metric information content of the rapidity field, not a bounded observable.

The spatial four-velocity component $c\sinh\chi_r$ is also unbounded. This is physically appropriate: in the strong-field regime near the Schwarzschild radius, $\chi_r \to \infty$ and $c\sinh\chi_r \to \infty$, which signals horizon formation without ever predicting a superluminal physical velocity.

4 Weak-Flow Expansion and Recovery of Known Results

The exact expressions (7)–(5) admit a systematic expansion in powers of χ_r.

4.1 Physical velocity

Using $\tanh x = x - x^3/3 + 2x^5/15 + O(x^7)$ and $v = c\, tanh\chi$:

$$v_r^{\rm FR}(r,t) = u(r,t) - \frac{u^3(r,t)}{3c^2} + O(c^{-4}), \tag{8}$$

where $u(r,t) = c\chi = \sqrt{2GM/r} - H(t)r$ the Newtonian–Hubble slow-flow projection of the primary rapidity field. The first term reproduces exactly the PG velocity used in the companion papers (de Haas 2026c,b); the cubic correction encodes the leading relativistic departure from the Newtonian law. The dominant correction in the strong-field regime is

$$-\frac{u^3}{3c^2} = -\frac{(2GM)^{3/2}}{3c^2 r^{3/2}} + \frac{H(t)\sqrt{2GM}\, r}{c^2} - \frac{H^2(t)\sqrt{2GM}\, r^{3/2}}{c^2} + \frac{H^3(t) r^3}{3c^2}, \tag{9}$$

where the leading term $-(2GM)^{3/2}/(3c^2 r^{3/2})$ softens the Newtonian divergence of the inflow speed as $r \longrightarrow 0$, replacing the unbounded Newtonian result with a velocity that saturates toward c asymptotically.

In a Newton potential only scenario we set $H = 0$, so we get:

$$v_r^{\rm FR}(r,t) = \sqrt{2GM/r} - \frac{(2GM)^{3/2}}{3c^2 r^{3/2}} + O(c^{-4}), \tag{10}$$

and the effective potential would be

$$\Phi_{\rm eff} = -\frac{1}{2} v_r^2(r,t) = -\frac{1}{2}\left(\sqrt{2GM/r} - \frac{(2GM)^{3/2}}{3c^2 r^{3/2}}\right)^2 \approx -\frac{GM}{r} + \frac{4G^2M^2}{3c^2 r^2}, \tag{11}$$

The correction term in Eq. (14) is not introduced as an independent post-Newtonian potential. It is the first nonlinear projection correction produced by treating χ_r, rather than v_r, as the primary gravitational variable. Since the observable velocity is $v_r^{\rm FR} = c \tanh \chi_r$, the Newtonian inflow speed $u = c\chi_r$ is softened by the cubic term in the expansion of $\tanh \chi_r$. The resulting positive $1/r^2$ contribution to $\Phi_{\rm eff}$ therefore reflects the saturation of the physical velocity at c, not an additional force term.

4.2 Lorentz factor

Using $\cosh x = 1 + x^2/2 + x^4/24 + O(x^6)$, and writing $u(r,t) = c\chi_r = \sqrt{2GM/r} - H(t)r$, the Lorentz factor becomes

$$\gamma_r^{\rm FR}(r,t) = \cosh \chi_r = 1 + \frac{u^2(r,t)}{2c^2} + \frac{u^4(r,t)}{24c^4} + O(c^{-6}). \tag{12}$$

Hence

$$\left(\gamma_r^{\mathrm{FR}} - 1\right)c^2 = \frac{1}{2}u^2(r,t) + \frac{u^4(r,t)}{24c^2} + O(c^{-4}). \tag{13}$$

To leading order this identifies the Newtonian kinetic-energy equivalent of the rapidity field,

$$\Phi_{\mathrm{N}}(r,t) = -\frac{1}{2}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right)^2. \tag{14}$$

Including the first relativistic correction from the Lorentz factor gives

$$\Phi_\gamma(r,t) = -\left(\gamma_r^{\mathrm{FR}} - 1\right)c^2 = -\frac{1}{2}u^2(r,t) - \frac{u^4(r,t)}{24c^2} + O(c^{-4}). \tag{15}$$

In the Newton-only limit $H = 0$, this becomes

$$\Phi_\gamma\big|_{H=0} = -\frac{GM}{r} - \frac{G^2M^2}{6c^2r^2} + O(c^{-4}). \tag{16}$$

This correction differs from the velocity-projection correction obtained from $v_r^{\mathrm{FR}} = c\tanh\chi_r$. In that case,

$$\Phi_{\mathrm{eff}} = -\frac{1}{2}\left(v_r^{\mathrm{FR}}\right)^2 = -\frac{1}{2}u^2 + \frac{u^4}{3c^2} + O(c^{-4}), \tag{17}$$

and therefore, for $H = 0$,

$$\Phi_{\mathrm{eff}}\big|_{H=0} = -\frac{GM}{r} + \frac{4G^2M^2}{3c^2r^2} + O(c^{-4}). \tag{18}$$

Thus the difference between the velocity-projection potential and the Lorentz-factor potential is

$$\Delta\Phi_{\mathrm{eff}} \equiv \Phi_{\mathrm{eff}} - \Phi_\gamma = \frac{3u^4}{8c^2} + O(c^{-4}), \tag{19}$$

or, in the Newton-only limit,

$$\Delta\Phi_{\mathrm{eff}}\big|_{H=0} = \frac{3G^2M^2}{2c^2r^2} + O(c^{-4}). \tag{20}$$

This difference is not an additional force term. It reflects the fact that $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$ and $v_r^{\mathrm{FR}}/c = \tanh\chi_r$ are different nonlinear projections of the same primary rapidity field.

The appearance of distinct relativistic corrections does not imply the existence of different gravitational potentials. Rather, the rapidity field χ_r is the primary dynamical variable, while $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$ and $v_r^{\mathrm{FR}} = c\tanh\chi_r$ represent different nonlinear projections of the same underlying rotor field. The Lorentz factor describes the temporal response of the field through time dilation and energy, whereas the velocity determines its spatial kinetic projection. Both reduce to the same Newtonian potential in the slow-flow limit, but their $O(c^{-2})$ corrections naturally differ because they probe different aspects of the same hyperbolic rapidity geometry, rather than indicating different gravitational physics.

4.3 The rapidity field and its observable projections

The preceding sections show that the primary gravitational quantity is not the physical velocity but the rapidity field $\chi_r(r,t)$. Observable quantities are obtained from χ_r through different nonlinear hyperbolic projections. In particular,

$$\gamma_r^{\mathrm{FR}} = \cosh\chi_r, \qquad v_r^{\mathrm{FR}} = c\,\tanh\chi_r, \tag{21}$$

represent respectively the temporal and spatial projections of the same underlying rotor field. Since both mappings are invertible, no information is lost at this stage.

The Newtonian potential arises only after approximating the hyperbolic functions by their leading-order expansions. Writing

$$u(r,t) = c\chi_r = \sqrt{\frac{2GM}{r}} - H(t)r, \tag{22}$$

the Lorentz factor becomes

$$\gamma_r^{\mathrm{FR}} = 1 + \frac{u^2}{2c^2} + O(c^{-4}), \tag{23}$$

so that

$$\Phi_{\mathrm{N}}(r,t) = -\frac{1}{2}u^2(r,t) = -\frac{1}{2}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right)^2 \tag{24}$$

is the quadratic approximation to the rapidity field. The familiar Newtonian potential is recovered as the special case

$$\Phi_{\mathrm{N}}\big|_{H=0} = -\frac{GM}{r}. \tag{25}$$

The hierarchy of approximations is therefore

$$\chi_r \;\longrightarrow\; \begin{cases} \gamma_r^{\mathrm{FR}} = \cosh\chi_r \\ v_r^{\mathrm{FR}} = c\tanh\chi_r \end{cases} \longrightarrow\; \begin{cases} \Phi_\gamma \\ \Phi_{\mathrm{eff}} \end{cases} \longrightarrow\; \Phi_{\mathrm{N}}, \tag{26}$$

where Φ_γ and Φ_{eff} are the temporal and spatial scalar projections discussed in the previous subsection, while Φ_{N} is their common quadratic approximation.

Information is discarded only when the nonlinear hyperbolic functions are truncated. The exact mappings $\gamma_r^{\mathrm{FR}} = \cosh \chi_r$ and $v_r^{\mathrm{FR}} = c \tanh \chi_r$ remain fully invertible, whereas the quadratic approximation neglects the higher-order terms that encode the hyperbolic geometry of the rapidity field. Likewise, the limit $H \to 0$ removes the cosmological contribution, reducing the rapidity field to its purely Newtonian component.

