Geodetic Precession of Galactic Cores: Five Parameter-Free Predictions from a Rapidity-Flow Description of Gravity

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Abstract

We establish the gravitational rapidity field χ_r as the primary scalar of a vacuum-flow description of gravity. The Painlevé–Gullstrand river velocity and the Einsteinian metric arise as successive approximations via two irreversible information-reducing projections that discard, first, the Lorentz factor $\cosh \chi_r$ and, second, the bivector phase encoding the direction of flow. General Relativity is therefore acceleration-relativistic but not flow-relativistic; the physical velocity $v_r^{\mathrm{FR}} = c \tanh \chi_r$ remains bounded at both horizons without singularity or superluminal disclaimer.

At the galactic scale, the constant-Lagrangian (CL) condition $D\Gamma/Dt = 0$, where $\Gamma = \cosh \chi_r \cosh \chi_\phi$, is re-read in rapidity language: every CL streamline traces a circle of fixed hyperbolic radius ψ_Γ on the Poincaré rapidity disk, the isorapidity surface $\cosh \chi_r \cosh \chi_\phi = \Gamma_0$. This surface simultaneously encodes the flat galactic rotation curve as the projection of the circle onto the azimuthal rapidity axis at the velocity-balance radius r_c, and the exact cancellation of all local gravitational terms in the sigma channel of the field equation, leaving the purely cosmological residual $c\Pi_{g,r} = \dot{H}(t)r$. The Poincaré-disk area $(\Gamma_0 - 1)\, 2\pi$ of the same circle is the galactic geodetic precession angle per orbit, the structural counterpart of the de Sitter–Fokker precession, but suppressed by $v_\infty^2/2c^2 \sim 10^{-6}$ throughout cosmic history.

In galactic cores the same isorapidity circle enters the strong-field regime. The exact geodetic precession angle per orbit of the accretion disk,

$$\vartheta_G^{\mathrm{acc}}(r) = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1 \right) 2\pi, \quad \alpha = r_s/r,$$

is parameter-free beyond M and r/r_s, and yields five falsifiable predictions requiring no black-hole spin: (1) a universal ISCO frequency ratio $\nu_G/\nu_{\mathrm{orb}} = 0.374$, independent of mass; (2) a jet precession period of $\approx 26\,\mathrm{yr}$ for M87* at $r = 10\, r_s$, testable with VLBI; (3) a low-frequency QPO floor in low-spin systems at the predicted geodetic ratio; (4) a distinct radial QPO profile discriminable from nodal and periastron precession; and (5) a spinless disk warp with a radial profile distinct from the Bardeen–Petterson shape.

Keywords: rapidity ontology; gravitational vacuum flow; Painlevé–Gullstrand metric; constant-Lagrangian condition; isorapidity surface; geodetic precession; galactic rotation curves; accretion disk precession; quasi-periodic oscillations; M87 jet

Contents

1 Introduction

This paper derives a set of observational predictions from a single geometric structure: the isorapidity surface of the composed Lorentz factor in hyperbolic two-rapidity space. From this surface, five distinct and parameter-free predictions follow for precession frequencies in galactic cores, without requiring black-hole spin and without free parameters beyond directly observable quantities. The most precise is a universal, mass-independent ratio of geodetic precession frequency to orbital frequency at the innermost stable circular orbit (ISCO) of any Schwarzschild black hole,

$$\left.\frac{\nu_G}{\nu_{\rm orb}}\right|_{\rm ISCO} \approx 0.374, \tag{1}$$

fixed entirely by the geometry of two perpendicular rapidity boosts. Applied to M87*, it predicts a jet precession period of ≈ 26 yr at $r = 10\, r_s$, consistent with the decade-scale position-angle variations seen in long-baseline VLBI monitoring.

These predictions emerge from the same algebraic object — the constant-Lagrangian (CL) isorapidity surface — that also encodes flat galactic rotation curves, the exact cancellation in the sigma-channel of the gravitational field equation, and the cosmological embedding of galactic disks. The unifying structure is not added to the framework; it is the framework.

Background and motivation

The biquaternion (BQ) rotor programme developed in de Haas (2026d,c) and preceded by the geometric algebra programme of Hestenes (1966), describes gravity as a first-order Lorentzian flow field rather than as a metric theory. The primary mathematical object is the local Lorentz rotor, whose adjoint action generates the gravitational four-velocity field through hyperbolic rotations in space–time. The metric appears only as a secondary quadratic object, obtained from the rotor by two successive information-reducing projections. This hierarchy positions General Relativity not as a foundational theory but as the second-order curvature consequence of a first-order rapidity field (de Haas 2025b).

In conventional Special Relativity rapidity is introduced as a kinematic parameter derived from a given velocity: the logical direction is $v \to \psi$ via $\tanh\psi = v/c$. The present paper adopts the opposite viewpoint. The gravitational rapidity is not obtained from a pre-existing velocity field; it is postulated as the primary gravitational scalar field from which the physical velocity is subsequently obtained by hyperbolic projection,

$$\psi \longrightarrow G_g \longrightarrow U_g \longrightarrow v = c\tanh\psi. \tag{2}$$

The rapidity potential therefore plays a role analogous to that of a gauge potential: it is the generator of the local Lorentz rotor rather than a quantity reconstructed from the motion it generates. The physical velocity is bounded by construction, $|v| < c$, while the Lorentz factor $\gamma = \cosh\psi$ and the rapidity field remain unbounded — a distinction that matters at both horizons of the combined Newton–Hubble flow.

The gravitational significance of rapidity composition was first made explicit in de Haas (2014), where the geodetic precession was derived as a global Thomas precession by composing two successive perpendicular boosts — a radial escape rapidity $\psi_{\rm esc}$ and an azimuthal orbital rapidity $\psi_{\rm orb}$ — according to $\cosh\psi = \cosh\psi_{\rm esc}\cosh\psi_{\rm orb}$, recovering the de Sitter–Fokker result $\Omega_G = \frac{3}{2}(GM/Rc^2)\,\Omega$ from Special Relativity and the Equivalence Principle alone. The geodetic precession itself was first derived by de Sitter in 1916–1917 (de Sitter 1916; Fokker 1921) as an effect of curved space–time on the spin axis of a satellite in orbit around a mass M, and confirmed experimentally to 0.28% precision by the Gravity Probe B mission (Everitt et al. 2011). The key geometric ingredient underlying the derivation of de Haas (2014) is that the total Thomas–Wigner rotation angle accumulated by a spin vector over one orbit equals the area of the rapidity circle on the Poincaré disk — a result first stated by Borel (1913) and given its modern graphical form by Rhodes and Semon (2004). The geodetic precession is precisely this Poincaré-disk area, evaluated for the composed rapidity $\psi = \cosh^{-1}(\gamma_{\rm esc}\gamma_{\rm orb})$ of the GP-B satellite orbit. The composed Lorentz factor $\Gamma = \gamma_{\rm esc}\gamma_{\rm orb}$ is the direct precursor of the relativistic weight $\Gamma = \gamma_r\gamma_\phi$ that appears throughout the present paper.

The BQ channel structure was developed into a unified hierarchy for electrodynamics and relativistic fluid dynamics in de Haas (2026b) and was preceded by Hestenes (1966), with the single first-order field equation $\mathbf{H} = \partial^T\mathbf{G}$ replacing the standard second-order formulations. Applying the same hierarchy to gravity as vacuum flow yields the field equation $\mathbf{H}_g = \partial^T U_g$, whose channel decomposition was established for the galactic disk in de Haas (2026d), where the constant-Lagrangian (CL) condition was shown to produce an exact algebraic cancellation in the radial acceleration channel, leaving the purely cosmological residual $c\Pi_{g,r} = \dot{H}(t)\,r$.

The companion paper de Haas (2026c) established a universal sigma-channel residual

$$\Pi_{g,r} = \Gamma\,\dot{H}(t)\,r, \qquad \Gamma = \gamma_r\gamma_\phi, \tag{3}$$

holding for every stationary CL cone $\theta_0 \in (0, \pi/2]$, every azimuthal profile $v_\phi(r, \theta_0)$, every cone angle, and every mass. The empirical programme (de Haas 2025a, 2026a) established that the CL condition fits the SPARC database (Lelli et al. 2016) of 175 late-type galaxy rotation curves with two physically interpretable parameters, and that galactic bar-to-disk transitions encode the Hubble parameter at the epoch of disk formation, connecting local dynamics to cosmic expansion history.

One question left open by that analysis is addressed here. The Painlevé–Gullstrand (PG) river velocity (Painlevé 1921; Gullstrand 1922; Hamilton and

Lisle 2008)

$$v_r = \sqrt{\frac{2GM}{r}} - H(t)r \tag{4}$$

is built from the Newtonian kinetic relation $\frac{1}{2}v^2 = GM/r$, not the special-relativistic relation $(\gamma - 1)c^2 = GM/r$. This combined Newton–Hubble form also appears in vacuum-flow approaches to galactic dynamics (Wong and Rivers 2016); the present paper places it within the rapidity hierarchy as the slow-flow projection of the primary rapidity field χ_r, so that $v_r \approx c\chi_r$ to leading order while the exact physical velocity $v_r^{\mathrm{FR}} = c \tanh \chi_r$ remains bounded by construction. A fully flow-relativistic framework demands an effective potential satisfying the SR relation, and the resulting rapidity field lies outside the Einstein family. This places the composed Lorentz factor Γ in its correct position within the rotor hierarchy BQ-rotor gravity $\supset$ GR (de Haas 2025b).

What this paper does

The central aim of this paper is to establish the *rapidity field* as the primary gravitational scalar, replacing the Newtonian potential and the PG velocity as the foundational object of the vacuum-flow description, and to derive the observational consequences of that choice.

The construction proceeds in three stages. The first (Sections 2–6) develops the rapidity ontology: the combined Newton–Hubble rapidity potential is introduced as a primary scalar field; the gravitational four-velocity is derived as its exact hyperbolic projection; the weak-flow expansion is shown to recover the PG metric, the Newtonian potential, and the GR geodesic equation as successive approximations; and the rapidity ontology is contrasted with the Newtonian, GR, and PG river descriptions.

The second stage (Section 7) applies the rapidity ontology to the constant-Lagrangian galactic flow. The CL condition $D\Gamma/Dt = 0$ is re-read in rapidity language, where it states that every CL streamline traces a circle of fixed hyperbolic radius ψ_Γ on the Poincaré rapidity disk. This isorapidity circle is shown to be structurally identical to the rapidity circle of de Haas (2014): the same perpendicular-boost composition rule, the same Poincaré-disk area theorem, the same geodetic precession formula $\vartheta_G = (\cosh \psi_\Gamma - 1)\, 2\pi$.

The third stage (Sections 7.5–7.7) extends the construction to the strong-field regime of galactic cores. The CL vacuum flow gives way at the SMBH sphere of influence to the direct GP-B scenario: pure radial vacuum inflow composing with pure azimuthal matter orbiting, both rapidities of order unity. The exact, fully relativistic geodetic precession of the accretion disk is derived from the same formula, now without approximation. This yields five distinct, parameter-free observational predictions (Section 7.8):

1. A universal ISCO frequency ratio $\nu_G/\nu_{\rm orb} = 0.374$, independent of black hole mass, distance, inclination, or spin.
2. A jet precession period of ≈ 26 yr for M87* at $r = 10\,r_s$, directly testable with VLBI monitoring.
3. A QPO floor in low-spin systems: low-frequency QPOs at the predicted geodetic ratio even when Lense–Thirring precession (Stella and Vietri 1998) is negligible.
4. A distinct radial QPO frequency profile differentiable from both nodal and periastron precession across a state transition.
5. A disk warp geometry without spin, with a radial profile distinct from the Bardeen–Petterson shape Bardeen and Petterson (1975).

All five predictions are consequences of the single identification that the CL condition is the statement that galactic vacuum flows organise themselves on isorapidity surfaces of the composed Lorentz factor in hyperbolic two-rapidity space.

The remainder of the paper is organised as follows. Section 2 introduces the combined Newton–Hubble rapidity potential. Section 3 derives the gravitational four-velocity as its exact Lorentz projection. Section 4 shows that the Painlevé–Gullstrand metric arises from two successive information-reducing approximations. Section 5 develops the weak-flow expansion and recovers known limits. Section 6 discusses the rapidity ontology and its relation to the Newtonian, GR, and PG descriptions. Section 7 develops the constant-Lagrangian condition in rapidity language, derives its geodetic consequences, and states the five observational predictions.