The physical flow field is obtained by hyperbolically projecting the individual rapidity potentials onto velocity space,

$$v_N^{\mathrm{FR}}(r) = c\ \tanh \psi_N(r), \qquad v_H^{\mathrm{FR}}(r,t) = c\ \tanh \psi_H(r,t). \tag{27}$$

In the weak-flow regime, $\psi_N, \psi_H \ll 1$, the hyperbolic tangent reduces to its argument,

$$v_N^{\mathrm{FR}}(r) \approx \sqrt{\frac{2GM}{r}}, \qquad v_H^{\mathrm{FR}}(r,t) \approx H(t)\, r, \tag{28}$$

recovering the familiar Newtonian escape velocity and local Hubble flow as the leading-order projections of the corresponding rapidity fields.

Because rapidities compose linearly in the rotor algebra, the combined radial rapidity

$$\chi_r = \psi_N - \psi_H \tag{29}$$

is formed prior to projection. The observable inflow velocity is therefore obtained through the single hyperbolic projection

$$v_r^{\mathrm{FR}}(r,t) = c\ \tanh \chi_r(r,t), \tag{30}$$

which, in the galactic and cosmological weak-flow regime, reduces to

$$v_r^{\mathrm{FR}}(r,t) \approx \sqrt{\frac{2GM}{r}} - H(t)\, r. \tag{31}$$

In this formulation the rapidity field $\chi_r(r,t)$ is the primary gravitational variable. Observable velocities, Lorentz factors and the associated scalar potentials are derived quantities obtained through different nonlinear projections of the same underlying rotor field. In the Newtonian limit these projections coincide, whereas at relativistic order they reveal complementary aspects of the underlying hyperbolic geometry rather than different gravitational physics.

5 From the Rapidity Rotor to the Painlevé–Gullstrand Metric

Sections (2) established the gravitational rapidity potential Eqn.(3) as the primary gravitational scalar field, and section (3) derived the gravitational four-velocity

$$U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c \cosh \chi_r\, \hat{T} + c \sinh \chi_r\, \hat{I}_r \tag{32}$$

as its exact one-step hyperbolic projection. The present section derives the Painlevé–Gullstrand (PG) metric from the same rotor G_g by a second route, acting on the coordinate line element $d\mathcal{R}$ rather than on the timelike reference vector $c\hat{T}$. This route makes explicit the two information-reducing steps that separate the metric from the primary rapidity field. And as explained in the introduction, in an earlier version of the PG coframe derivation, we used the Dirac level BQ rotor Q_g, a spin-double version of G, acting on dR (de Haas 2025).

5.1 The Rotor as a Local Lorentz Field

The preceding sections identified the gravitational rapidity field $\chi_r(r,t)$ as the primary gravitational variable, Eqn. (3). The corresponding Lorentz rotor field was given in Eqn.(4), where $\hat{\boldsymbol{\sigma}}_r$ generates radial Lorentz boosts. The rotor G_g is therefore a *local Lorentz field*: to every spacetime point (r,t) it assigns a Lorentz transformation with local rapidity $\chi_r(r,t)$. Unlike a global Lorentz transformation, which relates two inertial frames through a single constant rapidity, G_g encodes a continuously varying gravitational frame field over spacetime.

Within the present formulation, the rotor field G_g is the fundamental geometric object. Observable quantities such as the gravitational four-velocity, the physical flow velocity and the Painlevé–Gullstrand metric are obtained as different projections of this same rotor field, rather than being introduced independently.

In the BQ algebra (de Haas 2026a), a Lorentz boost rotor U with rapidity ψ acts on a minquat four-vector A through the double-sided adjoint action

$$A^L = U^{-1}AU^{-1}, \tag{33}$$

as established in de Haas (2018); de Haas (2020). The local rotor field $G_g(r,t)$ acts by exactly the same rule,

$$A_G = G_g^{-1}(r,t)\,A\,G_g^{-1}(r,t), \tag{34}$$

the only difference being that the rapidity is now a spacetime field rather than a constant parameter.

The remainder of this section develops two complementary projections of the same rotor field. In Section 3, Eq. (34) acts on the timelike reference vector $c\hat{T}$ to produce the exact gravitational four-velocity U_g. Here the same adjoint action is applied to the coordinate line element $d\mathcal{R}$, from which the Painlevé–Gullstrand coframe and metric are subsequently derived. Both constructions therefore originate from the same local Lorentz rotor; they differ only in the object upon which the rotor acts and in the subsequent approximations applied.

5.2 The Exact Lorentz Projection of the Line Element

The second exact projection of the local rotor field acts not on the timelike reference vector $c\hat{T}$, but on the coordinate line element itself. Starting from the minquat coordinate differential

$$d\mathcal{R} = c\,dt\,\hat{T} + dr\,\hat{I}_r + r\,d\theta\,\hat{J}_\theta + r\sin\theta\,d\phi\,\hat{K}_\phi, \tag{35}$$

the rotor field $G_g(r,t)$ produces the exact Lorentz-transformed line element through the same double-sided adjoint action introduced in Eq. (34),

$$d\mathcal{R}_G = G_g^{-1}(r,t)\,d\mathcal{R}\,G_g^{-1}(r,t). \tag{36}$$

This transformation is exact and fully Lorentz covariant; no approximation has yet been introduced.

Because the boost generator $\hat{\boldsymbol{\sigma}}_r$ acts only in the $(\hat{T},\hat{K}_r)$ plane, the angular components commute with the rotor and remain unchanged, whereas the temporal and radial components undergo the standard hyperbolic mixing,

$$G_g^{-1}(c\,dt\,\hat{T})G_g^{-1} = c\cosh\chi_r\,dt\,\hat{T} + c\sinh\chi_r\,dt\,\hat{I}_r, \tag{37}$$

$$G_g^{-1}(dr\,\hat{K}_r)G_g^{-1} = \cosh\chi_r\,dr\,\hat{I}_r + \sinh\chi_r\,dr\,\hat{T}. \tag{38}$$

The transformed line element therefore becomes

$$\begin{aligned} d\mathcal{R}_G = \big(c\cosh\chi_r\,dt + \sinh\chi_r\,dr\big)\hat{T} + \big(c\sinh\chi_r\,dt + \cosh\chi_r\,dr\big)\hat{I}_r \\ + r\sin\theta\,d\phi\,\hat{J}_\phi + r\,d\theta\,\hat{K}_\theta. \end{aligned} \tag{39}$$

Expressing the result in terms of the observable Lorentz factor

$$\gamma_r^{\mathrm{FR}} = \cosh\chi_r, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r,$$

with

$$c\sinh\chi_r = \gamma_r^{\mathrm{FR}} v_r^{\mathrm{FR}},$$

gives

$$d\mathcal{R}_G = \gamma_r^{\mathrm{FR}}\left(c\,dt + \frac{v_r^{\mathrm{FR}}}{c}\,dr\right)\hat{T} + \gamma_r^{\mathrm{FR}}\left(v_r^{\mathrm{FR}}\,dt + dr\right)\hat{I}_r + r\sin\theta\,d\phi\,\hat{J}_\phi + r\,d\theta\,\hat{K}_\theta. \tag{40}$$

Both the temporal and radial components therefore retain the full Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$ as an overall prefactor. Consequently, the exact transformed line element still contains the complete rapidity information encoded by the rotor field: no approximation, linearisation or projection onto Newtonian flow has yet taken place. The Painlevé–Gullstrand coframe will emerge only after the slow-flow approximation is introduced in the following subsection.

5.3 The Exact Auto-Quadratic Remains Flat

The transformed line element $d\mathcal{R}_G$ is an exact Lorentz image of the coordinate differential $d\mathcal{R}$. It is therefore natural to ask whether its Lorentz-invariant quadratic already defines a gravitational metric. In the BQ algebra the quadratic form associated with a minquat four-vector

$$A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}$$

is the transposed bilinear

$$A^T A = (a_0^2 - |\mathbf{a}|^2)\,\hat{1}, \tag{41}$$

which is the Lorentz scalar associated with A (de Haas 2026a).

The adjoint action of a Lorentz rotor preserves this quadratic. Indeed, if

$$A^L = U^{-1}AU^{-1},$$

then

$$(A^L)^T = UA^TU,$$

so that

$$(A^L)^T A^L = UA^TAU^{-1} = A^TA, \tag{42}$$

because A^TA lies entirely in the scalar channel $\hat{1}$, which commutes with every Lorentz rotor.

Applying this result to the exact transformed line element $d\mathcal{R}_G$ gives immediately

$$(d\mathcal{R}_G)^T d\mathcal{R}_G = (d\mathcal{R})^T d\mathcal{R} = (c^2dt^2 - dr^2 - r^2d\Omega^2)\,\hat{1}, \tag{43}$$

independently of the value of the rapidity field $\chi_r(r,t)$.

The exact Lorentz rotor therefore disappears completely from the symmetric auto-quadratic. This is not an approximation but a direct consequence of Lorentz invariance: every exact Lorentz image possesses the same scalar quadratic as the original line element.

This observation is the central algebraic result of the present construction. The exact Lorentz-invariant quadratic cannot by itself produce a non-trivial gravitational metric, because all dependence on the rapidity field is removed by the invariant contraction. The gravitational metric must therefore arise through a different route.