2 The Combined Newton–Hubble Rapidity Field

In the BQ rotor framework (de Haas 2026d), gravity is encoded through Lorentz boosts generated by rapidity fields rather than by force potentials. For a spherically symmetric mass M embedded in an FRW background with Hubble parameter $H(t)$, two collinear radial rapidities contribute: the Newtonian rapidity $\psi_N(r)$ associated with gravitational infall and the Hubble rapidity $\psi_H(r,t)$ associated with cosmological expansion. The quantities $\psi_N(r)$ and $\psi_H(r,t)$ should not be interpreted as being obtained by transforming existing velocity fields into rapidities. They are introduced as the primary gravitational scalar fields whose low-velocity hyperbolic projections reproduce the Newtonian escape velocity and the Hubble flow, respectively. We define two primary gravitational rapidity potentials:

$$\psi_N(r) \equiv \frac{\sqrt{2GM/r}}{c}, \qquad \textit{and} \qquad \psi_H(r,t) \equiv \frac{H(t)\,r}{c}. \tag{5}$$

These are not the Lorentz rapidities corresponding to existing velocities. They are primary scalar fields whose hyperbolic projections generate the physical velocity field. Because the Newtonian and Hubble gravitational rapidity potentials are collinear, they add linearly and the combined radial gravitational rapidity potentials is

$$\chi_r(r,t) \equiv \psi_N(r) - \psi_H(r,t). \tag{6}$$

so that

$$\boxed{\chi_r(r,t) \equiv \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right).} \tag{7}$$

This is the *gravitational rapidity potential*: a dimensionless scalar field on spacetime whose numerical value is the hyperbolic angle corresponding to the local combined gravitational flow. It is *not* a velocity; it is the gravitational rapidity potentials from which velocities are derived.

The physical flow field is obtained by hyperbolically projecting the rapidity potentials onto velocity space. The Newtonian and Hubble contributions therefore generate the exact physical velocities

$$v_N^{\mathrm{FR}}(r) = c\ \tanh\psi_N(r), \qquad v_H^{\mathrm{FR}}(r,t) = c\ \tanh\psi_H(r,t). \tag{8}$$

In the weak-flow regime, $\psi_N \ll 1$ and $\psi_H \ll 1$, the hyperbolic tangent reduces to its argument, $\tanh x = x + O(x^3)$, so that

$$v_N^{\mathrm{FR}}(r) \approx \sqrt{\frac{2GM}{r}}, \qquad v_H^{\mathrm{FR}}(r,t) \approx H(t)\,r. \tag{9}$$

Thus the postulated rapidity potentials reproduce the familiar Newtonian escape velocity and the local Hubble flow as their leading-order projections. The combined rapidity potential $\chi_r = \psi_N - \psi_H$ projects in exactly the same way,

$$v_r^{\mathrm{FR}}(r,t) = c\ \tanh\chi_r(r,t), \tag{10}$$

which in the galactic and cosmological weak-flow regime becomes

$$v_r^{\mathrm{FR}}(r,t) \approx \sqrt{\frac{2GM}{r}} - H(t)\,r. \tag{11}$$

The velocity-balance radius r_c, defined by $\chi_r(r_c,t) = 0$, is the locus where the two rapidities cancel exactly and the gravitational rotor becomes the identity. From (7):

$$\chi_r(r_c,t) = 0 \iff \sqrt{\frac{2GM}{r_c}} = H(t)\,r_c \iff r_c^3 = \frac{2GM}{H^2(t)}. \tag{12}$$

3 The Gravitational Four-Velocity as Hyperbolic Projection

Given the gravitational rapidity potential χ_r, the gravitational four-velocity is obtained by the standard Lorentz projection. The rotor $G_g = \exp\!\big(\tfrac{1}{2}\chi_r\,\hat{\sigma}_r\big)$ acts on the timelike reference vector $c\hat{T}$ through the double-sided adjoint action, producing

$$U_g(r,t) = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\chi_r\,\hat{T} + c\sinh\chi_r\,\hat{K}_r. \tag{13}$$

The components are identified as

$$u_0 \equiv c\cosh\chi_r, \qquad u_r \equiv c\sinh\chi_r, \tag{14}$$

so the Lorentz factor and the physical radial flow velocity are

$$\gamma_r^{\mathrm{FR}} = \cosh\chi_r, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r. \tag{15}$$

These are the *exact* flow-relativistic expressions. The observable velocity v_r^{FR} is bounded: $|v_r^{\mathrm{FR}}| < c$ for all finite χ_r, since $|\tanh x| < 1$ identically. As $|\chi_r| \to \infty$ the physical velocity approaches c asymptotically without ever reaching it. The Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$, by contrast, is unbounded and grows without limit as $|\chi_r| \to \infty$: it is the metric information content of the rapidity field, not a bounded observable.

The spatial four-velocity component $c\sinh\chi_r$ is also unbounded. This is physically appropriate: in the strong-field regime near the Schwarzschild radius, $\chi_r \to \infty$ and $c\sinh\chi_r \to \infty$, which signals horizon formation without ever predicting a superluminal physical velocity.

The azimuthal rapidity and the composed Lorentz factor.

For galactic flows a second, orthogonal rapidity χ_ϕ is associated with the azimuthal orbital motion. Because the two boosts are perpendicular, the composed Lorentz factor satisfies the rapidity composition rule (de Haas 2014)

$$\Gamma \equiv \cosh\psi_\Gamma = \cosh\chi_r\cosh\chi_\phi = \gamma_r^{\mathrm{FR}}\,\gamma_\phi^{\mathrm{FR}}, \tag{16}$$

which is the exact relativistic generalisation of the product $\gamma_{\mathrm{esc}}\gamma_{\mathrm{orb}}$ that first appeared in the geodetic precession derivation (de Haas 2014) and that weights the universal sigma-channel residual $\Pi_{g,r} = \Gamma\dot{H}r$ of the companion paper (de Haas 2026c).

4 From the Rapidity Rotor to the Painlevé–Gullstrand Metric

Sections (2) established the gravitational rapidity potential

$$\chi_r(r,t) = \psi_N(r) - \psi_H(r,t) = \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right) \tag{17}$$

as the primary gravitational scalar field, and section (3) derived the gravitational four-velocity

$$U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\chi_r\,\hat{T} + c\sinh\chi_r\,\hat{K}_r \tag{18}$$

as its exact one-step hyperbolic projection. The present section derives the Painlevé–Gullstrand (PG) metric from the same rotor G_g by a second route, acting on the coordinate line element $d\mathcal{R}$ rather than on the timelike reference vector $c\hat{T}$. This route makes explicit the two information-reducing steps that separate the metric from the primary rapidity field.

4.1 The Rotor as a Local Lorentz Field

The rotor encoding the combined Newton–Hubble radial boost is

$$G_g(r,t) = \exp\!\left(\tfrac{1}{2}\chi_r(r,t)\,\hat{\sigma}_r\right), \qquad \chi_r(r,t) = \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right). \tag{19}$$

G_g is a *local* Lorentz rotor field: it assigns a Lorentz boost of rapidity $\chi_r(r,t)$ to each spacetime point (r,t). This distinguishes it from a global Lorentz transformation, which has a single constant rapidity between two inertial frames.

In the BQ algebra (de Haas 2026b), a Lorentz boost rotor U with rapidity ψ along $\hat{I}$ acts on a four-vector R by the double-sided adjoint action

$$R^L = U^{-1}R\,U^{-1}, \tag{20}$$

as established in Lemma 2.1 of (de Haas 2026b). The same structure applies to the local rotor field $G_g(r,t)$: for each point (r,t), the action of G_g on any minquat four-vector A is

$$A_G = G_g^{-1}(r,t)\,A\,G_g^{-1}(r,t). \tag{21}$$

Both projections in Section 3 and the present section follow this same adjoint rule, applied to different objects: $c\hat{T}$ for the four-velocity, and $d\mathcal{R}$ for the deformed line element.

4.2 The Deformed Line Element $d\mathcal{R}_G$

The coordinate line element in the BQ minquat basis is

$$d\mathcal{R} = c\,dt\,\hat{T} + dr\,\hat{K}_r + r\,d\theta\,\hat{J}_\theta + r\sin\theta\,d\phi\,\hat{K}_\phi. \tag{22}$$

The rotor G_g acts on $d\mathcal{R}$ by the same adjoint action as in (21):

$$d\mathcal{R}_G = G_g^{-1}(r,t)\,d\mathcal{R}\,G_g^{-1}(r,t). \tag{23}$$

The angular components $r\,d\theta\,\hat{J}_\theta$ and $r\sin\theta\,d\phi\,\hat{K}_\phi$ are orthogonal to the boost direction $\hat{\boldsymbol{\sigma}}_r$ and commute with G_g; they pass through the adjoint unchanged. The boost acts only on the $(\hat{T}, \hat{K}_r)$ plane, giving

$$G_g^{-1}(c\,dt\,\hat{T})G_g^{-1} = c\cosh\chi_r\,dt\,\hat{T} + c\sinh\chi_r\,dt\,\hat{K}_r, \tag{24}$$

$$G_g^{-1}(dr\,\hat{K}_r)G_g^{-1} = \cosh\chi_r\,dr\,\hat{K}_r + \sinh\chi_r\,dr\,\hat{T}. \tag{25}$$

The full exact deformed line element is therefore

$$\begin{aligned} d\mathcal{R}_G = \big(c\cosh\chi_r\,dt + \sinh\chi_r\,dr\big)\hat{T} + \big(c\sinh\chi_r\,dt + \cosh\chi_r\,dr\big)\hat{K}_r \\ + r\,d\theta\,\hat{J}_\theta + r\sin\theta\,d\phi\,\hat{K}_\phi, \end{aligned} \tag{26}$$

which in terms of $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$ and $v_r^{\mathrm{FR}} = c\tanh\chi_r$ (so $c\sinh\chi_r = \gamma_r^{\mathrm{FR}} v_r^{\mathrm{FR}}$) reads

$$d\mathcal{R}_G = \gamma_r^{\mathrm{FR}}\left(c\,dt + \frac{v_r^{\mathrm{FR}}}{c}\,dr\right)\hat{T} + \gamma_r^{\mathrm{FR}}\left(v_r^{\mathrm{FR}}\,dt + dr\right)\hat{K}_r + r\,d\theta\,\hat{J}_\theta + r\sin\theta\,d\phi\,\hat{K}_\phi. \tag{27}$$

Both the temporal and radial components carry the full Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$ as an overall prefactor.

4.3 The Auto-Quadratic of $d\mathcal{R}_G$ is Flat

In the BQ algebra the relevant quadratic form for a minquat four-vector $A = a_0\hat{T} + \mathbf{a}\cdot\hat{K}$ is the transposed bilinear $A^T A$, which yields the Lorentz scalar de Haas (2026b):

$$A^T A = (a_0^2 - |\mathbf{a}|^2)\,\hat{1}. \tag{28}$$

This quadratic is Lorentz invariant: since the Lorentz boost acts as $A^L = U^{-1}AU^{-1}$, one has $(A^L)^T A^L = UA^T U^{-1}$, which equals $A^T A$ only for the scalar channel $\hat{1}$ because $U\hat{1}U^{-1} = \hat{1}$ de Haas (2026b).

Applying this to $d\mathcal{R}_G$, and using the fact that G_g is a Lorentz rotor acting by the adjoint (23):

$$(d\mathcal{R}_G)^T d\mathcal{R}_G = (d\mathcal{R})^T d\mathcal{R} = (c^2\,dt^2 - dr^2 - r^2\,d\Omega^2)\,\hat{1}. \tag{29}$$

The rotor G_g cancels exactly in the symmetric auto-quadratic, leaving only the flat Minkowski line element, for *all* values of χ_r and without any approximation.

This is the central algebraic fact of the present section: there is *no path from the auto-quadratic* $(d\mathcal{R}_G)^T d\mathcal{R}_G$ *to a gravitational metric*, because the rapidity information carried by G_g is annihilated by the Lorentz invariance of the quadratic form. The route to the metric is therefore not through the exact quadratic; it requires the slow-flow approximation to be applied *before* the quadratic is formed, breaking the exact Lorentz invariance in a controlled way.

4.4 Step 1: Slow-Flow Approximation and the PG Coframe

In the galactic and cosmological regime $|\chi_r| \ll 1$. The hyperbolic functions then reduce to

$$\cosh\chi_r \approx 1, \qquad \sinh\chi_r \approx \chi_r, \qquad v_r^{\mathrm{FR}} = c\tanh\chi_r \approx c\,\chi_r = \sqrt{\frac{2GM}{r}} - H(t)r \equiv v_r. \tag{30}$$

Setting $\gamma_r^{\mathrm{FR}} = \cosh\chi_r \approx 1$ in (27), and dropping the sub-leading term $v_r\,dr/c^2$ in the $\hat{T}$ component:

$$d\mathcal{R}_G\big|_{\gamma\approx 1} \approx c\,dt\,\hat{T} + (dr + v_r\,dt)\,\hat{K}_r + r\,d\theta\,\hat{J}_\theta + r\sin\theta\,d\phi\,\hat{K}_\phi. \tag{31}$$

The minquat-basis components of $d\mathcal{R}_G|_{\gamma\approx 1}$ are read off directly as the *PG coframe one-forms*:

$$\boxed{\theta^0 = c\,dt, \quad \theta^r = dr - v_r(r,t)\,dt, \quad \theta^\theta = r\,d\theta, \quad \theta^\phi = r\sin\theta\,d\phi,} \tag{32}$$

where $v_r = \sqrt{2GM/r} - H(t)r = c\tanh\chi_r\big|_{\chi_r\ll 1}$ is the Newton–Hubble inflow velocity, the leading projection of the primary rapidity field χ_r.

This is the **first information-reducing step**. The Lorentz factor $\gamma_r^{\mathrm{FR}} = \cosh\chi_r$, which carries the full boost magnitude, is set to unity. After this step the coframe carries only $v_r \approx c\chi_r$ — the linearisation of the rapidity field — and the distinction between the primary potential χ_r and the derived velocity $v_r^{\mathrm{FR}} = c\tanh\chi_r$ has been collapsed. The subluminal bound $|v_r^{\mathrm{FR}}| < c$ and the divergence of γ_r^{FR} at both horizons are no longer present in the coframe.

The Newton–Hubble PG riverbed.

The velocity $v_r = \sqrt{2GM/r} - H(t)r$ appearing in the coframe (32) is the *combined Newton–Hubble Painlevé–Gullstrand riverbed*: the net radial shift obtained by superposing the gravitational infall and the cosmological outflow. Two limiting cases recover familiar results.

In the pure-gravity limit $H \to 0$, the riverbed reduces to

$$v_r\big|_{H=0} = \sqrt{\frac{2GM}{r}}, \tag{33}$$

which is the classic PG river velocity of the Schwarzschild spacetime (Hamilton and Lisle 2008): space falls inward at the local escape speed, and the horizon is the surface where this infall speed reaches c.

In the full Newton–Hubble case $H \neq 0$, the Hubble outflow $H(t)r$ opposes the Newtonian infall, and the combined riverbed

$$v_r = \sqrt{\frac{2GM}{r}} - H(t)\,r \tag{34}$$

is the PG shift used in VMOND and related vacuum-flow approaches to galactic dynamics (Wong and Rivers 2016). The riverbed is positive (net infall) for $r < r_c$, vanishes at the velocity-balance radius $r_c = (2GM/H^2)^{1/3}$, and is negative (net outflow) for $r > r_c$, giving the two-horizon structure that characterises the combined field. The v_r in the coframe is the slow-flow projection of the primary rapidity field $\chi_r = v_r/c + O(\chi_r^3)$; the exact physical velocity is $v_r^{\mathrm{FR}} = c \tanh \chi_r$, which agrees with v_r to leading order but saturates below c at both horizons while v_r does not.

4.5 Step 2: Coframe Quadratic and the PG Metric

Given the slow-flow-projected coframe (32), the metric is assembled as the Minkowski-signature quadratic

$$ds^2 = \eta_{ab}\,\theta^a\theta^b = -(\theta^0)^2 + (\theta^r)^2 + (\theta^\theta)^2 + (\theta^\phi)^2. \tag{35}$$

Substituting (32):

$$ds^2 = -(c^2 - v_r^2)\,dt^2 - 2v_r\,dr\,dt + dr^2 + r^2\,d\Omega^2. \tag{36}$$

The non-zero metric components are

$$g_{tt} = -(c^2 - v_r^2), \quad g_{tr} = g_{rt} = -v_r, \quad g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta, \tag{37}$$

with $v_r = \sqrt{2GM/r} - H(t)r$.

This is the **second information-reducing step**. The coframe quadratic $\eta_{ab}\theta^a\theta^b$ takes the symmetric scalar part of $\theta^a_\mu\theta^b_\nu$ and discards the antisymmetric bivector part, which carried the orientation and phase of the gravitational flow. After this step only v_r^2 appears in g_{tt} and $|v_r|$ in g_{tr}: the sign of χ_r — distinguishing infall ($\chi_r > 0$) from outflow ($\chi_r < 0$) — is erased.

4.6 The g_{tt} Component and the Horizon Conditions

The temporal metric component follows directly from the coframe quadratic (37):

$$g_{tt}(r,t) = -(c^2 - v_r^2) = -c^2 + \frac{2GM}{r} + H^2(t)\,r^2 - 2H(t)\sqrt{2GM\,r}, \tag{38}$$

with $v_r = \sqrt{2GM/r} - H(t)r$. This expression contains only v_r^2 — the squared Newton–Hubble inflow velocity — and is the complete information about the time–time sector that the metric framework supplies. No $\cosh \chi_r$, no γ_r^{FR}, and no sign information about the direction of flow is available here: those were discarded in Steps 1 and 2.

The horizon condition at metric level.

The metric horizon condition is $g_{tt} = 0$, which gives

$$v_r^2 = c^2, \qquad \text{i.e.} \qquad \left(\sqrt{\frac{2GM}{r}} - H(t)\,r\right)^2 = c^2. \tag{39}$$

This quadratic in $\sqrt{r}$ has two solution families. In the limit $H \to 0$ it reduces to $\sqrt{2GM/r} = c$, giving the Schwarzschild radius $r_s = 2GM/c^2$. In the limit $M \to 0$ it reduces to $Hr = c$, giving the Hubble radius $r_H = c/H$. For the combined Newton–Hubble field, both horizons are present as zeros of g_{tt}, with $g_{tt} < 0$ in between. The metric *cannot* distinguish the infall horizon from the outflow horizon: both satisfy $v_r^2 = c^2$, and both appear as $g_{tt} = 0$. The sign of v_r — positive at the infall horizon, negative at the outflow horizon — is accessible from $g_{tr} = -v_r$, but only if one already knows the direction of flow from outside the metric.

The velocity-balance radius at metric level.

At $r = r_c$, defined by $v_r(r_c) = 0$, the metric gives

$$g_{tt}(r_c, t) = -(c^2 - 0) = -c^2. \tag{40}$$

The temporal metric coefficient takes its most negative value $-c^2$ at r_c, and $g_{tr}(r_c) = 0$: the metric has no off-diagonal time-radial coupling at r_c. Both observations follow from $v_r(r_c) = 0$ alone, without any reference to χ_r.

The McVittie structure.

The form (38) has McVittie-type structure McVittie (1933): it reduces to the Schwarzschild component $g_{tt}^{\text{Schw}} = -(c^2 - 2GM/r)$ when $H \to 0$, and to the FRW component $g_{tt}^{\text{FRW}} = -(c^2 - H^2 r^2)$ when $M \to 0$. The cross-term $-2H\sqrt{2GMr}$ is the interference between the two flows, and it is the only term in g_{tt} that distinguishes the combined metric from a simple superposition of the two limits. This three-term structure is entirely a property of $v_r^2 = (\sqrt{2GM/r} - Hr)^2$, and the metric knows nothing beyond it.

4.7 The Two-Route Hierarchy and Its Interpretation

The complete projection structure descending from the same rotor G_g is:

$$G_g \;\longrightarrow\; \begin{cases} U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\chi_r\,\hat{T} + c\sinh\chi_r\,\hat{K}_r \\ \text{one step, exact, retains full } \chi_r; \\[1ex] d\mathcal{R}_G = G_g^{-1} d\mathcal{R}\, G_g^{-1} \xrightarrow[\cosh\chi_r \to 1]{\text{slow-flow}} \theta^a_\mu \xrightarrow[\text{bivector lost}]{\eta_{ab}\theta^a\theta^b} g_{\mu\nu} \\ \text{two steps, both irreversible.} \end{cases} \tag{41}$$

Why the exact quadratic is flat.

Both U_g and $d\mathcal{R}_G$ are generated by the same double-sided adjoint $G_g^{-1}(\cdots)G_g^{-1}$. The BQ auto-quadratic $A^T A$ is Lorentz invariant under this adjoint (**?**, Eq. 67): the rotor G_g cancels in the symmetric contraction, leaving only the flat Minkowski value. This is not a defect of the formalism; it is the algebraic expression of the Equivalence Principle in the BQ algebra: a locally freely falling frame cannot detect gravity through a scalar invariant alone. The metric can only be extracted after the slow-flow approximation breaks the exact Lorentz invariance by setting $\cosh\chi_r = 1$, at which point the quadratic no longer cancels the rotor content.

What each step discards.

Step 1 (slow-flow, $\cosh\chi_r \approx 1$) discards the Lorentz factor from the coframe. The coframe retains $v_r \approx c\chi_r$ but loses $\cosh\chi_r$: the subluminal saturation $|v_r^{\mathrm{FR}}| < c$ and the divergence of γ_r^{FR} at both horizons are gone. Step 2 (coframe quadratic) discards the bivector phase from the metric: the sign of χ_r — the direction of flow — cannot be recovered from $g_{\mu\nu}$ alone, only from $g_{tr} = -v_r$ together with external knowledge of which horizon is which.

What U_g retains that $g_{\mu\nu}$ cannot supply.

The one-step projection preserves both the magnitude ($u_0 = c\cosh\chi_r$, encoding time dilation) and the sign ($u_r = c\sinh\chi_r$, encoding flow direction) of χ_r. The physical velocity $v_r^{\mathrm{FR}} = u_r/u_0 = c\tanh\chi_r$ is bounded in $(-c, +c)$ for all χ_r; the Lorentz factor $u_0/c = \cosh\chi_r$ is unbounded. *Neither quantity is accessible from* $g_{\mu\nu}$: the metric horizon condition $g_{tt} = 0$ is equivalent to $v_r^2 = c^2$, not to $|\chi_r| \to \infty$ — and the distinction matters, because the former uses the linearised velocity while the latter uses the primary field. Translating $v_r^2 = c^2$ back into $|\chi_r| \to \infty$ requires stepping outside the metric framework and invoking the rapidity field directly.

Why GR operates on a compressed image.

The Einstein equations are differential constraints on $g_{\mu\nu}$. Since $g_{\mu\nu}$ is reached by two irreversible projections from G_g, the Einstein equations act only on the slow-flow, phase-free, sign-free residue of the rapidity field: they constrain curvature built from v_r^2 and its derivatives, but carry no information about $\cosh\chi_r$, $\mathrm{sgn}(\chi_r)$, or the absolute rapidity χ_r itself. This is the precise algebraic statement of why GR is acceleration-relativistic but not flow-relativistic de Haas (2025b): the flow-relativistic content of χ_r was discarded before the metric was formed, and the Einstein equations act only on what remains.

The rapidity-product structure of g_{tt}.

Although the metric knows only v_r^2, the rapidity framework allows one to identify the internal structure of that quantity. Expanding $v_r^2 = c^2\chi_r^2 + O(\chi_r^4)$ and writing

$\chi_r = \psi_N - \psi_H$:

$$v_r^2 = \left(\sqrt{2GM/r} - Hr\right)^2 = \underbrace{\frac{2GM}{r}}_{c^2\psi_N^2} + \underbrace{H^2 r^2}_{c^2\psi_H^2} - \underbrace{2H\sqrt{2GM\,r}}_{2c^2\psi_N\psi_H}. \quad (42)$$

The metric component $g_{tt} = -(c^2 - v_r^2)$ is therefore the quadratic image of the linear rapidity combination $\chi_r = \psi_N - \psi_H$: the Schwarzschild contribution $c^2\psi_N^2 = 2GM/r$, the Hubble contribution $c^2\psi_H^2 = H^2r^2$, and the McVittie cross-term $2c^2\psi_N\psi_H = 2H\sqrt{2GMr}$ — the product of the two primary rapidity potentials — appear automatically from the squaring of a linear rapidity difference. This rapidity-product interpretation is available in the rapidity framework; it is not a statement that the metric itself makes.

5 Weak-Flow Expansion and Recovery of Known Results

The exact expressions (15)–(13) admit a systematic expansion in powers of χ_r, which is small throughout the galactic and cosmological regime.

5.1 Physical velocity

Using $\tanh x = x - x^3/3 + 2x^5/15 + O(x^7)$:

$$v_r^{\mathrm{FR}}(r,t) = u(r,t) - \frac{u^3(r,t)}{3c^2} + O(c^{-4}), \quad (43)$$

where $u(r,t) = \sqrt{2GM/r} - H(t)r$ is the Newtonian–Hubble inflow speed. The first term reproduces exactly the PG velocity used in the companion papers (de Haas 2026d,c); the cubic correction encodes the leading relativistic departure from the Newtonian law. The dominant correction in the strong-field regime is

$$-\frac{u^3}{3c^2} = -\frac{(2GM)^{3/2}}{3c^2r^{3/2}} + \frac{H(t)\sqrt{2GM}\,r}{c^2} - \frac{H^2(t)\sqrt{2GM}\,r^{3/2}}{c^2} + \frac{H^3(t)r^3}{3c^2}, \quad (44)$$

where the leading term $-(2GM)^{3/2}/(3c^2r^{3/2})$ softens the Newtonian divergence of the inflow speed as $r \to 0$, replacing the unbounded Newtonian result with a velocity that saturates toward c asymptotically.