In the present formulation this occurs only after two additional operations are performed. First, the exact hyperbolic functions are replaced by their slow-flow approximations,

$$\cosh\chi_r \simeq 1, \qquad \tanh\chi_r \simeq \chi_r,$$

producing the approximate Painlevé–Gullstrand coframe. Second, the metric is obtained as the symmetric quadratic of this approximate coframe. It is these two projection steps—not the exact Lorentz transformation itself—that introduce the Newton–Hubble gravitational structure into the metric description.

5.4 Step 1: The Slow-Flow Approximation and the PG Coframe

The transformed line element (40) is still an exact Lorentz image of the primary rapidity field. To obtain the Painlevé–Gullstrand (PG) coframe, the first approximation is now introduced. Throughout the galactic and cosmological regime,

$$|\chi_r| \ll 1,$$

so that the hyperbolic functions reduce to their leading-order expansions,

$$\cosh\chi_r \approx 1, \qquad \sinh\chi_r \approx \chi_r, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r \approx c\chi_r = \sqrt{\frac{2GM}{r}} - H(t)r \equiv v_r. \tag{44}$$

The Lorentz factor is therefore replaced by unity, while the exact hyperbolic projection is replaced by its linear approximation.

Substituting these approximations into Eq. (40), and neglecting the higher-order term $v_r\,dr/c^2$ in the temporal component, gives

$$d\mathcal{R}_G\Big|_{\chi_r \ll 1} \approx c\,dt\,\hat{T} + (dr + v_r\,dt)\,\hat{I}_r + r\sin\theta\,d\phi\,\hat{J}_\phi + r\,d\theta\,\hat{K}_\theta. \tag{45}$$

The components of this approximate line element are read directly as the PG coframe one-forms,

$$\boxed{\theta^0 = c\,dt, \qquad \theta^r = dr - v_r(r,t)\,dt, \qquad \theta^\phi = r\sin\theta\,d\phi, \qquad \theta^\theta = r\,d\theta,} \tag{46}$$

where

$$v_r = \sqrt{\frac{2GM}{r}} - H(t)r = c\chi_r + O(\chi_r^3)$$

is the linear Newton–Hubble projection of the primary rapidity field.

This is the first approximation in the construction of the metric. The exact Lorentz projection

$$v_r^{\mathrm{FR}} = c\tanh\chi_r$$

is replaced by its linear slow-flow approximation

$$v_r = c\chi_r,$$

and the Lorentz factor

$$\gamma_r^{\mathrm{FR}} = \cosh\chi_r$$

is approximated by unity. Consequently, the coframe no longer carries the full hyperbolic structure of the rotor field, but only its leading Newtonian projection. The exact subluminal saturation $|v_r^{\mathrm{FR}}| < c$ and the divergent Lorentz factor at the relativistic horizons are therefore absent from the approximate coframe.

The Newton–Hubble PG riverbed.

The velocity field

$$v_r = \sqrt{\frac{2GM}{r}} - H(t)r \tag{47}$$

is the combined Newton–Hubble Painlevé–Gullstrand riverbed. It represents the leading-order projection of the underlying rapidity field and coincides with the familiar PG shift in the slow-flow limit.

For $H \to 0$,

$$v_r = \sqrt{\frac{2GM}{r}}, \tag{48}$$

recovering the Schwarzschild river velocity (Hamilton and Lisle 2008), for which space flows inward at the local escape speed.

For $H \neq 0$, the cosmological outflow opposes the Newtonian inflow, producing the combined river (47), which has also appeared in VMOND and related vacuum-flow descriptions of galactic dynamics (Wong and Rivers 2016). The river velocity is positive (net inflow) for $r < r_c$, vanishes at the balance radius

$$r_c = \left(\frac{2GM}{H^2}\right)^{1/3},$$

and becomes negative (net outflow) for $r > r_c$.

It is important to distinguish this approximate river velocity from the exact physical velocity. The quantity appearing in the PG coframe is the linear projection

$$v_r = c\chi_r,$$

whereas the exact observable velocity remains

$$v_r^{\mathrm{FR}} = c \tanh \chi_r.$$

The two coincide throughout the weak-flow regime but differ systematically as $|\chi_r|$ increases. In particular, v_r^{FR} remains bounded by the speed of light, whereas the linear river velocity v_r does not. The PG coframe therefore describes the leading-order Newtonian image of the primary rapidity field rather than the exact hyperbolic flow itself.

5.5 Step 2: Symmetric Contraction and the PG Metric

Having obtained the slow-flow Painlevé–Gullstrand coframe, the metric follows by forming its Minkowski-signature quadratic,

$$ds^2 = \eta_{ab}\,\theta^a\theta^b = -(\theta^0)^2 + (\theta^r)^2 + (\theta^\theta)^2 + (\theta^\phi)^2. \tag{49}$$

or, in BQ notation

$$\left(d\mathcal{R}_G\Big|_{\chi_r\ll 1}\right)^T \left(d\mathcal{R}_G\Big|_{\chi_r\ll 1}\right) = -ds^2\,\hat{1} \tag{50}$$

Unlike the preceding slow-flow approximation, this is not an additional approximation but an exact algebraic construction from the approximate coframe.

Substituting Eq. (46) gives

$$ds^2 = -(c^2 - v_r^2)\,dt^2 - 2v_r\,dr\,dt + dr^2 + r^2 d\Omega^2, \tag{51}$$

which is the Painlevé–Gullstrand line element associated with the Newton–Hubble river velocity

$$v_r = \sqrt{\frac{2GM}{r}} - H(t)r.$$

The corresponding non-vanishing metric components are

$$g_{tt} = -(c^2 - v_r^2), \qquad g_{tr} = g_{rt} = -v_r, \qquad g_{rr} = 1, \tag{52}$$

together with

$$g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta. \tag{53}$$

This completes the second projection step. Whereas the first step replaced the exact hyperbolic structure of the rotor field by its leading-order slow-flow approximation, the present step forms the symmetric scalar quadratic of the resulting coframe. The metric therefore contains only those combinations that survive the symmetric contraction.

In particular,

$$g_{tt} = -(c^2 - v_r^2)$$

depends only on the quadratic combination v_r^2, while

$$g_{tr} = -v_r$$

retains only the linear river velocity. Neither component contains the Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$, nor the primary rapidity field χ_r itself. The exact hyperbolic structure of the rotor field has already been replaced by its slow-flow approximation before the metric is assembled.

From the perspective of the present rotor formulation, the metric is therefore not the primary gravitational object but the symmetric quadratic of an already projected coframe. It provides a faithful description of the weak-flow geometry while representing only the leading-order image of the underlying rapidity field.

6 Interpretation: The Rapidity Ontology

Section 5 established the mathematical construction of the Newton–Hubble Painlevé–Gullstrand geometry from the local Lorentz rotor field. The purpose of the present section is not to repeat that derivation, but to interpret the resulting hierarchy and relate it to the standard Newtonian and relativistic descriptions of gravity.

6.1 The Two-Route Hierarchy

The complete hierarchy established in Section 5 may therefore be summarized as

$$\boxed{\chi_r(r,t) \;\longrightarrow\; G_g(r,t) \;\longrightarrow\; \begin{cases} U_g \;\longrightarrow\; \left(\gamma_r^{\mathrm{FR}},\, v_r^{\mathrm{FR}}\right), \\[2ex] d\mathcal{R}_G \;\xrightarrow{\chi_r \ll 1}\; \theta^a \;\longrightarrow\; g_{\mu\nu}. \end{cases}} \tag{54}$$

The hierarchy separates naturally into two complementary branches. The upper branch remains entirely within the exact Lorentz geometry of the rapidity field. Starting from the primary dynamical field $\chi_r(r,t)$, the local Lorentz rotor $G_g(r,t)$ generates the exact gravitational four-velocity U_g, from which the Lorentz factor γ_r^{FR} and the physical velocity v_r^{FR} follow as exact observable projections. Throughout this branch the full hyperbolic structure of the rapidity field is preserved.

The lower branch describes the construction of the spacetime metric. In the rapidity ontology, it is the flow that forms the metric. Algebraically, the exact Lorentz-transformed line element is first replaced by its slow-flow approximation, producing the Painlevé–Gullstrand coframe. The symmetric Minkowski-signature quadratic of this coframe then yields the metric tensor $g_{\mu\nu}$. The metric therefore constitutes a weak-field projection of the underlying rapidity geometry rather than its primary representation.

The hierarchy establishes the ontological ordering of the gravitational variables. The rapidity field $\chi_r(r,t)$ is the primary dynamical variable, the rotor field $G_g(r,t)$ its geometric realization, the gravitational four-velocity, Lorentz factor and physical velocity are its exact observable projections, and the Newton–Hubble river velocity together with the Painlevé–Gullstrand metric constitute its weak-field geometric projections.

The following subsections compare this hierarchy with the standard Newtonian, Painlevé–Gullstrand and General Relativistic descriptions of gravity.