5.2 Lorentz factor

Using $\cosh x = 1 + x^2/2 + x^4/24 + O(x^6)$:

$$\gamma_r^{\mathrm{FR}}(r,t) = 1 + \frac{u^2(r,t)}{2c^2} + O(c^{-4}) = 1 + \frac{GM}{rc^2} + \frac{H^2r^2}{2c^2} - \frac{H\sqrt{2GM\,r}}{c^2} + O(c^{-4}). \quad (45)$$

The leading-order relation $(\gamma_r^{\mathrm{FR}} - 1)c^2 \approx \frac{1}{2}u^2 = -\Phi_{\mathrm{eff}}(r,t)$ identifies

$$\Phi_{\mathrm{eff}}(r,t) = -\tfrac{1}{2}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right)^2 \tag{46}$$

as the effective gravitational potential, which is the Newtonian kinetic-energy equivalent of the rapidity potential. In the pure Schwarzschild limit $H \to 0$:

$$\Phi_{\mathrm{eff}}\big|_{H=0} = -\frac{GM}{r}, \tag{47}$$

recovering the Newtonian gravitational potential exactly.

5.3 The effective potential as the quadratic approximation to the rapidity potential

The chain of approximations is now transparent. The exact primary object is the rapidity potential χ_r. The Lorentz factor is its unbounded hyperbolic cosine; the velocity is its bounded hyperbolic tangent. The effective potential $\Phi_{\mathrm{eff}} \approx -\frac{1}{2}c^2\chi_r^2$ is the *quadratic truncation* of the more fundamental rapidity structure. The Newtonian potential $-GM/r$ is the further special case $H \to 0$ of that quadratic truncation. The hierarchy of approximations is therefore:

$$\chi_r \xrightarrow{\cosh} \gamma_r^{\mathrm{FR}} \xrightarrow{\chi_r^2/2} 1 - \Phi_{\mathrm{eff}}/c^2 \xrightarrow{H\to 0} 1 + GM/(rc^2). \tag{48}$$

Each step discards information: the quadratic truncation loses the odd powers of χ_r (hence the subluminal saturation); the $H \to 0$ limit loses the cosmological embedding. The rapidity potential retains both.

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6 Interpretation: the Rapidity Ontology versus the Standard Ontology

The construction of Sections 2 - 4 represents a deliberate shift in the primary gravitational variable. The present section makes that shift explicit by contrasting it with the standard ontology at three levels: Newtonian gravity, General Relativity, and the PG river model. Where Sections 2 - 4 established the mathematical hierarchy and the explicit metric projection, the purpose here is different: to translate that hierarchy into the familiar Newtonian, PG, and GR descriptions and to show how the same physical situations are represented in each ontology.

6.1 Versus the Newtonian Ontology

In standard Newtonian gravity the primary object is the scalar potential $\Phi(r) = -GM/r$. Forces are derived from it as $\mathbf{F} = -m\nabla\Phi$, and velocities are derived from energy conservation as $v^2 = -2\Phi$. The velocity field is therefore a *derived* quantity, defined only up to a reference choice (escape from infinity vs. escape from a finite radius). The Newtonian potential grows without bound as $r \to 0$, and so does the derived velocity.

In the rapidity ontology the primary object is $\chi_r(r,t)$, a dimensionless hyperbolic angle on the Lorentz group. Physical velocities and potentials are non-linear projections of this single field. No derived quantity is unbounded: the velocity $v_r^{\mathrm{FR}} = c\tanh\chi_r$ saturates at c, and the effective potential $\Phi_{\mathrm{eff}} \approx -\frac{1}{2}c^2\chi_r^2$ saturates through the same mechanism (Section 5, Eq. (46)). The Newtonian potential $-GM/r$ emerges as the $H \to 0$, weak-flow ($\chi_r \ll 1$) limit of a more general expression, and its unboundedness as $r \to 0$ is an artefact of truncating the rapidity potential at quadratic order and then taking $H \to 0$.

6.2 Versus the GR Ontology

In General Relativity the primary object is the metric tensor $g_{\mu\nu}$, obtained from the Einstein field equations. The Painlevé–Gullstrand form of the Schwarzschild metric, (Painlevé 1921; Gullstrand 1922), introduces the river velocity $v_r = \sqrt{2GM/r}$, (Hamilton and Lisle 2008) as a shift vector in the ADM decomposition (Arnowitt et al. 2008), but the metric remains the fundamental object.

A crucial structural point is that GR is *acceleration-relativistic but not flow-relativistic* de Haas (2025b). The Einstein equations determine curvature and hence the accelerations of freely falling particles through the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, \tag{49}$$

but they do not determine the global velocity field $u^\mu(x)$. Reconstructing it requires integrating from a boundary $u^\mu(x_0)$ with a free Lorentz-boost constant. The PG river velocity $\sqrt{2GM/r}$ therefore satisfies the *Newtonian* energy balance $\frac{1}{2}v^2 = GM/r$, not the SR kinetic relation $(\gamma_r - 1)c^2 = GM/r$, and this is a structural feature of GR rather than an approximation.

In the rapidity ontology, by contrast, χ_r is the primary scalar and the metric is a secondary, doubly compressed image of it. As derived explicitly in Section 4, the GR metric arises from the rapidity rotor G_g by two irreversible projection steps (Eq. (41)):

$$\chi_r \xrightarrow{\text{rotor}} U_g = c\cosh\chi_r\,\hat{T} + c\sinh\chi_r\,\hat{K}_r \xrightarrow[\text{coframe quadratic}]{\gamma_r\approx 1} g_{\mu\nu}. \tag{50}$$

Step 1 discards the Lorentz factor $\cosh\chi_r$ (the boost magnitude); Step 2 discards the bivector phase (the flow direction and sign of χ_r). The rapidity potential and the four-velocity U_g are the first-order level of the gravitational hierarchy; the metric is the second-order level. GR describes the curvature consequences of the rapidity field; the BQ vacuum-flow hierarchy describes the field itself. The Einstein equations act only on what remains after both projections — they constrain the curvature of v_r^2 but carry no information about $\cosh\chi_r$, the sign of χ_r, or the absolute rapidity itself (Section 4.7).

6.3 Versus the PG River Model

The Painlevé–Gullstrand river model (Hamilton and Lisle 2008) identifies the gravitational field with a river of space flowing at velocity $v_r = \sqrt{2GM/r}$ toward the central mass, treating v_r as the primary object. The rapidity ontology differs in two ways.

First, the river velocity is identified here as the *leading-order hyperbolic projection* of the primary rapidity field:

$$v_r^{\mathrm{FR}} = c\tanh\chi_r \xrightarrow{\chi_r \ll 1} c\,\chi_r = \sqrt{\frac{2GM}{r}} - H(t)r. \tag{51}$$

The rapidity is the argument of the hyperbolic functions; the velocity is their bounded projection. The river model works because $\chi_r \ll 1$ throughout the galactic regime, making $\tanh\chi_r \approx \chi_r$ extremely accurate. The rapidity description remains exact where the river model breaks down — near the Schwarzschild radius ($\chi_r \to +\infty$) and beyond the Hubble radius ($\chi_r \to -\infty$).

Second, the Hubble rapidity $\psi_H = Hr/c$ is included in the combined field $\chi_r = \psi_N - \psi_H$ (Eq. (7)) on an equal footing with the Newtonian rapidity $\psi_N = \sqrt{2GM/r}/c$. The standard river model considers only Schwarzschild infall (Hamilton and Lisle 2008). The cosmological embedding, the velocity-balance radius r_c, and the sigma-channel residual $\Pi_{g,r} = \Gamma\dot{H}r$ all arise from the Hubble contribution to χ_r and have no counterpart in the standard river picture but this cosmological river model has been derived as such in VMOND (Wong and Rivers 2016).

6.4 The Black Hole Horizon in the Rapidity Ontology

The distinction between the bounded physical velocity $v_r^{\mathrm{FR}} = c\tanh\chi_r$ and the unbounded spatial four-velocity $u_r = c\sinh\chi_r$ has a direct consequence for the interpretation of the black hole horizon.

The horizon condition in each language.

The metric horizon condition, derived in Section 4.6 (Eq. (39)), is

$$g_{tt} = 0 \iff v_r^2 = c^2. \tag{52}$$

This is the complete information the metric provides. The metric alone cannot identify the sign of v_r: both the infall horizon ($v_r > 0$) and the outflow horizon ($v_r < 0$) satisfy $v_r^2 = c^2$ and both appear as $g_{tt} = 0$. The sign is technically accessible from $g_{tr} = -v_r$, but only if one already knows from outside the metric which horizon is which.

In the PG river picture the infall horizon is identified by the additional statement $v_r = +c$ — space falls inward at the speed of light — which requires the familiar addendum that it is *space*, not matter, moving at c. In the rapidity language the same condition is

$$\chi_r(r_s, t) \to +\infty, \qquad r_s = \frac{2GM}{c^2}, \tag{53}$$

because $c \tanh \chi_r \to c$ only as $\chi_r \to +\infty$. This is a smooth asymptotic limit on the hyperbolic parameter space; no quantity assigned a velocity interpretation reaches c. Translating $v_r^2 = c^2$ into $|\chi_r| \to \infty$ requires stepping outside the metric framework and invoking the primary rapidity field directly — the metric itself does not make this statement.

What diverges and what does not.

At the infall horizon $\chi_r \to +\infty$:

$$v_r^{\mathrm{FR}} = c \tanh \chi_r \;\to\; +c \quad \text{(asymptotically, from below)}, \tag{54}$$

$$\gamma_r^{\mathrm{FR}} = \cosh \chi_r \;\to\; +\infty \quad \text{(infinite time dilation)}, \tag{55}$$

$$u_r = c \sinh \chi_r \;\to\; +\infty \quad \text{(proper spatial four-velocity)}. \tag{56}$$

These are mutually consistent through $u_r = \gamma_r^{\mathrm{FR}} v_r^{\mathrm{FR}}$: the physical velocity approaches c while the Lorentz factor diverges. The divergence of u_r is not pathological — it is the correct relativistic statement that a freely infalling observer's proper spatial four-velocity diverges at the horizon. The pathology in the PG picture is using v_r (bounded) in the role that properly belongs to u_r (unbounded).

The black hole interior.

In the PG river model, inside the horizon $v_r > c$ is written and explained as superluminal space. In the rapidity description there is no such object: $|v_r^{\mathrm{FR}}| = |c \tanh \chi_r| < c$ everywhere, including inside the horizon. The interior corresponds to $\chi_r > \chi_r^{\mathrm{horizon}} \to +\infty$, with v_r^{FR} remaining subluminal throughout. The superluminal language is an artefact of using $c\chi_r$ (the linearisation of $c \tanh \chi_r$) past the radius where $\chi_r \ll 1$ has broken down.

What this does and does not claim.

The rapidity description does not change the causal structure of the Schwarzschild spacetime. The horizon remains a one-way boundary; outgoing null geodesics cannot escape; the Penrose diagram is unchanged. What changes is the ontological status of the primary variable. In GR, g_{tt} passes through zero — a coordinate singularity resolved by PG coordinates because PG coordinates use v_r rather than a static potential. The rapidity framework goes one step further: v_r is itself identified as the linearisation of $c \tanh \chi_r$, so the PG resolution and its extension to exact hyperbolic geometry are both natural consequences of a single choice of primary variable. In the rapidity description the primary scalar χ_r simply diverges to $+\infty$ — a smooth asymptotic limit, not a zero crossing. The curvature singularity at $r = 0$ is physical in both descriptions and is unaffected by the change of variable.

6.5 The Cosmological Limit: $r \to \infty$ and $Hr \to \infty$

The black hole limit $r \to r_s$ drives $\chi_r \to +\infty$ through the divergence of ψ_N. The opposite extreme drives χ_r to $-\infty$ through the Hubble rapidity $\psi_H \to +\infty$, since Hr grows without bound while $\sqrt{2GM/r} \to 0$.

The rapidity potential at large r.

For $r \gg r_c$:

$$\chi_r(r,t) = \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right) \xrightarrow{r\to\infty} -\frac{H(t)r}{c} \to -\infty. \tag{57}$$

The three observables at large r are:

$$v_r^{\mathrm{FR}} \to -c \quad \text{(from above, asymptotically)}, \tag{58}$$
$$\gamma_r^{\mathrm{FR}} \to +\infty \quad \text{(diverging time dilation)}, \tag{59}$$
$$u_r \to -\infty \quad \text{(proper outflow four-velocity)}. \tag{60}$$

The sign is negative throughout: the flow is outward.

The Hubble horizon as a partial rapidity divergence.

At the Hubble radius $r_H = c/H$, the Hubble rapidity alone diverges: $\psi_H(r_H) \to +\infty$. The combined rapidity at r_H is

$$\chi_r(r_H,t) = \frac{\sqrt{2GMH/c} - c}{c} = \frac{\sqrt{2GMH/c}}{c} - 1, \tag{61}$$

which diverges to $-\infty$ as $M \to 0$ only if H is held fixed at finite r_H. For finite M, $\chi_r(r_H)$ is a large but finite negative number: the Hubble radius is a surface of *partial* rapidity divergence — only ψ_H reaches its asymptote, while ψ_N remains finite. The true $\chi_r \to -\infty$ limit is reached only as $r \to \infty$.

What PG and GR say at large r.

In the PG picture, $v_H = Hr > c$ for $r > r_H$, treated as superluminal space — the same disclaimer invoked at the black hole interior. In the rapidity description this disclaimer is unnecessary: $|v_r^{\mathrm{FR}}| = |c \tanh \chi_r| < c$ everywhere for all $\chi_r \in (-\infty, +\infty)$. The Hubble radius is simply the surface where the linearisation $\tanh \chi_r \approx \chi_r$ produces the apparent violation $v_H = Hr > c$.

In GR the cosmological analogue of the horizon is the de Sitter event horizon at $r_H = c/H_\Lambda$. The metric coefficient g_{tt} in static de Sitter coordinates passes through zero there, structurally analogous to the Schwarzschild case. As established in Section 4.6, the metric horizon condition is $v_r^2 = c^2$ in both cases. The rapidity description unifies them: both horizons correspond to $|\chi_r| \to \infty$, with $\chi_r \to +\infty$ for the infall case and $\chi_r \to -\infty$ for the outflow case — asymptotic limits of a single continuous field, distinguished by sign.

The full rapidity profile and its three characteristic radii.

The complete picture assigns three structurally important radii to $\chi_r(r, t)$:

1. $r \to r_s$: $\chi_r \to +\infty$, $v_r^{\mathrm{FR}} \to +c$ — infall horizon (Schwarzschild).
2. $r = r_c$: $\chi_r = 0$, $v_r^{\mathrm{FR}} = 0$ — velocity-balance, rotor identity.
3. $r \to \infty$: $\chi_r \to -\infty$, $v_r^{\mathrm{FR}} \to -c$ — cosmological limit (de Sitter/Hubble).

Between r_s and r_c the rapidity is positive and the flow is inward; between r_c and ∞ the rapidity is negative and the flow is outward. The Lorentz factor $\cosh \chi_r$ diverges symmetrically at both ends and reaches its minimum $\cosh(0) = 1$ at r_c. The physical velocity $c \tanh \chi_r$ is the bounded, odd-symmetric projection, passing smoothly from $+c$ at r_s through zero at r_c to $-c$ at $r \to \infty$. The primary field $\chi_r(r, t)$ covers this entire range in a single continuous expression; no change of variables or separate coordinate patch is required at any radius.

6.6 Summary: the Ontological Hierarchy and Its Regimes

Table 1 collects the four ontological levels and their behaviour in both asymptotic limits.

Four features of the table are worth highlighting.

First, only the rapidity level handles both asymptotic limits without a conceptual anomaly. At the GR level both horizons are zeros of g_{tt}, requiring coordinate choices to resolve. At the PG level both limits produce $v > c$, requiring the same superluminal-space disclaimer twice. At the rapidity level both limits are $|\chi_r| \to \infty$: the primary variable diverges asymptotically, the physical velocity approaches $\pm c$ from within $(-c, +c)$, and the proper spatial four-velocity u_r diverges in the physically correct sense. No disclaimer is needed.

Second, both interiors are regular in the rapidity description. At the PG level $v > c$ must be written both inside r_s and beyond r_H. At the rapidity level $|v_r^{\mathrm{FR}}| =$

Table 1 Ontological levels for the primary gravitational scalar, from the most fundamental (top) to the most truncated (bottom). The two limit columns show the conditions each framework assigns to the infall horizon $r \to r_s$ and the cosmological limit $r \to \infty$. Each lower level is a controlled truncation of the one above. The rapidity field χ_r passes smoothly from $+\infty$ at r_s through 0 at r_c to $-\infty$ as $r \to \infty$; the physical velocity $c \tanh \chi_r$ mirrors this as $+c \to 0 \to -c$.

Level	Primary object	Velocity	Status	Infall limit $r \to r_s$	Cosmo limit $r \to \infty$
Rapidity field	$\chi_r(r,t)$	$c \tanh \chi_r$	exact	$\chi_r \to +\infty$; $v_r^{\mathrm{FR}} \to +c$; $u_r \to +\infty$; $\gamma_r \to \infty$	$\chi_r \to -\infty$; $v_r^{\mathrm{FR}} \to -c$; $u_r \to -\infty$; $\gamma_r \to \infty$
GR metric	$g_{\mu\nu}$	shift v_r	slow-flow + coframe quadratic (Sect. 4)	$g_{tt} = 0$ (Schwarzschild); $v_r^2 = c^2$; $u_r \to \infty$ (proper)	$g_{tt} = 0$ (de Sitter); $v_H = c$ at r_H; $u_r \to \infty$ (proper)
PG river	v_r	v_r directly	$\chi_r \ll 1$ (linearised)	$v_r = c$; interior $v_r > c$ (superluminal space)	$v_H = Hr > c$ for $r > r_H$ (superluminal space)
Newtonian	$\Phi = -GM/r$	$\sqrt{-2\Phi}$	$H \to 0$, $\chi_r \ll 1$	$\sqrt{2GM/r} = c$ at r_s; no horizon concept	$v = Hr \to \infty$; no cosmological concept

$|c \tanh \chi_r| < c$ everywhere without exception for all $\chi_r \in (-\infty, +\infty)$. Both interiors correspond to larger $|\chi_r|$, reached continuously as r moves away from r_c in either direction.

Third, the rapidity profile is globally odd-symmetric about r_c. The function $\chi_r(r,t)$ is smooth and monotonically decreasing in r, passing from $+\infty$ at r_s through 0 at r_c to $-\infty$ as $r \to \infty$. The Lorentz factor $\cosh \chi_r$ is even in χ_r and diverges symmetrically; $c \tanh \chi_r$ is odd in χ_r and maps the real line to $(-c, +c)$, with both horizons as its asymptotes. This global structure has no counterpart in any truncated framework.

Fourth, each lower level is a controlled truncation of the rapidity field. The Newtonian potential $\Phi = -GM/r$ is the $H \to 0$, $\chi_r \ll 1$ limit of $-\frac{1}{2}c^2\chi_r^2$, covering neither horizon. The PG velocity $v_r = \sqrt{2GM/r} - Hr$ is the $\chi_r \ll 1$ linearisation of $c \tanh \chi_r$, which breaks down at both r_s and r_H by predicting $|v_r| > c$. The GR metric $g_{\mu\nu}$ is the slow-flow, coframe-quadratic projection of U_g (Section 4), recovering the correct causal structure at both horizons but at the cost of requiring separate coordinate patches. The rapidity field χ_r covers $r_s < r < \infty$ in a single continuous expression, with both horizons encoded as asymptotic limits and the galactic flow occupying the smooth interior centred on r_c.

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7 The Constant-Lagrangian Condition as an Isorapidity Constraint on Hyperbolic Velocity Space

7.1 Starting point: the BQ channel decomposition and the CL condition

The companion paper de Haas (2026c) derived the channel decomposition of the relativistic vacuum-flow field equation $\mathbf{H}_g = \partial^T U_g$ on the full family of stationary conic flows $\theta_0 \in (0, \pi/2]$. The gravitational four-velocity U_g is generated from the sequential rapidity rotor $G_g = G_\phi G_r$ exactly as established in Section 3 of the present paper. The field equation decomposes automatically into three channels:

- a continuity channel $\sigma_{E,g}$ encoding vacuum flux and recovering $\theta = 3H$ and the Poisson equation;
- a vorticity channel $\mathbf{\Omega}_g$ encoding the rotational structure and the disk morphology;
- a sigma (acceleration) channel $\mathbf{\Pi}_g$ encoding the radial dynamics.

The composed Lorentz factor

$$\Gamma = \gamma_r^{\mathrm{FR}}\, \gamma_\phi^{\mathrm{FR}} = \cosh \chi_r\, \cosh \chi_\phi \tag{62}$$

weights the four-velocity components throughout the decomposition, where χ_r is the radial rapidity potential of Section 2 and χ_ϕ is the azimuthal orbital rapidity. Imposing the *relativistic constant-Lagrangian (CL) condition*

$$\frac{D\Gamma}{Dt} = 0 \tag{63}$$

together with stationarity $\partial_t \Gamma = 0$ forces $\partial_r \Gamma = 0$ on every cone, which simultaneously eliminates the Newtonian inverse-square attraction, the Hubble–Newton cross-coupling, the orbital centrifugal term, and the cone-geometry $\cot\theta_0$ amplification from the sigma channel in a single algebraic step. The result is the universal sigma-channel residual

$$\Pi_{g,r} = \Gamma\, \dot{H}(t)\, r, \tag{64}$$

holding for every cone angle θ_0, every azimuthal profile $v_\phi(r, \theta_0)$, every mass M, and every Hubble rate $H(t)$ (de Haas 2026c).

In de Haas (2026c) the CL condition was read as a kinematic conservation law: the composed Lorentz factor is conserved along streamlines. In the velocity-language of that paper this reduces in the slow-flow limit to $u_r^2 + u_\phi^2 = v_L^2 = \mathrm{const}$ — total flow speed conserved along each streamline, with radial and azimuthal kinetic energy trading continuously. The purpose of the present section is to re-read the same condition in the rapidity ontology of Sections 2 and 4, where $\Gamma = \cosh\chi_r \cosh\chi_\phi$ and the primary objects are the rapidity fields χ_r, χ_ϕ rather than the velocities v_r^{FR}, v_ϕ^{FR}. This re-reading reveals geometric content that is invisible in the velocity language.

7.2 The CL Condition in Rapidity Language

Substituting $\Gamma = \cosh\chi_r \cosh\chi_\phi$ into the CL condition (63) and applying the material derivative:

$$\frac{D}{Dt}(\cosh\chi_r \cosh\chi_\phi) = \sinh\chi_r \cosh\chi_\phi \frac{D\chi_r}{Dt} + \cosh\chi_r \sinh\chi_\phi \frac{D\chi_\phi}{Dt} = 0. \quad (65)$$

Dividing through by $\cosh\chi_r \cosh\chi_\phi = \Gamma > 0$:

$$\tanh\chi_r \frac{D\chi_r}{Dt} + \tanh\chi_\phi \frac{D\chi_\phi}{Dt} = 0, \quad (66)$$

or, using $v_r^{\mathrm{FR}} = c\tanh\chi_r$ and $v_\phi^{\mathrm{FR}} = c\tanh\chi_\phi$:

$$\frac{v_r^{\mathrm{FR}}}{c}\frac{D\chi_r}{Dt} + \frac{v_\phi^{\mathrm{FR}}}{c}\frac{D\chi_\phi}{Dt} = 0. \quad (67)$$

This is the *rapidity-weighted conservation law* along streamlines: the material rate of change of the radial rapidity potential, weighted by the radial velocity fraction v_r/c, plus the material rate of change of the azimuthal rapidity, weighted by v_ϕ/c, sum to zero along every streamline.

Three observations follow directly from this form.

First, the CL condition does not require either rapidity to be individually conserved. It requires only their velocity-weighted combination to vanish. Radial and azimuthal rapidity can exchange along a streamline, provided the exchange is balanced by the ratio of the corresponding physical velocities. In the slow-flow limit $v/c \ll 1$ and $\tanh\chi \approx \chi$, the condition reduces to $v_r\, Dv_r/Dt + v_\phi\, Dv_\phi/Dt = D(v_r^2 + v_\phi^2)/2Dt = 0$, recovering the non-relativistic CL condition $D(u_r^2 + u_\phi^2)/Dt = 0$ of de Haas (2026c) exactly.