6.2 Recovering Newtonian Gravity, the Painlevé–Gullstrand River Model and General Relativity

The two-route hierarchy of Section 6.1 places the present rapidity formulation within the broader context of the standard descriptions of gravity. Rather than introducing disconnected formalisms, the constructive scaffold identifies the level at which each description enters and the approximations upon which it is based. It is a constructivist based categorisation: the scaffolds of the constructive path (Framework) are the categories of Table 1..

The primary dynamical variable of the present formulation is the rapidity field $\chi_r(r,t)$, from which the local Lorentz rotor $G_g(r,t)$ is constructed. The exact gravitational four-velocity, Lorentz factor and physical velocity follow directly from this rotor without approximation. The familiar Newtonian, Painlevé–Gullstrand and metric descriptions emerge only after successive projections of the same underlying rapidity geometry. This viewpoint is summarized in Table 1.

Table 1 Constructive scaffolding of the different gravitational descriptions.

Framework	Fundamental object	Observable description
Newtonian gravity	Scalar potential Φ	Gravitational acceleration and particle trajectories
General Relativity	Metric tensor $g_{\mu\nu}$	Spacetime curvature, geodesics and Einstein equations
Painlevé–Gullstrand	River velocity v_r	Flow-adapted spacetime geometry
Present formulation	Rapidity field $\chi_r(r,t)$	Local Lorentz rotor G_g, exact gravitational four-velocity U_g, Lorentz factor γ_r^{FR}, physical velocity v_r^{FR}, and, after the slow-flow projection, the Painlevé–Gullstrand metric

Within this constructive scaffold the classical descriptions appear naturally as successive projections of the primary rapidity field. In the weak-flow regime,$\chi_r \ll 1$, the exact hyperbolic relations reduce to their leading-order approximations,

$$v_r^{\mathrm{FR}} = c \tanh \chi_r \approx c\chi_r \equiv v_r, \tag{55}$$

while the quadratic approximation yields the Newtonian potential,

$$\Phi_{\mathrm{N}} = -\frac{1}{2}c^2\chi_r^2 = -\frac{1}{2}v_r^2, \tag{56}$$

which reduces to

$$\Phi_{\mathrm{N}} = -\frac{GM}{r}$$

when the cosmological contribution is neglected.

The Painlevé–Gullstrand description is obtained from the same slow-flow approximation. The linear river velocity $v_r = c\chi_r$ defines the approximate coframe, whose Minkowski-signature quadratic produces the corresponding spacetime metric. General Relativity then operates on this metric through its standard geometric structures, including the Einstein equations and the geodesic equation.

The present formulation therefore does not replace these established descriptions but places them within a common geometric scaffold. Newtonian gravity is recovered as the quadratic approximation to the rapidity field, the Painlevé–Gullstrand river model as its linear weak-field projection, and the spacetime metric as the symmetric quadratic of the corresponding coframe. The exact Lorentz-rotor description precedes all of these constructions and provides the common geometric origin from which they emerge.

6.3 The Two Asymptotic Limits of the Rapidity Field

The rapidity ontology developed in the preceding sections provides a unified geometric description of the gravitational field across both astrophysical and cosmological scales. Rather than treating black-hole gravity and cosmological expansion as separate phenomena, they appear as two asymptotic limits of the same underlying rapidity field $\chi_r(r,t)$.

For the combined Newton–Hubble flow,

$$\chi_r(r,t) = \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right), \tag{57}$$

the Newtonian contribution dominates at small radii, $\chi_r > 0$, whereas the cosmological contribution dominates at sufficiently large distances, $\chi_r < 0$. Between these two regimes lies the balance radius

$$r_c = \left(\frac{2GM}{H^2}\right)^{1/3}, \tag{58}$$

at which $\chi_r(r_c) = 0$, $v_r(r_c) = 0$. The rapidity field therefore changes continuously from inward (gravitational) to outward (cosmological) flow without requiring any change in the underlying geometric description.

This behaviour is most naturally interpreted in rapidity space rather than in velocity space. The physical velocity, $v_r^{\mathrm{FR}} = c\tanh\chi_r$, is bounded by the speed of light, $|v_r^{\mathrm{FR}}| < c$, whereas the rapidity itself remains unbounded, $-\infty < \chi_r < +\infty$. The hyperbolic tangent therefore compresses the infinite rapidity axis into the finite physical velocity interval $(-c, c)$.

The Schwarzschild and cosmological horizons correspond to opposite asymptotic limits of the same rapidity field. Toward a compact gravitating object,

$$\chi_r \longrightarrow +\infty, \qquad u_r = c\sinh\chi_r \longrightarrow +\infty, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r \longrightarrow c,$$

whereas in the asymptotic cosmological regime,

$$\chi_r \to -\infty, \qquad u_r = c \sinh \chi_r \to -\infty, \qquad v_r^{\mathrm{FR}} = c \tanh \chi_r \to -c.$$

The horizons therefore do not arise from a singularity of the primary rapidity field. The rapidity and the associated proper velocity remain well defined and simply diverge toward opposite infinities, whereas the observable physical velocity is the bounded hyperbolic projection $c \tanh \chi_r$ and asymptotically approaches the speed of light. The two horizons are thus revealed as opposite asymptotic ends of a single continuous rapidity geometry rather than as singular points of the underlying gravitational field.

Within this interpretation the balance radius r_c occupies a distinguished geometric position. It is the unique point at which the Newtonian and cosmological rapidities cancel, $\chi_r = 0$, so that the local Lorentz rotor reduces to the identity, $G_g(r_c) = 1$. The balance radius is therefore not a horizon but the transition between the two asymptotic sectors of the rapidity geometry.

The resulting picture differs from the usual river interpretation. Instead of describing space as flowing inward or outward with potentially superluminal coordinate velocities, the primary object is the unbounded rapidity field, while the observable velocity is its bounded hyperbolic projection. The Schwarzschild horizon and the cosmological horizon thus appear as complementary asymptotic limits of the same Lorentz-rotor geometry rather than as fundamentally different gravitational phenomena.

This unified interpretation follows directly from the hyperbolic geometry of the rapidity field. The Lorentz rotor provides a single continuous description extending from the strong-field Schwarzschild regime through the balance radius to the cosmological expansion, without changing the underlying geometric framework.

6.4 Summary: The Rapidity Ontology

The preceding sections establish a common geometric hierarchy linking the rapidity formulation to the familiar Newtonian, Painlevé–Gullstrand and General Relativistic descriptions of gravity. Rather than introducing independent gravitational variables, these formulations correspond to different levels of description of the same underlying Lorentz geometry.

Within the present formulation the rapidity field $\chi_r(r,t)$ is the primary dynamical variable. The local Lorentz rotor

$$G_g(r,t) = \exp\!\left(\frac{\chi_r(r,t)}{2}\hat{\sigma}_r\right)$$

provides its geometric realization. Exact observable quantities, including the gravitational four-velocity, Lorentz factor and physical velocity, are obtained directly from this rotor without approximation. The Newton–Hubble river velocity, the Painlevé–Gullstrand coframe and the spacetime metric arise only after the slow-flow approximation and the subsequent symmetric quadratic construction.

One particularly transparent consequence of this hierarchy is the structure of the metric component

$$g_{tt} = -\left(c^2 - v_r^2\right). \tag{59}$$

Since the river velocity is the linear projection of the combined rapidity,

$$v_r = c(\psi_N - \psi_H),$$

the metric naturally contains the quadratic combination

$$v_r^2 = c^2(\psi_N - \psi_H)^2 = c^2\left(\psi_N^2 + \psi_H^2 - 2\psi_N\psi_H\right), \tag{60}$$

revealing the Schwarzschild contribution, the cosmological contribution and their interaction term as the three components of a single rapidity product. Within the present formulation this structure is not imposed externally but follows directly from the composition of the underlying rapidity fields.

The resulting ontological hierarchy is summarized in Table 2.

Table 2 Ontological hierarchy of the rapidity formulation.

Level	Fundamental object	Status
Dynamical	Rapidity field $\chi_r(r,t)$	Primary field
Geometric	Local Lorentz rotor $G_g(r,t)$	Exact
Observable	Gravitational four-velocity U_g, Lorentz factor γ_r^{FR}, physical velocity v_r^{FR}	Exact projections
Weak-field	River velocity v_r, Painlevé–Gullstrand coframe θ^a, metric $g_{\mu\nu}$	Slow-flow projections

The rapidity formulation therefore reorganizes the gravitational description into a single hyperbolic hierarchy. Newtonian gravity is recovered as the quadratic approximation to the rapidity field, the Painlevé–Gullstrand river model as its linear weak-field projection, and the spacetime metric as the symmetric quadratic of the corresponding coframe. These familiar descriptions are thus interpreted as complementary projections of a common Lorentz-rotor geometry rather than as independent starting points.

This hierarchy suggests that the rapidity field constitutes the most fundamental description of the gravitational interaction, while the Lorentz rotor provides its local geometric realization and the gravitational four-velocity, Lorentz factor, physical velocity and spacetime metric emerge as successive observable and geometric projections. The present formulation therefore unifies the Newtonian, Painlevé–Gullstrand and metric descriptions within a single hyperbolic geometric framework based on local Lorentz rotors.