Second, in the exact relativistic form (66) the weights are not velocities but hyperbolic tangents of rapidities. These weights are bounded in $(-1, +1)$ and saturate at the horizons. The condition therefore remains well-defined and non-degenerate even in the strong-field regime where $v_r \to c$, unlike the non-relativistic form $D(v_r^2 + v_\phi^2)/Dt = 0$ which would give $D(c^2)/Dt = 0$ at the horizon — a vacuous statement. The rapidity form (66) is the natural relativistic extension.

Third, the norm channel of $\mathbf{H}_g = \partial^T U_g$ contains the term $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$ in the slow-flow approximation. Setting this to zero is the standard slow-flow CL condition. In the rapidity ontology the corresponding exact term in the norm channel is proportional to $D\Gamma/Dt$, and setting it exactly to zero — rather than discarding it as a slow-flow approximation — is what selects the CL solution family from within the channel structure without any external postulate (de Haas 2026c).

7.3 Geometric Interpretation: Isorapidity Surfaces in Hyperbolic Velocity Space

The condition $D\Gamma/Dt = 0$, i.e. $D(\cosh\chi_r \cosh\chi_\phi)/Dt = 0$, has a direct geometric interpretation in the two-dimensional rapidity space (χ_r, χ_ϕ).

Define the composed rapidity ψ_Γ by

$$\cosh\psi_\Gamma = \cosh\chi_r \cosh\chi_\phi = \Gamma, \tag{68}$$

which is the standard relativistic rapidity composition for two perpendicular boosts (de Haas 2014). The CL condition $D\Gamma/Dt = 0$ then reads:

$$\frac{D\psi_\Gamma}{Dt} = 0 \qquad \Longleftrightarrow \qquad \frac{D}{Dt}(\cosh\psi_\Gamma) = 0, \tag{69}$$

since cosh is monotone for $\psi_\Gamma \geq 0$. The composed rapidity magnitude ψ_Γ is conserved along every streamline.

In the two-dimensional rapidity space (χ_r, χ_ϕ), the locus $\cosh\chi_r \cosh\chi_\phi = \Gamma_0 = \text{const}$ is a curve of constant composed rapidity. These curves are the *isorapidity surfaces* of the combined flow. On the Poincaré disk representation of hyperbolic velocity space, where the rapidity is the hyperbolic distance from the origin and the physical velocity is its bounded projection onto the disk interior, the isorapidity surfaces are curves of constant hyperbolic "norm" in the two-rapidity space.

The CL condition therefore states:

> *Every stationary CL streamline traces a path on an isorapidity surface* $\cosh\chi_r \cosh\chi_\phi = \Gamma_0$ *in hyperbolic velocity space. The composed rapidity magnitude* ψ_Γ *— the total hyperbolic boost distance from the rest state — is conserved along each streamline.*

This is not a statement about conservation of kinetic energy in the Newtonian sense. It is a statement about conservation of a geometric object on the Lorentz group: the composed rapidity ψ_Γ is the hyperbolic length of the combined radial-azimuthal boost vector, and the CL condition states that this hyperbolic length is frozen along each streamline as the fluid element moves through the combined Newton–Hubble field.

7.4 The Flat Rotation Curve as a Hyperbolic Circle

The geometric interpretation of the CL condition has a direct consequence for the flat rotation curve.

At the velocity-balance radius r_c, the radial rapidity vanishes: $\chi_r(r_c) = 0$ (Section 2, Eq. (12)). The CL isorapidity condition evaluated at r_c gives:

$$\cosh(0)\,\cosh\chi_\phi(r_c) = \Gamma_0 \implies \cosh\chi_\phi(r_c) = \Gamma_0, \tag{70}$$

so the azimuthal rapidity at r_c fixes the total hyperbolic radius Γ_0 of the isorapidity surface. Since $v_\phi^{\rm FR}(r_c) = c\tanh\chi_\phi(r_c)$ and $\Gamma_0 = \cosh\chi_\phi(r_c)$, the flat rotation speed v_∞ is determined by Γ_0 through:

$$v_\infty \approx v_\phi(r_c) = c\tanh\chi_\phi(r_c) = c\sqrt{1-\Gamma_0^{-2}}. \tag{71}$$

The entire galactic flow from r_s to r_c to large r lies on the single isorapidity surface $\cosh\chi_r\cosh\chi_\phi = \Gamma_0$ in rapidity space. As r increases from r_s, χ_r decreases from $+\infty$ toward 0; the CL condition forces χ_ϕ to increase compensatingly so that their combined cosh-product stays constant. At r_c, $\chi_r = 0$ and $\chi_\phi = \psi_\Gamma$ carries the full rapidity budget. Beyond r_c, χ_r becomes negative (outflow) and χ_ϕ must decrease again; the isorapidity surface curves back toward the origin in rapidity space.

This is the rapidity-language statement of the flat rotation curve: the flow does not maintain constant orbital velocity because of a coincidental energy balance between Newtonian gravity, the Hubble flow, and the orbital motion. It maintains constant orbital velocity because the entire CL orbit traces a *single hyperbolic circle* of radius Γ_0 in the two-rapidity space (χ_r, χ_ϕ), and the flatness of the rotation curve is the projection of this hyperbolic circle onto the azimuthal rapidity axis.

7.5 What the Sigma Channel Residual Measures

With $\partial_r\Gamma = 0$ enforced, the sigma channel residual (64) takes the form $\Pi_{g,r} = \Gamma\dot{H}r$. So,

$$\Pi_{g,r} = \Gamma\,\dot{H}\,r = \cosh\chi_r\cosh\chi_\phi\,\dot{H}\,r. \tag{72}$$

This has a transparent reading in the isorapidity picture. The factor $\Gamma = \cosh\psi_\Gamma$ is the hyperbolic cosine of the composed rapidity — the radial distance from the origin on the isorapidity surface in the cosh direction. The factor $\dot{H}\,r$ is the rapidity acceleration of the FRW background at position r. The sigma channel residual is therefore the *FRW rapidity-tidal signal weighted by the hyperbolic radius of the isorapidity surface on which the flow is organised*.

Two consequences follow.

First, in the de Sitter limit $\dot{H} = 0$, the sigma channel residual vanishes identically for the entire astrophysical hierarchy simultaneously — thin disks, thick disks, spheroids, AGN jets, and relativistic accretion disks — because all of them live on isorapidity surfaces of their respective Γ_0, and the FRW rapidity acceleration to which those surfaces couple has vanished. The flow is in exact relativistic self-freefall.

Second, the de Sitter result $\Pi_{g,r} = 0$ is not a consequence of the specific shape of the isorapidity surface (i.e. not a consequence of the particular value of Γ_0). It is a consequence of the FRW background rapidity having zero second derivative. This means that *any* CL flow — regardless of its azimuthal profile, its mass, or its morphological type — decouples from the cosmological dynamics in the de Sitter limit. The isorapidity surface structure determines the Lorentz weight Γ in the residual but not whether the residual is zero or non-zero: that is determined entirely by the background rapidity $\psi_F(t)$.

7.6 The CL Isorapidity Circle as a Galactic Geodetic Precession

The isorapidity structure established in Section 7.4 is not merely a kinematic convenience: it is the direct galactic counterpart of the geodetic precession derived for a satellite gyroscope in de Haas (2014). This subsection makes that structural identity explicit.

Recap: the GP-B case

In de Haas (2014) the geodetic precession of a gyroscope in a free-fall orbit is derived as a global Thomas precession using two perpendicular rapidity boosts. A radial escape rapidity $\psi_{\rm esc}$, acquired by an elevator in free fall from rest at infinity to orbital radius R, is composed with an azimuthal orbital rapidity $\psi_{\rm orb}$. Because the two boosts are perpendicular, the composed rapidity ψ satisfies

$$\cosh\psi = \cosh\psi_{\rm esc}\,\cosh\psi_{\rm orb} = \gamma_{\rm esc}\,\gamma_{\rm orb}. \tag{73}$$

On the Poincaré rapidity disk the escape pod traces a circle of radius ψ during one orbit. The total Thomas–Wigner rotation angle accumulated per revolution equals the area of that circle in hyperbolic geometry,

$$A = (\cosh\psi - 1)\,2\pi, \tag{74}$$

giving the geodetic precession rate (de Haas 2014)

$$\Omega_G = (\cosh\psi - 1)\,\Omega = (\gamma_{\rm esc}\,\gamma_{\rm orb} - 1)\,\Omega \approx \tfrac{3}{2}\,\frac{GM}{Rc^2}\,\Omega. \tag{75}$$

The hyperbolic area enclosed by the rapidity circle on the Poincaré disk *is* the geodetic precession angle.

The CL galaxy: the same circle

The constant-Lagrangian condition established in de Haas (2026c) and re-read in rapidity language in Section 7 is

$$\frac{D}{Dt}\big(\cosh\chi_r\,\cosh\chi_\phi\big) = 0 \quad\Longleftrightarrow\quad \cosh\psi_\Gamma = \cosh\chi_r\,\cosh\chi_\phi = \Gamma_0 = \text{const.} \tag{76}$$

This is *mathematically identical* to the perpendicular-boost composition rule of de Haas (2014): the composed rapidity ψ_Γ defined by $\cosh\psi_\Gamma = \Gamma_0$ is conserved along every CL streamline, and every streamline therefore traces a hyperbolic circle of radius ψ_Γ on the Poincaré disk in the two-rapidity plane (χ_r, χ_ϕ).

The rôle of the velocity-balance radius r_c in fixing Γ_0 is transparent. By definition, $\chi_r(r_c) = 0$ (the radial rapidity vanishes where infall and Hubble outflow balance exactly). Evaluating (76) at r_c gives

$$\cosh(0)\, \cosh\chi_\phi(r_c) = \Gamma_0 \quad \Longrightarrow \quad \cosh\chi_\phi(r_c) = \Gamma_0, \tag{77}$$

so Γ_0 is simply the azimuthal Lorentz factor at r_c. At that radius the isorapidity circle carries its full rapidity budget in the azimuthal direction alone; r_c is the point on the circle closest to the origin in rapidity space (the "equator" where $\chi_r = 0$, $\chi_\phi = \psi_\Gamma$). The flat rotation speed is therefore determined directly by Γ_0:

$$v_\infty \approx v_\phi(r_c) = c \tanh\chi_\phi(r_c) = c\sqrt{1 - \Gamma_0^{-2}}. \tag{78}$$

Conversely, an observed flat rotation speed v_∞ fixes the hyperbolic radius of the isorapidity surface as

$$\Gamma_0 = \left(1 - \frac{v_\infty^2}{c^2}\right)^{-1/2}, \tag{79}$$

i.e. Γ_0 is the Lorentz factor of the flat-rotation speed. For typical spiral galaxies $v_\infty \sim 10^{-3}c$, giving $\Gamma_0 - 1 \sim v_\infty^2/2c^2 \sim 10^{-6}$, so the isorapidity circle has a very small but non-zero hyperbolic radius.

The galactic geodetic precession

Every vectorial quantity that undergoes parallel transport along a CL streamline — a jet direction, a disk normal, the spin axis of a compact object embedded in the flow — accumulates a Thomas–Wigner rotation equal to the area enclosed by the isorapidity circle:

$$A_{\text{galaxy}} = (\cosh\psi_\Gamma - 1)\, 2\pi = (\Gamma_0 - 1)\, 2\pi. \tag{80}$$

The associated galactic geodetic precession rate, referred to the pattern frequency Ω_{pat} characterising the flow (e.g. the outer-disk angular velocity), is

$$\Omega_G^{\text{gal}} = (\Gamma_0 - 1)\, \Omega_{\text{pat}} \approx \frac{v_\infty^2}{2c^2}\, \Omega_{\text{pat}}. \tag{81}$$

The suppression factor $v_\infty^2/2c^2 \sim 10^{-6}$ makes this effect tiny for individual rotation periods, but it is geometrically unavoidable: the hyperbolic structure of the isorapidity circle guarantees a non-zero enclosed area whenever $\Gamma_0 > 1$, i.e. whenever the flat rotation speed is non-zero.

The CL condition is precisely the statement that this circle exists and that its radius ψ_Γ is frozen along streamlines. It is not an additional postulate imported from the geodetic precession literature: it is the same perpendicular-boost composition rule of de Haas (2014), now applied to the stationary two-rapidity field (χ_r, χ_ϕ) of the galactic vacuum flow rather than to a periodic satellite orbit. In the satellite case the circle is closed by the orbital period; in the galaxy it is closed by the CL stationarity condition $\partial_r \Gamma_0 = 0$. The area formula is the same in both cases.

Structural comparison

Table 2 collects the parallel between the two cases.

Table 2 Structural parallel between the GP-B geodetic precession (de Haas 2014) and the galactic CL isorapidity precession (de Haas 2026c). Both arise from the same perpendicular-boost composition rule on the Poincaré rapidity disk; they differ only in what closes the rapidity circle.

Feature	GP-B satellite	CL galaxy
Radial rapidity	$\psi_{\rm esc}$ (free fall from ∞)	χ_r (Newton–Hubble field)
Azimuthal rapidity	$\psi_{\rm orb}$ (orbital velocity)	χ_ϕ (orbital velocity)
Composition rule	$\cosh\psi = \gamma_{\rm esc}\gamma_{\rm orb}$	$\cosh\psi_\Gamma = \gamma_r^{\rm FR}\gamma_\phi^{\rm FR} = \Gamma_0$
Poincaré disk locus	circle, radius ψ	isorapidity circle, radius ψ_Γ
Area enclosed	$(\cosh\psi - 1)\,2\pi$	$(\Gamma_0 - 1)\,2\pi$
Precession rate	$(\gamma_{\rm esc}\gamma_{\rm orb} - 1)\,\Omega$	$(\Gamma_0 - 1)\,\Omega_{\rm pat}$
Circle closure	Orbital period in real space	CL condition $D\Gamma_0/Dt = 0$
Radius anchor	Orbital radius R	Balance radius r_c, where $\chi_r(r_c) = 0$
Γ_0 fixed by	$v_{\rm esc}$, $v_{\rm orb}$ at R	$v_\infty = c\tanh\chi_\phi(r_c)$

Both entries in the last row of the precession-rate row reduce to the familiar weak-field limit: the satellite result gives $(3/2)(GM/Rc^2)\,\Omega$ to first order (de Haas 2014), and the galactic result gives $(v_\infty^2/2c^2)\,\Omega_{\rm pat}$ to leading order. The common origin is the Poincaré-disk area theorem for parallel transport in hyperbolic rapidity space.

7.7 Geodetic Precession of the Accretion Disk: the Fully Relativistic GP-B Scenario

The analysis of Section 7.6 concerned the CL vacuum flow across the full galactic extent, where both the radial and azimuthal rapidities are small ($\chi_r, \chi_\phi \ll 1$) and the geodetic precession per orbit is suppressed by $v_\infty^2/2c^2 \sim 10^{-6}$. As shown there, the galactic geodetic precession remains negligible throughout cosmic history: it is set permanently by the weak-field character of galactic dynamics and accumulates to only millidegrees per galaxy over a Hubble time. The natural place to look for non-negligible geodetic precession is therefore the galactic core, where the same isorapidity circle transitions from its weak-field ($\chi_r \ll 1$) tail into the strong-field regime ($\chi_r \to \infty$) near the central SMBH.

In this section we leave the CL vacuum-flow framework and step into the complementary scenario that is the direct strong-field counterpart of the derivation in de Haas (2014): pure radial vacuum inflow of space meeting pure orbital matter motion on an accretion disk around a Schwarzschild black hole. The result is the *exact, fully relativistic geodetic precession of the accretion disk*, obtained from the same Poincaré-disk area theorem without any weak-field approximation.

Departure from the CL framework

The CL condition $D\Gamma/Dt = 0$ governs stationary galactic vacuum flows in which radial and azimuthal motion are coupled along the same streamlines. Near a black hole accretion disk the situation is structurally different: the *vacuum* (space itself) flows purely radially inward at the Painlevé–Gullstrand escape velocity, while *matter* on the disk orbits azimuthally at a velocity set by gravity. These are two physically distinct objects — the inward-flowing metric and the orbiting material — and their rapidity boosts are perpendicular in exactly the same sense as the two boosts in the GP-B derivation of de Haas (2014). The scenario is therefore not a CL flow but the gravitational GP-B scenario itself, transplanted from the weak-field environment of a low-Earth orbit into the strong-field region $r \sim r_s$ of a black hole.

The two rapidities

At coordinate radius r outside a Schwarzschild black hole of mass M (with $r_s = 2GM/c^2$ and $\alpha \equiv r_s/r$), the two perpendicular rapidity boosts are:

Radial inflow rapidity (vacuum).

The PG river flows inward with the escape speed $v_{\rm esc}(r) = c\sqrt{\alpha}$, but in the fully relativistic rapidity framework the primary object is the gravitational rapidity potential χ_r, defined by

$$v_{\rm esc}^{\rm FR}(r) = c\tanh\chi_r(r), \quad \chi_r(r) = \tanh^{-1}\sqrt{\alpha} = \tanh^{-1}\sqrt{\frac{r_s}{r}}. \tag{82}$$

The associated Lorentz factor is $\gamma_r^{\rm FR} = \cosh\chi_r = (1-\alpha)^{-1/2}$, which diverges as $r \to r_s$.

Azimuthal orbital rapidity (matter).

For matter in a circular Keplerian orbit the virial orbital speed is $v_{\rm orb}^2 = GM/r = \frac{1}{2}c^2\alpha$, giving

$$v_{\rm orb}^{\rm FR}(r) = c\tanh\chi_\phi(r), \quad \chi_\phi(r) = \tanh^{-1}\sqrt{\frac{\alpha}{2}} = \tanh^{-1}\sqrt{\frac{r_s}{2r}}. \tag{83}$$

The associated Lorentz factor is $\gamma_\phi^{\rm FR} = \cosh\chi_\phi = (1-\alpha/2)^{-1/2}$. At the ISCO ($r = 3r_s$, $\alpha = \frac{1}{3}$) one has $v_{\rm orb} = c/\sqrt{6} \approx 0.408c$, so $\chi_\phi({\rm ISCO}) \approx 0.431$.

The composed rapidity and the geodetic precession angle

Because the radial inflow and the azimuthal orbit are perpendicular boosts, their composed rapidity satisfies (cf. eq. (20) of de Haas 2014 and eq. (13) of the present paper)

$$\cosh\psi_{\rm acc}(r) = \cosh\chi_r(r)\,\cosh\chi_\phi(r) = \gamma_r^{\rm FR}\,\gamma_\phi^{\rm FR} = \frac{1}{\sqrt{1-\alpha}\,\sqrt{1-\alpha/2}}. \tag{84}$$

This is the exact expression, valid at all radii $r > r_s$. The geodetic precession angle accumulated by any spin vector (disk normal, magnetic axis, jet base direction) during one complete orbit of the accreting matter is the Poincaré-disk area of the circle with hyperbolic radius $\psi_{\rm acc}$:

$$\vartheta_G^{\rm acc}(r) = \big(\cosh\psi_{\rm acc}(r) - 1\big)\,2\pi = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1\right)2\pi. \tag{85}$$

The geodetic precession angular velocity is

$$\Omega_G^{\rm acc}(r) = \big(\cosh\psi_{\rm acc}(r) - 1\big)\,\Omega_{\rm orb}(r), \quad \Omega_{\rm orb}(r) = \sqrt{\frac{GM}{r^3}}. \tag{86}$$

No approximation has been made. The only inputs are M, r, and the exact hyperbolic composition of the two perpendicular rapidities.

Weak-field recovery

In the weak-field limit $\alpha \ll 1$:

$$\cosh\psi_{\rm acc} - 1 \approx \tfrac{3}{4}\alpha = \frac{3GM}{2rc^2}, \tag{87}$$

giving $\Omega_G^{\rm acc} \approx \frac{3}{2}(GM/rc^2)\,\Omega_{\rm orb}$, which is the standard geodetic precession rate of de Haas (2014) (eq. (29) of that paper). The strong-field formula (86) is the exact, unapproximated extension of that result to the accretion-disk regime.

A mass-independent prediction at the ISCO

A key feature of eq. (86) is that the ratio $\Omega_G^{\rm acc}/\Omega_{\rm orb} = \cosh\psi_{\rm acc} - 1$ depends only on the dimensionless radius $\alpha = r_s/r$, not on M separately. At the ISCO ($\alpha = 1/3$) this ratio is

$$\left.\frac{\Omega_G^{\rm acc}}{\Omega_{\rm orb}}\right|_{\rm ISCO} = \frac{1}{\sqrt{(1-\frac{1}{3})(1-\frac{1}{6})}} - 1 = \frac{1}{\sqrt{\frac{2}{3}\cdot\frac{5}{6}}} - 1 \approx 0.374, \tag{88}$$

independent of the black hole mass. This is a parameter-free, universal prediction: *for any Schwarzschild black hole, the geodetic precession frequency at the ISCO is* 37.4% *of the orbital frequency.* Applied across the full population of black hole X-ray binaries and AGN, the same ratio 0.374 predicts a geodetic QPO frequency that scales with orbital frequency alone, with no free parameters.

Numerical values at key radii

Table 3 collects the geodetic precession angle per orbit and the mass-independent frequency ratio at four characteristic radii.

Table 3 Exact geodetic precession angle per orbit $\vartheta_G^{\mathrm{acc}}$ and mass-independent precession-to-orbital frequency ratio $\Omega_G^{\mathrm{acc}}/\Omega_{\mathrm{orb}}$ at selected radii for a Schwarzschild black hole. $\alpha = r_s/r$; the ISCO is at $r = 3r_s$, the photon orbit at $r = 1.5\, r_s$. No approximation is used. The ratio in the final column is universal: it holds for stellar-mass and supermassive black holes alike.

Radius r/r_s	α	$\cosh\psi_{\mathrm{acc}} - 1$	$\vartheta_G^{\mathrm{acc}}$ (deg/orbit)	$\Omega_G^{\mathrm{acc}}/\Omega_{\mathrm{orb}}$
100	0.010	0.0075	2.7°	0.0075
10	0.100	0.076	27°	0.076
3 (ISCO)	0.333	0.374	135°	**0.374**
1.5 (phot. orb.)	0.667	$\to \infty$	$\to 360°$	$\to \infty$

At $r = 100\, r_s$ the precession recovers the weak-field limit. At the ISCO the disk normal precesses by 135° per orbit — more than one third of a full turn. This is no longer a perturbative effect: the Poincaré disk circle with radius ψ_{acc} has grown to macroscopic hyperbolic size, and the geodetic precession is an $O(1)$ fraction of the orbital motion.

Physical interpretation and distinction from Lense–Thirring precession

This is not Lense–Thirring precession. The geodetic precession derived here operates for a *Schwarzschild* (non-rotating) black hole; it requires no spin of the central object. Lense–Thirring (frame-dragging) precession is a separate, additive effect that arises only when the black hole spins. The present effect is the direct strong-field generalisation of the gravitational Thomas precession of de Haas (2014): it is caused by the combination of radial vacuum inflow and azimuthal matter orbiting, mediated by the hyperbolic composition rule for perpendicular rapidity boosts.

The precessing object is the disk as a whole. Any spin-bearing structure anchored to the orbital plane — the disk normal, the jet launch direction, the magnetic dipole axis of an accreting neutron star — is subject to this precession. It is not a single-particle effect but a property of the geometry at orbital radius r, accumulated once per orbit.

Observable signatures

The large precession angles at $r \lesssim 10\, r_s$ open three realistic observational windows.

Low-frequency quasi-periodic oscillations (QPOs).

The geodetic precession frequency $\nu_G^{\rm acc} = \Omega_G^{\rm acc}/2\pi$ is a well-defined function of r and M alone (eq. (86)). At the ISCO the universal ratio (88) gives $\nu_G = 0.374\, \nu_{\rm orb}$ regardless of mass. For a stellar-mass black hole ($M = 10\, M_\odot$), $\nu_{\rm orb}(\text{ISCO}) \approx 220$ Hz, giving $\nu_G \approx 82$ Hz — squarely in the range of observed low-frequency QPOs in X-ray binaries (0.1–450 Hz). For a supermassive black hole ($M = 10^8\, M_\odot$), the same ratio gives a geodetic precession period of ≈ 16 hr at the ISCO, consistent with observed quasi-periodic variability in AGN and Seyfert nuclei. Current models for these QPOs invoke Lense–Thirring precession of a tilted inner disk (the relativistic precession model of Stella and Vietri 1998), but the present geodetic mechanism operates without any black-hole spin, produces a *universal* frequency ratio at the ISCO, and has a distinct radial profile of $\Omega_G/\Omega_{\rm orb}$ that in principle differentiates it from the Lense–Thirring prediction.

Jet precession in AGN: the M87 case.

The jet launch direction is anchored in the inner accretion disk. If the disk normal undergoes geodetic precession at a rate $\Omega_G^{\rm acc}(r_{\rm launch})$, the jet direction precesses at the same rate, producing a helical or wobbling jet morphology. For the PG stagnation–launch zone identified for M87 in the companion paper (de Haas 2026c) at $r_{\rm launch} \sim 5$–$30\, r_s$, the geodetic frequency ratio from Table 3 is $\Omega_G/\Omega_{\rm orb} \approx 0.010$–$0.076$. With $M_{\rm BH} \approx 6.5 \times 10^9\, M_\odot$ for M87*, the orbital period at $r = 10\, r_s$ is approximately 2 yr, giving a geodetic precession period of $\sim 1/(0.076 \times \nu_{\rm orb}) \approx 26$ yr. This is directly in the observed range of M87 jet position-angle variations (10–30 yr timescales seen in long-baseline VLBI monitoring), without requiring black-hole spin. The precession angle per orbital period at the launch zone is 10°–5°, consistent with the observed helical structure and persistent wobble of the M87 jet.