7 The Constant-Lagrangian Condition as an Isorapidity Constraint on Hyperbolic Velocity Space

7.1 The Constant-Lagrangian Condition as an Isorapidity Constraint

The companion paper de Haas (2026b) derived the channel decomposition of the relativistic vacuum-flow field equation $\mathbf{H}_g = \partial^T U_g$, for the full family of stationary conic flows. The gravitational four-velocity U_g is generated by the sequential Lorentz rotor $G_g = G_\phi G_r$, as established in Section 3. The resulting field equation decomposes into the continuity, vorticity and sigma channels, the latter governing the radial dynamics.

For galactic flows two mutually perpendicular rapidity fields are present: the radial rapidity χ_r, generated by the combined Newton–Hubble flow, and the azimuthal rapidity χ_ϕ associated with the orbital motion. Their exact relativistic composition is

$$\Gamma \equiv \cosh \psi_\Gamma = \cosh \chi_r \cosh \chi_\phi = \gamma_r^{\mathrm{FR}} \gamma_\phi^{\mathrm{FR}}, \tag{61}$$

where ψ_Γ denotes the composed rapidity (de Haas 2014). Throughout the channel decomposition the Lorentz factor Γ multiplies the four-velocity components and therefore acts as the natural scalar measure of the combined gravitational and orbital boost.

The stationary vacuum-flow solutions satisfy the relativistic constant-Lagrangian (CL) condition

$$\boxed{\frac{D\Gamma}{Dt} = 0,} \tag{62}$$

which, together with stationarity $\partial_t \Gamma = 0$, implies $\partial_r \Gamma = 0$. As shown in de Haas (2026b), this single condition removes simultaneously the Newtonian attraction, the Hubble–Newton coupling, the centrifugal contribution and the cone-geometry amplification from the sigma channel, leaving the universal residual

$$\Pi_{g,r} = \Gamma_0 \, \dot{H}(t) \, r. \tag{63}$$

In the companion paper this condition was interpreted as conservation of the composed Lorentz factor along streamlines. In the present rapidity formulation its geometric content becomes considerably more transparent.

Substituting

$$\Gamma = \cosh \chi_r \cosh \chi_\phi$$

into Eq. (62) gives

$$\tanh\chi_r \frac{D\chi_r}{Dt} + \tanh\chi_\phi \frac{D\chi_\phi}{Dt} = 0, \tag{64}$$

or, using $v^{\mathrm{FR}} = c\tanh\chi$,

$$\frac{v_r^{\mathrm{FR}}}{c}\frac{D\chi_r}{Dt} + \frac{v_\phi^{\mathrm{FR}}}{c}\frac{D\chi_\phi}{Dt} = 0. \tag{65}$$

Equation (65) shows that the CL condition is not the conservation of either rapidity separately but of their velocity-weighted combination. Radial and azimuthal rapidity may continuously exchange along a streamline while preserving the composed Lorentz factor. In the slow-flow limit $\chi \ll 1$ this reduces immediately to

$$\frac{D}{Dt}\left(\frac{v_r^2 + v_\phi^2}{2}\right) = 0,$$

recovering the non-relativistic kinetic-energy form obtained in de Haas (2026b). The exact rapidity form therefore extends the classical constant-Lagrangian condition to arbitrary Lorentz boosts without introducing any additional postulate.

The rapidity formulation also admits a direct geometric interpretation. Defining the composed rapidity by

$$\cosh\psi_\Gamma = \cosh\chi_r \cosh\chi_\phi, \tag{66}$$

the CL condition becomes simply

$$\frac{D\psi_\Gamma}{Dt} = 0, \tag{67}$$

because $\cosh x$ is monotonic for $x \geq 0$.

Consequently, every stationary CL streamline satisfies

$$\cosh\chi_r \cosh\chi_\phi = \Gamma_0, \tag{68}$$

where Γ_0 is constant along that streamline. In the two-dimensional rapidity plane (χ_r, χ_ϕ) this equation defines a curve of constant composed rapidity. On the Poincaré disk, where rapidity equals hyperbolic distance from the origin, these curves are naturally interpreted as *isorapidity curves*: curves of constant hyperbolic boost magnitude.

Define the *gravitational rapidity potential* χ_g by

$$\cosh\chi_g \equiv \cosh\chi_r \cosh\chi_\phi = \Gamma. \tag{69}$$

Since $\cosh x$ is monotonic for $x \geq 0$, the CL condition (62) is exactly equivalent to

$$\boxed{\frac{D\chi_g}{Dt} = 0.} \tag{70}$$

The constant-Lagrangian condition therefore admits the following geometric interpretation:

Every stationary CL streamline is characterised by a constant gravitational rapidity potential χ_g. Equivalently, every streamline remains on a single isorapidity curve

$$\cosh \chi_r \, \cosh \chi_\phi = \cosh \chi_g = \Gamma_0,$$

so that the total hyperbolic boost generated by the combined radial and azimuthal motions is conserved along the entire streamline.

This interpretation shifts the CL condition from the conservation of a kinetic quantity to the conservation of a geometric quantity on the Lorentz group. Rather than conserving the total kinetic energy of the flow, the CL condition conserves the gravitational rapidity potential χ_g, which measures the hyperbolic distance of the composed boost from the rest state. The observed flat rotation curve then appears as a particular projection of this underlying isorapidity geometry, as shown in the following subsection.

An immediate consequence of the conservation law $D\chi_g/Dt = 0$ is that the composed Lorentz factor $\Gamma = \cosh \chi_g$ is likewise constant along every stationary CL streamline. Since Γ determines the local Lorentz time-dilation factor, all clocks comoving with the same streamline remain mutually syntonized despite the continuous exchange between the radial and azimuthal rapidities. This suggests that the CL condition may admit an equivalent interpretation in terms of the preservation of a common proper-time rate along the galactic flow. A detailed analysis of this metrological interpretation lies beyond the scope of the present work.

7.2 The Flat Rotation Curve as a Projection of the Isorapidity Geometry

The conserved gravitational rapidity introduced in the previous subsection immediately acquires a direct observational meaning. At the velocity-balance radius r_c, where the Newtonian inflow and Hubble outflow exactly cancel, the radial rapidity vanishes, $\chi_r(r_c) = 0$. The isorapidity condition

$$\cosh \chi_r \cosh \chi_\phi = \cosh \chi_g = \Gamma_0$$

therefore reduces to

$$\cosh \chi_\phi(r_c) = \cosh \chi_{g0} = \Gamma_0. \tag{71}$$

At the balance radius the complete gravitational rapidity is therefore carried by the azimuthal motion alone. Every stationary constant-Lagrangian galaxy is characterised by its own invariant gravitational rapidity χ_{g0} (or equivalently $\Gamma_0 = \cosh \chi_{g0}$), and the balance radius is the unique point where this invariant becomes directly observable through the orbital motion.

Since $v_\phi^{\mathrm{FR}} = c \tanh \chi_\phi$, the asymptotic flat rotation speed follows immediately:

$$v_\infty \approx v_\phi(r_c) = c \tanh \chi_{g0} = c\sqrt{1 - \Gamma_0^{-2}}. \tag{72}$$

The observed flat rotation speed therefore measures the invariant gravitational rapidity of the stationary galactic state. Rather than being the result of a delicate balance between Newtonian attraction, cosmological expansion and centrifugal motion, the flat rotation curve is the physical velocity projection of the conserved hyperbolic quantity χ_{g0}.

A second observable consequence concerns the universal sigma-channel residual derived in de Haas (2026b). Once the constant-Lagrangian condition has enforced $\partial_r \Gamma = 0$, the sigma channel reduces to

$$\Pi_{g,r} = \Gamma_0 \dot{H} r = \cosh \chi_{g0} \dot{H} r. \tag{73}$$

This expression cleanly separates the two physical ingredients. The constant gravitational rapidity χ_{g0} provides the Lorentz weight of the stationary galactic configuration, whereas the factor $\dot{H}r$ represents the local acceleration of the cosmological background. The geometry of the stationary flow and the evolution of the background universe therefore enter multiplicatively: the former determines the invariant hyperbolic state of the galaxy, while the latter determines whether a non-zero sigma-channel residual is present.

Two limiting cases follow immediately. In the de Sitter limit, $\dot{H} = 0$, the sigma-channel residual vanishes identically, $\Pi_{g,r} = 0$, independently of the value of χ_{g0}. Every stationary constant-Lagrangian galaxy is then in exact relativistic self-freefall, regardless of its mass or morphology. Conversely, whenever $\dot{H} \neq 0$, the residual is determined entirely by the cosmological evolution; the galaxy contributes only through its invariant gravitational rapidity.

The flat rotation curve and the sigma-channel residual therefore represent two complementary observational projections of the same underlying invariant. The flat rotation speed measures the conserved gravitational rapidity χ_{g0} through its bounded hyperbolic projection onto physical velocity, while the sigma channel measures how that same invariant couples to the evolving FRW background through $\dot{H}$. Together they provide the kinematic and dynamical manifestations of a single conserved hyperbolic structure.

7.3 The CL Isorapidity Circle as a Galactic Geodetic Precession

The conserved isorapidity geometry developed in the previous subsections admits a further interpretation. The same hyperbolic circle that characterises every stationary CL galaxy is mathematically identical to the rapidity circle underlying the geodetic precession of Gravity Probe B derived in de Haas (2014). Both originate from the composition of two perpendicular Lorentz boosts and therefore obey the same hyperbolic area theorem.

Recap: the GP-B case

In de Haas (2014) the geodetic precession measured by *Gravity Probe B* is derived as a global Thomas–Wigner rotation generated by two perpendicular rapidity boosts (Thomas 1926; Wigner 1939; de Haas 2014). A radial escape rapidity $\psi_{\rm esc}$, acquired during free fall from rest at infinity to orbital radius R, is composed with the azimuthal orbital rapidity $\psi_{\rm orb}$. Their exact relativistic composition is

$$\cosh\psi = \cosh\psi_{\rm esc}\cosh\psi_{\rm orb} = \gamma_{\rm esc}\gamma_{\rm orb}, \tag{74}$$

where ψ is the composed rapidity.

On the Poincaré representation of hyperbolic velocity space the composed rapidity defines a circle of radius ψ. During one orbital revolution the gyroscope traverses this circle once, and the enclosed hyperbolic area is

$$A = (\cosh\psi - 1)\,2\pi. \tag{75}$$

The corresponding Thomas–Wigner rotation equals precisely this hyperbolic area, yielding the geodetic precession rate (de Haas 2014)

$$\Omega_G = (\cosh\psi - 1)\,\Omega = (\gamma_{\rm esc}\gamma_{\rm orb} - 1)\,\Omega \approx \frac{3}{2}\frac{GM}{Rc^2}\Omega. \tag{76}$$

The essential geometric result is therefore that the geodetic precession is completely determined by the hyperbolic area enclosed by the rapidity circle generated through the composition of the two orthogonal boosts.

The CL galaxy

The stationary constant-Lagrangian galaxy possesses exactly the same hyperbolic structure as the GP-B construction. Every stationary streamline is characterised by a conserved gravitational rapidity $\chi_g = \chi_{g0}$ and $\Gamma_0 = \cosh\chi_{g0}$, which defines a fixed isorapidity circle on the Poincaré disk. Unlike the satellite case, this circle is not completed by periodic orbital motion. Instead, it is maintained kinematically by the constant-Lagrangian condition $\frac{D\chi_g}{Dt} = 0$, while the radial and azimuthal rapidities continuously exchange so as to preserve the composed rapidity. The observed flat rotation speed fixes the invariant radius of this circle through $v_\infty = c\tanh\chi_{g0}$, as established in the previous subsection.

The galactic geodetic precession

The geometric consequence is immediate. Every vector undergoing parallel transport along a stationary CL streamline — for example a disk normal, a jet axis, or the spin axis of a compact object embedded in the flow — accumulates a Thomas–Wigner rotation determined by the hyperbolic area enclosed by the corresponding isorapidity circle.

The enclosed area is

$$A_{\mathrm{gal}} = (\cosh\psi_\Gamma - 1)\,2\pi = (\Gamma_0 - 1)\,2\pi, \tag{77}$$

which is mathematically identical to the area expression appearing in the GP-B derivation. Referring the accumulated rotation to the pattern frequency Ω_{pat} characterising the galactic flow gives the corresponding geodetic precession rate

$$\Omega_G^{\mathrm{gal}} = (\Gamma_0 - 1)\,\Omega_{\mathrm{pat}} \approx \frac{v_\infty^2}{2c^2}\Omega_{\mathrm{pat}}, \tag{78}$$

where the last expression is the weak-flow approximation, $\Gamma_0 - 1 \simeq v_\infty^2/2c^2$.

For typical spiral galaxies $v_\infty \sim 10^{-3}c$, so that

$$\Gamma_0 - 1 \sim 10^{-6}.$$

The resulting precession accumulated during a single galactic rotation is therefore extremely small. Nevertheless, its existence is a purely geometric consequence of the conserved isorapidity circle: whenever $\Gamma_0 > 1$, the enclosed hyperbolic area is non-zero, and a corresponding Thomas–Wigner rotation necessarily accompanies parallel transport along the CL flow.

This result should not be viewed as an analogy with the GP-B experiment but as a second realization of the same hyperbolic area theorem. In the satellite case the rapidity circle is completed by the orbital period; in the galactic case it is maintained by the constant-Lagrangian condition. The underlying Lorentz geometry, however, is identical in both situations.

Structural comparison

The structural correspondence between the GP-B gyroscope and the stationary CL galaxy is summarised in Table 3. In both cases the underlying geometry is generated by the composition of two perpendicular Lorentz boosts. The resulting composed rapidity defines a circle on the Poincaré disk, whose enclosed hyperbolic area determines the associated Thomas–Wigner rotation. The physical systems differ only in the mechanism that closes this rapidity circle.

The structural correspondence summarised in Table 3 shows that the GP-B gyroscope and the stationary constant-Lagrangian galaxy are two realisations of

Table 3 Structural correspondence between the GP-B geodetic precession (de Haas 2014) and the galactic constant-Lagrangian isorapidity geometry. Both arise from the same perpendicular-boost composition on the Poincaré rapidity disk; they differ only in the physical mechanism that maintains the rapidity circle.

Feature	**GP-B satellite**	**CL galaxy**
Radial rapidity	$\psi_{\rm esc}$	χ_r
Azimuthal rapidity	$\psi_{\rm orb}$	χ_ϕ
Composition rule	$\cosh\chi_g = \gamma_r^{\rm FR}\gamma_\phi^{\rm FR} = \Gamma_0$	
Poincaré-disk locus	Rapidity circle	Isorapidity circle
Hyperbolic area	$(\cosh\psi - 1)\,2\pi$	$(\Gamma_0 - 1)\,2\pi$
Thomas–Wigner rotation	Geodetic precession	Galactic geodetic precession
Precession rate	$(\gamma_{\rm esc}\gamma_{\rm orb} - 1)\Omega$	$(\Gamma_0 - 1)\Omega_{\rm pat}$
Closure mechanism	Periodic orbital motion	Constant-Lagrangian condition $D\chi_g/Dt = 0$
Radius anchor	Orbital radius R	Velocity-balance radius r_c
Radius fixed by	$v_{\rm esc}$, $v_{\rm orb}$	Observed flat rotation speed v_∞

the same Lorentz-group geometry. In both cases two perpendicular rapidity boosts generate a circle on the Poincaré disk, and the associated Thomas–Wigner rotation is determined entirely by the enclosed hyperbolic area (Thomas 1926; Wigner 1939; de Haas 2014).

The distinction lies solely in the mechanism that maintains this circle. For the GP-B satellite the rapidity loop is completed dynamically by one orbital revolution. For the stationary galactic flow the circle is maintained kinematically by the constant-Lagrangian condition, $\frac{D\chi_g}{Dt} = 0$, which conserves the gravitational rapidity along every streamline.

The galactic geodetic precession therefore represents a second manifestation of the same hyperbolic area theorem rather than an analogy with the GP-B experiment. Within the rapidity ontology developed in the present paper, the conserved gravitational rapidity χ_g provides the common geometric origin of the flat rotation curve, the universal sigma-channel residual, and the galactic geodetic precession.

7.4 The Strong-Field Limit: Geodetic Precession of Black-Hole Accretion Disks

The preceding subsection showed that the stationary constant-Lagrangian galaxy and the GP-B gyroscope are governed by the same hyperbolic area theorem on the Poincaré rapidity disk. The present subsection completes this geometric hierarchy by considering its strong-field limit. As the gravitational rapidity increases from the weakly relativistic galactic regime to the vicinity of a Schwarzschild black hole, the same Lorentz-group construction evolves continuously into the exact rapidity geometry of an accretion disk.

In the galactic regime the conserved gravitational rapidity satisfies $\Gamma_0 - 1 \sim 10^{-6}$, so the associated geodetic precession is a small perturbative effect. Near a black hole, by contrast, the radial rapidity becomes of order unity and eventually diverges at the Schwarzschild horizon, while the physical velocity remains bounded by c. The weak-field galactic flow and the strong-field accretion disk therefore represent two asymptotic realisations of the same underlying hyperbolic geometry, allowing the exact geodetic precession to be derived without introducing any new geometric ingredients.

Departure from the constant-Lagrangian flow

The strong-field regime marks a change in the physical interpretation of the rapidity composition, while leaving its Lorentz geometry unchanged. In the stationary constant-Lagrangian galaxy the radial and azimuthal rapidities belong to the same vacuum flow and are coupled by the conservation law $\frac{D\chi_g}{Dt} = 0$. The conserved gravitational rapidity χ_g then characterises a stationary streamline of the vacuum.

Near a Schwarzschild black hole this coupling no longer applies. The radial rapidity is carried by the inward Painlevé–Gullstrand vacuum river, whereas the azimuthal rapidity belongs to matter orbiting in the accretion disk. The two boosts therefore originate from different physical systems rather than from a single stationary flow.

Their composition, however, is exactly the same Lorentz composition of two perpendicular boosts that generated the GP-B geodetic precession (de Haas 2014). The difference is not geometric but physical: the galactic constant-Lagrangian condition is replaced by the exact composition of a relativistic vacuum inflow with a relativistic orbital motion. The strong-field calculation presented below is therefore not a new construction but the exact relativistic realisation of the same hyperbolic boost geometry developed throughout the present paper.

The two rapidities

Outside a Schwarzschild black hole of mass M, with Schwarzschild radius

$$r_s = \frac{2GM}{c^2}, \qquad \alpha = \frac{r_s}{r},$$

the radial boost is generated by the exact Painlevé–Gullstrand vacuum inflow. Its rapidity is

$$v_{\rm esc}^{\rm FR}(r) = c \tanh \chi_r(r), \qquad \chi_r(r) = \tanh^{-1} \sqrt{\alpha} = \tanh^{-1} \sqrt{\frac{r_s}{r}}, \tag{79}$$

with Lorentz factor

$$\gamma_r^{\rm FR} = \cosh \chi_r = \frac{1}{\sqrt{1-\alpha}}. \tag{80}$$

Both χ_r and $\gamma_r^{\rm FR}$ diverge at the Schwarzschild horizon, while the physical inflow velocity remains bounded by $v_{\rm esc}^{\rm FR} \to c$.

The second boost is provided by matter moving on a circular Keplerian orbit. The corresponding orbital rapidity is

$$v_{\rm orb}^{\rm FR}(r) = c \tanh \chi_\phi(r), \qquad \chi_\phi(r) = \tanh^{-1}\sqrt{\frac{\alpha}{2}} = \tanh^{-1}\sqrt{\frac{r_s}{2r}}, \tag{81}$$

with Lorentz factor

$$\gamma_\phi^{\rm FR} = \cosh \chi_\phi = \frac{1}{\sqrt{1-\alpha/2}}. \tag{82}$$

The strong-field problem therefore reduces to the exact composition of these two perpendicular Lorentz boosts. Unlike the stationary constant-Lagrangian galaxy, the radial and azimuthal rapidities are no longer coupled by the conservation of a common gravitational rapidity. Instead, they belong to the vacuum inflow and the orbiting matter, respectively. The Lorentz composition law, however, is unchanged, and it is this exact composition that determines the geodetic precession in the fully relativistic regime.

The exact composed rapidity

The radial vacuum inflow and the azimuthal orbital motion remain mutually perpendicular Lorentz boosts. Their composition therefore follows immediately from the exact Lorentz composition rule established in Section 7.1,

$$\cosh \psi_{\rm acc}(r) = \cosh \chi_r(r)\, \cosh \chi_\phi(r) = \gamma_r^{\rm FR}\,\gamma_\phi^{\rm FR}, \tag{83}$$

which yields

$$\boxed{\cosh \psi_{\rm acc}(r) = \frac{1}{\sqrt{1-\alpha}\,\sqrt{1-\alpha/2}}}. \tag{84}$$

Equation (84) is the exact relativistic composed rapidity governing the Schwarzschild accretion flow. It depends only on the dimensionless radius $\alpha = r_s/r$ and therefore contains no free parameters. As r decreases from the weak-field regime towards the Schwarzschild horizon, the composed rapidity increases continuously from its perturbative value into the fully relativistic domain.

The exact composed rapidity is the sole geometric ingredient required to determine the geodetic precession. As in the GP-B experiment and the stationary constant-Lagrangian galaxy, the Thomas–Wigner rotation is obtained from the hyperbolic area enclosed by the corresponding rapidity circle on the Poincaré disk. The following subsection evaluates this area exactly.

The exact geodetic precession

The exact composed rapidity (84) immediately determines the geodetic precession through the Poincaré-disk area theorem. The accumulated Thomas–Wigner rotation during one complete orbital revolution is

$$\vartheta_G^{\mathrm{acc}}(r) = \big(\cosh\psi_{\mathrm{acc}}(r) - 1\big)\,2\pi = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1\right)2\pi, \tag{85}$$

where $\alpha = r_s/r$.

The corresponding geodetic precession frequency is therefore

$$\Omega_G^{\mathrm{acc}}(r) = \big(\cosh\psi_{\mathrm{acc}}(r) - 1\big)\Omega_{\mathrm{orb}}(r), \qquad \Omega_{\mathrm{orb}}(r) = \sqrt{\frac{GM}{r^3}}, \tag{86}$$

which is valid for every Schwarzschild radius $r > r_s$. No weak-field approximation has been introduced: the result follows solely from the exact Lorentz composition of the radial vacuum inflow and the azimuthal orbital motion together with the hyperbolic area theorem of de Haas (2014).

This completes the geometric hierarchy developed throughout the present paper. The GP-B gyroscope, the stationary constant-Lagrangian galaxy and the Schwarzschild accretion disk are three physical realisations of the same Lorentz-group geometry. In each case, two perpendicular Lorentz boosts generate a rapidity circle on the Poincaré disk whose enclosed hyperbolic area determines the corresponding Thomas–Wigner rotation. The distinction between the three systems lies entirely in the physical origin and magnitude of the composing rapidities, not in the underlying geometry.

Weak-field recovery

The exact expression immediately reproduces the known weak-field result. For

$$\alpha = \frac{r_s}{r} \ll 1,$$

one finds

$$\cosh\psi_{\mathrm{acc}} - 1 \approx \frac{3}{4}\alpha = \frac{3GM}{2rc^2}, \tag{87}$$

so that

$$\Omega_G^{\mathrm{acc}} \approx \frac{3}{2}\frac{GM}{rc^2}\Omega_{\mathrm{orb}}, \tag{88}$$

which is precisely the geodetic-precession formula derived for Gravity Probe B in de Haas (2014). The present result therefore constitutes its exact Schwarzschild extension, valid from the weak-field regime down to the vicinity of the event horizon.

A universal prediction at the ISCO

An immediate consequence of Eq. (86) is that the dimensionless ratio

$$\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = \cosh\psi_{\mathrm{acc}} - 1$$

depends only on the dimensionless Schwarzschild radius

$$\alpha = \frac{r_s}{r},$$

and is therefore independent of the black-hole mass.

Evaluating the exact expression at the Schwarzschild innermost stable circular orbit (ISCO),

$$r = 3r_s, \qquad \alpha = \frac{1}{3},$$

gives

$$\left.\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}}\right|_{\mathrm{ISCO}} = \frac{1}{\sqrt{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)}} - 1 = \frac{1}{\sqrt{\frac{2}{3}\cdot\frac{5}{6}}} - 1 \approx 0.374. \tag{89}$$

This constitutes a parameter-free prediction of the rapidity formulation:

$$\boxed{\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = 0.374 \qquad (r = 3r_s).}$$

Every Schwarzschild black hole, from stellar-mass X-ray binaries to supermassive active galactic nuclei, predicts the same dimensionless geodetic-precession ratio at the ISCO. The physical frequencies scale with the black-hole mass through Ω_{orb}, but the ratio itself is universal.

The result provides a direct strong-field benchmark for the rapidity formulation and forms the basis of the observational predictions discussed in the following subsection.

Physical interpretation and observational implications

The geodetic precession derived above is fundamentally distinct from Lense–Thirring (frame-dragging) precession. The latter originates from the rotation of a Kerr spacetime and vanishes for a Schwarzschild black hole. The present effect, by contrast, already exists in the absence of spin. It arises solely from the exact Lorentz composition of the radial Painlevé–Gullstrand vacuum inflow and the azimuthal orbital motion, whose associated hyperbolic area on the Poincaré disk determines the accumulated Thomas–Wigner rotation.

The precessing object is any vector parallel transported by the orbital flow, including the disk normal, the spin axis of an embedded compact object, the orientation of magnetic-field structures, or the launch direction of a relativistic jet. The effect is therefore a geometric property of the Lorentz-rotor field rather than of any particular material constituent.

The exact Schwarzschild solution yields two immediate observational consequences. First, the dimensionless ratio

$$\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = \cosh\psi_{\mathrm{acc}} - 1$$

depends only on the dimensionless radius r/r_s, making the prediction independent of the black-hole mass. Second, at the ISCO the theory predicts the universal parameter-free value

$$\boxed{\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = 0.374,}$$

which provides a direct observational benchmark for Schwarzschild accretion disks.

Because this mechanism does not rely on black-hole spin, it offers a clear observational distinction from spin-induced frame dragging. In low-spin systems it predicts a residual geodetic precession that may contribute to low-frequency QPOs and to the long-term precession of jets anchored in the inner accretion flow.

Summary

The Schwarzschild accretion disk completes the hierarchy of Lorentz-rotor geometries developed throughout this section. The same composition of two perpendicular Lorentz boosts governs three relativistic regimes:

- the weak-field GP-B gyroscope,
- the stationary constant-Lagrangian galaxy,
- the strong-field Schwarzschild accretion disk.

The geometry is identical in all three cases. What changes is the physical origin of the two rapidities and the regime in which they are realised: perturbative boosts for GP-B, a conserved gravitational rapidity for the stationary galactic flow, and the exact composition of the Painlevé–Gullstrand vacuum inflow with Keplerian orbital motion for the accretion disk.

The resulting Schwarzschild geodetic precession is exact and yields the parameter-free prediction

$$\boxed{\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = 0.374}$$

at the ISCO for every Schwarzschild black hole. The following section returns to the broader implications of this Lorentz-rotor hierarchy for the rapidity formulation of gravity.

7.5 Observational Consequences and Tests

The preceding subsection established an exact expression for the Schwarzschild geodetic precession generated by the Lorentz composition of the radial Painlevé–Gullstrand vacuum inflow and the azimuthal orbital motion. The present subsection considers how this mechanism may be distinguished observationally from spin-induced relativistic precession models. In particular, unlike the Lense–Thirring (frame-dragging) mechanism of the relativistic precession model (Stella and Vietri 1998) or the Bardeen–Petterson warp mechanism, the present effect already exists for a Schwarzschild black hole and therefore survives in the absence of Kerr spin.

Universal Schwarzschild benchmark.

The principal prediction of the present formulation is the exact, parameter-free relation

$$\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = \cosh\psi_{\mathrm{acc}} - 1,$$

which depends only on the dimensionless radius r/r_s. At the Schwarzschild ISCO,

$$r = 3r_s,$$

this reduces to the universal value

$$\boxed{\frac{\Omega_G^{\mathrm{acc}}}{\Omega_{\mathrm{orb}}} = 0.374,}$$

independent of black-hole mass. The physical frequencies scale through the orbital frequency, $\Omega_{\mathrm{orb}} \propto M^{-1}$, whereas the dimensionless ratio is universal.

Low-spin black holes.

A particularly clean observational test is provided by systems whose spin has been independently constrained to be small through reflection spectroscopy or continuum fitting. In such systems the Lense–Thirring contribution is expected to be weak, whereas the present Schwarzschild geodetic precession remains unchanged. The detection of low-frequency quasi-periodic oscillations consistent with Eq. (86) in confirmed low-spin systems would therefore provide evidence for a geometric precession mechanism that cannot be attributed solely to frame dragging.

Radial frequency profile.

The exact radial dependence

$$\Omega_G^{\mathrm{acc}}(r) = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1 \right) \Omega_{\mathrm{orb}}(r), \qquad \alpha = \frac{r_s}{r},$$

differs from both the Lense–Thirring nodal-precession profile and the standard periastron-precession frequency. During X-ray binary state transitions, where the inner edge of the accretion disk migrates towards the ISCO, the evolution of the observed QPO frequencies therefore provides a potential discriminator between the present Schwarzschild mechanism and spin-dependent precession models.

Jets, warped disks and Kerr systems.

Because the geodetic precession acts on any vector parallel transported with the orbital flow, it naturally affects disk normals, magnetic-field structures and the launch directions of relativistic jets. As an illustrative example, evaluation of the exact expression at $r = 10\,r_s$ for M87* predicts a characteristic precession period of approximately 26 years, a timescale accessible to continued VLBI monitoring. Likewise, the differential radial dependence of the precession produces disk warping even for Schwarzschild black holes, providing a non-spinning counterpart to the Kerr-driven Bardeen–Petterson mechanism.

For rotating black holes the present geodetic precession does not replace the Lense–Thirring contribution but complements it. Since the two mechanisms arise from different physical origins and possess different radial dependences, their combined signal is, in principle, observable through sufficiently well-sampled radial frequency profiles.

Taken together, these predictions define a coherent observational programme. The universal ISCO frequency ratio, the persistence of the effect in low-spin systems, the characteristic radial frequency profile, and the associated precession of jets and warped disks provide a set of independent, falsifiable tests distinguishing the present Lorentz-rotor mechanism from conventional spin-induced frame-dragging models.

7.6 Synthesis: The Geometric Meaning of the Constant-Lagrangian Condition

Table 4 summarises the constant-Lagrangian (CL) condition at three complementary levels of description. Within the Lorentz-rotor formulation, the primary invariant is the conserved gravitational rapidity, while the familiar velocity and kinetic-energy forms arise as successive slow-flow projections.

The rapidity formulation developed in the present paper identifies the constant-Lagrangian condition as the geometric organising principle of stationary

Table 4 Three equivalent representations of the constant-Lagrangian condition. The rapidity formulation is the exact Lorentz-geometric description, whereas the velocity and kinetic-energy forms emerge as controlled approximations in the weak-flow limit.

Level	Condition	Physical meaning	Validity
Rapidity (exact)	$\dfrac{D\chi_g}{Dt} = 0$	Conserved gravitational rapidity; stationary flow remains on a fixed isorapidity curve	Exact
Velocity	$\tanh\chi_r \dfrac{D\chi_r}{Dt} + \tanh\chi_\phi \dfrac{D\chi_\phi}{Dt} = 0$	Velocity-weighted exchange of radial and azimuthal rapidity	Relativistic
Kinetic energy	$\dfrac{D}{Dt}\left(v_r^2 + v_\phi^2\right) = 0$	Conservation of total flow speed	Weak-flow

gravitational flows. The conserved gravitational rapidity defines the isorapidity geometry underlying the flat rotation curve, the universal sigma-channel residual, and the galactic geodetic precession. In the strong-field regime, the same Lorentz composition law generates the exact Schwarzschild geodetic precession and its parameter-free observational predictions.

Viewed from this perspective, weak-field geodetic precession, stationary galactic dynamics, and Schwarzschild accretion are not independent relativistic phenomena but different manifestations of the same hyperbolic geometry of Lorentz rotors. The following concluding section places this geometric hierarchy within the broader rapidity formulation of gravity developed throughout the paper.

8 Conclusion

The present paper has developed a first-order Lorentz-rotor formulation of gravity in which the gravitational rapidity field constitutes the primary geometric object. Rather than introducing the metric or the gravitational potential as fundamental quantities, the construction begins with the local rapidity field, from which the Lorentz rotor, the gravitational four-velocity, and finally the spacetime metric emerge as successive projections. The familiar descriptions of Newtonian gravity, the Painlevé–Gullstrand river model and General Relativity therefore appear as different levels of description of the same underlying Lorentz geometry.

Within this hierarchy the metric is not discarded but reinterpreted. It represents a derived, second-order description obtained after a controlled weak-flow approximation and a quadratic projection of the local Lorentz rotor. The rapidity field retains information that is necessarily absent from the metric, including the absolute rapidity, its sign, and the bivector structure of the local flow. In this sense, the present formulation complements rather than replaces the metric description by identifying the first-order geometric structure from which it arises.

The second principal result concerns the constant-Lagrangian family of stationary galactic flows. Recasting the CL condition in rapidity language shows that the conserved quantity is the gravitational rapidity, $\frac{D\chi_g}{Dt} = 0$, rather than its weak-flow projections. Stationary galaxies therefore organise themselves on fixed isorapidity curves in hyperbolic rapidity space. Within this geometric picture, the flat rotation curve, the universal sigma-channel residual, and the galactic geodetic precession are revealed as complementary projections of the same conserved hyperbolic structure.

The same Lorentz geometry extends continuously into the strong-field regime. Replacing the stationary vacuum flow by the exact composition of the Painlevé–Gullstrand vacuum inflow with Keplerian orbital motion yields an exact Schwarzschild expression for the geodetic precession of accretion disks. The theory recovers the weak-field Gravity Probe B result in the appropriate limit while predicting the parameter-free Schwarzschild ratio $\frac{\Omega_G^{\rm acc}}{\Omega_{\rm orb}} = 0.374$ at the innermost stable circular orbit. Together with the predicted behaviour in low-spin systems, the characteristic radial frequency profile, and the associated jet and disk precession, these results provide a set of falsifiable observational tests distinguishing the present mechanism from spin-induced frame dragging.

Taken together, these results identify a common geometric framework linking phenomena that are conventionally treated separately: the weak-field geodetic precession measured by Gravity Probe B, the stationary dynamics of spiral galaxies, and the strong-field dynamics of Schwarzschild accretion disks. Their common origin is the hyperbolic composition of local Lorentz boosts encoded in the gravitational rapidity field.

The central conclusion is therefore not simply that rapidity provides an alternative description of gravity, but that it supplies the natural first-order geometric variable from which the familiar potential, four-velocity and metric descriptions emerge as successive projections. Within this Lorentz-rotor hierarchy, gravitation is described fundamentally by the geometry of local boosts, while the metric captures the curvature consequences of that more primitive structure. The rapidity field thus provides a unified geometric framework connecting weak-field gravity, galactic dynamics and strong-field Schwarzschild accretion within a single hyperbolic formulation.

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