Accretion disk warping and neutron stars.

At different radii the geodetic precession rate $\Omega_G^{\rm acc}(r)$ follows a radial profile distinct from that of Lense–Thirring precession. In a differentially precessing disk this produces a warp whose shape encodes the black hole mass through eq. (85) without requiring a spin measurement — an alternative to the standard Bardeen–Petterson mechanism, which requires Kerr spin. For an accreting neutron star, the disk geodetic precession modulates the alignment between the disk normal and the neutron star's magnetic axis, producing periodic variations in the X-ray beaming pattern. The predicted geodetic frequency at the inner disk edge ($r \sim$ a few $\times r_s$) is in the 10–100 Hz range for typical neutron star masses, overlapping with observed horizontal-branch and lower QPOs whose physical origin remains debated.

Summary

The geodetic precession of the accretion disk around a Schwarzschild black hole is the exact, fully relativistic GP-B scenario: radial vacuum inflow (the PG river) provides the first perpendicular boost χ_r, and the orbital motion of matter provides the second χ_ϕ. Their hyperbolic composition $\cosh\psi_{\rm acc} = \cosh\chi_r \cosh\chi_\phi$ gives a Poincaré-disk circle whose area $(\cosh\psi_{\rm acc} - 1)\, 2\pi$ is the geodetic precession angle per orbit — the same formula as in de Haas (2014) but with both rapidities of order unity. The ISCO value $\Omega_G/\Omega_{\rm orb} \approx 0.374$ is mass-independent and parameter-free, providing a universal prediction across the full population of accreting black holes. Applied to M87, the formula predicts a jet precession period of ~ 26 yr at $r = 10\, r_s$, consistent with observed VLBI jet wobble timescales. The effect requires no spin, is complementary to and distinguishable from Lense–Thirring precession, and is most prominent — and most observable — in galactic cores, where the isorapidity circle transitions from the weak-field galactic tail ($\Gamma_0 - 1 \sim 10^{-6}$) to the strong-field accretion regime ($\cosh\psi_{\rm acc} - 1 \sim 0.374$ at the ISCO).

7.8 Distinct Observational Predictions

The geodetic precession derived in Section 7.7 differs structurally from the standard Lense–Thirring (frame-dragging) precession of the relativistic precession model (Stella and Vietri 1998) and from the Bardeen–Petterson warp mechanism, both of which require a spinning (Kerr) black hole and depend on the spin parameter a as a free quantity. The present effect operates for a Schwarzschild ($a = 0$) black hole, is driven entirely by the hyperbolic composition of the radial vacuum inflow rapidity χ_r and the azimuthal orbital rapidity χ_ϕ, and produces a set of predictions that are either parameter-free or depend only on the directly observable quantities M and r/r_s. We state these predictions explicitly so that they can serve as observational targets.

Prediction 1: universal, mass-independent ISCO frequency ratio.

For any accreting Schwarzschild black hole, the ratio of the geodetic precession frequency to the Keplerian orbital frequency at the ISCO ($r = 3r_s$, $\alpha = 1/3$) is

$$\left.\frac{\nu_G}{\nu_{\rm orb}}\right|_{\rm ISCO} = \cosh\psi_{\rm acc}\big|_{\alpha=1/3} - 1 = \frac{1}{\sqrt{\frac{2}{3}\cdot\frac{5}{6}}} - 1 \approx 0.374, \tag{89}$$

independent of black hole mass, distance, inclination, or spin. This is a single number predicted by the geometry of the perpendicular-boost composition, with no free parameters. For a $10\, M_\odot$ black hole this gives $\nu_G \approx 82$ Hz alongside

$\nu_{\text{orb}} \approx 220$ Hz; for a $10^8\, M_\odot$ AGN the same ratio gives a geodetic period of ≈ 16 hr alongside an orbital period of ≈ 6 hr. No existing precession model makes a parameter-free, mass-independent prediction of this kind at the ISCO.

Prediction 2: M87 jet precession period.

At $r = 10\, r_s$ around M87* ($M_{\text{BH}} \approx 6.5 \times 10^9\, M_\odot$), the Keplerian orbital period is ≈ 2 yr and the frequency ratio from Table 3 is $\Omega_G/\Omega_{\text{orb}} \approx 0.076$, giving a geodetic jet precession period of

$$T_G^{\text{M87}} \approx \frac{2\,\text{yr}}{0.076} \approx 26\,\text{yr}. \tag{90}$$

This is a specific, falsifiable prediction for the long-term position-angle variation of the M87 jet, observable with continued VLBI monitoring. The prediction uses only M_{BH} and the dimensionless radius $r/r_s = 10$; it requires no knowledge of the black-hole spin.

Prediction 3: QPO floor in low-spin systems.

In accreting systems where the black-hole spin is independently constrained to be near zero (for example from X-ray reflection spectroscopy or continuum fitting), Lense–Thirring precession should be negligible and the relativistic precession model predicts no significant low-frequency QPO. The present mechanism predicts that low-frequency QPOs should nevertheless persist, with the geodetic frequency given by

$$\nu_G(r) = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1 \right) \nu_{\text{orb}}(r), \quad \alpha = r_s/r, \tag{91}$$

and in particular $\nu_G \approx 0.374\, \nu_{\text{orb}}$ at the ISCO. The detection of low-frequency QPOs in confirmed low-spin black-hole systems at the predicted ratio would constitute evidence for the geodetic mechanism and could not be attributed to Lense–Thirring precession.

Prediction 4: distinct radial QPO frequency profile.

As the inner truncation radius of the accretion disk moves inward during an X-ray binary state transition, the geodetic QPO frequency follows the profile given by eq. (91). This profile is distinct from both the Lense–Thirring nodal precession profile ($\nu_{\text{LT}} \propto r^{-3}$ for slow spin) and the periastron precession profile. If QPO frequencies can be tracked as a function of inferred emission radius through a full state transition — as r varies from $\sim 10\, r_s$ to $\sim 3\, r_s$ — the radial curvature of eq. (91) provides a discriminating test between the geodetic and Lense–Thirring mechanisms without requiring an independent spin measurement.

Prediction 5: disk warp geometry without spin.

Differential geodetic precession across disk radii produces a warp with a radial shape determined by eq. (91): the inner regions precess faster than the outer regions, with the precession angle per orbit reaching 135° at the ISCO. In systems where a disk warp is directly imaged — AGN with VLBI water masers, warped megamaser disks — the warp geometry encodes this profile without requiring a spin measurement. The present warp shape differs from the Bardeen–Petterson profile (which requires Kerr spin and scales differently with radius) and could in principle be discriminated with sufficient angular resolution. The warp is driven by the same mechanism regardless of whether the black hole spins, making it present and potentially dominant in systems where the Bardeen–Petterson effect is suppressed.

Common feature: additive to Lense–Thirring.

In spinning black holes all five effects are present simultaneously: the geodetic precession derived here adds to the Lense–Thirring contribution. Since the two mechanisms have different radial profiles and different dependences on a, their combined signal is in general not degenerate with either alone. Disentangling them requires either an independent spin constraint or a sufficiently well-sampled radial frequency profile. For Schwarzschild black holes the geodetic precession is the only geometric precession mechanism, providing the cleanest test environment. The five predictions above are therefore most directly testable in systems with known or constrained low spin, and most precisely quantified for M87* through eq. (90).

7.9 Summary: Three Levels of the CL Condition and Their Geodetic Consequences

Table 4 collects the three levels at which the CL condition $D\Gamma/Dt = 0$ can be read, from the most familiar to the most fundamental.

Table 4 The constant-Lagrangian condition at three levels of description. Each lower level is a controlled approximation of the one above. The rapidity level is the exact and geometrically transparent form; the velocity and kinetic-energy levels are its slow-flow projections.

Level	Condition	Physical meaning	Status
Rapidity (exact)	$D(\cosh\chi_r \cosh\chi_\phi)/Dt = 0$	Composed rapidity magnitude ψ_Γ conserved along streamlines; flow traces isorapidity surface in (χ_r, χ_ϕ) space	Exact, all v/c
Velocity (slow-flow)	$\tanh\chi_r\, D\chi_r/Dt + \tanh\chi_\phi\, D\chi_\phi/Dt = 0$	Velocity-weighted rapidity rates balance along streamlines	$O(v^3/c^3)$ correction
Kinetic energy (non-relativistic)	$D(v_r^2 + v_\phi^2)/Dt = 0$	Total flow speed conserved; kinetic energy exchanges between radial and azimuthal motion	$\chi \ll 1$ limit

The rapidity level reveals features invisible at the lower levels, which the subsequent subsections have developed in two directions: the geometric structure of the CL flow itself, and its geodetic consequences across the full range of gravitational scales.

The flat rotation curve is a hyperbolic circle.

The CL flow traces the isorapidity surface $\cosh\chi_r \cosh\chi_\phi = \Gamma_0$ in two-rapidity space. The flat orbital speed is the projection of this surface onto the χ_ϕ axis at $r = r_c$, where $\chi_r = 0$ and $\cosh\chi_\phi(r_c) = \Gamma_0$; the flatness of the rotation curve is the radial flatness of the isorapidity surface near r_c in the χ_ϕ direction.

The sigma-channel cancellation is a gradient condition.

The elimination of all Newtonian, centrifugal, and Hubble cross-terms from the sigma channel is the single condition $\nabla\Gamma \cdot \hat{r} = 0$ — the radial gradient of Γ vanishes because the flow is confined to a surface of constant Γ. Seven pairwise cancellations in the velocity language collapse to one geometric statement in rapidity space, leaving the purely cosmological residual $c\,\Pi_{g,r} = \dot{H}(t)\,r$.

The isorapidity circle as a geodetic precession.

The same isorapidity surface $\cosh\psi_\Gamma = \Gamma_0$ is the Poincaré-disk circle whose hyperbolic area $(\Gamma_0 - 1)\,2\pi$ gives the Thomas–Wigner rotation angle accumulated by any spin vector parallel-transported along a CL streamline — the galactic geodetic precession (Section 7.6). This is structurally identical to the GP-B derivation of de Haas (2014): the same perpendicular-boost composition rule $\cosh\psi = \cosh\chi_r \cosh\chi_\phi$, the same Poincaré-disk area theorem, the same geodetic precession formula. At galactic scales both rapidities are small ($\chi_r, \chi_\phi \ll 1$) and the precession amplitude is $\Gamma_0 - 1 \approx v_\infty^2/2c^2 \sim 10^{-6}$, negligible throughout cosmic history. The BTFR has its unique origin not in this geodetic structure but in the CL boundary condition at $r = R$ combined with the identification of the bulge scale with r_c.

Galactic cores as the strong-field limit.

The isorapidity circle transitions continuously from its weak-field galactic tail ($\Gamma_0 - 1 \sim 10^{-6}$) to the strong-field accretion regime as r decreases into the SMBH sphere of influence. In the core the CL vacuum flow gives way to the direct GP-B scenario — pure radial vacuum inflow composing with pure azimuthal matter orbiting — and both rapidities grow to order unity (Section 7.7). At the ISCO the geodetic precession angle is 135° per orbit and the frequency ratio $\Omega_G/\Omega_{\rm orb} = 0.374$ is mass-independent and parameter-free.

Five distinct observational predictions.

The strong-field geodetic precession yields five predictions stated explicitly in Section 7.8, each distinguishable from Lense–Thirring precession and requiring no black-hole spin:

1. A universal ISCO frequency ratio $\nu_G/\nu_{\rm orb} = 0.374$, independent of mass, distance, inclination, and spin — the only parameter-free QPO prediction of this kind.
2. A jet precession period of ≈ 26 yr for M87* at $r = 10\, r_s$, directly testable with continued VLBI monitoring.
3. A QPO floor in low-spin systems: low-frequency QPOs at the predicted geodetic ratio even when Lense–Thirring precession is negligible.
4. A distinct radial QPO frequency profile $\nu_G(r) \propto [(1-\alpha)(1-\alpha/2)]^{-1/2} - 1$, differentiable from both nodal and periastron precession profiles across a state transition.
5. A disk warp geometry driven without spin, with a radial profile distinct from the Bardeen–Petterson shape, testable in VLBI water-maser systems.

In spinning black holes the geodetic and Lense–Thirring contributions are additive with different radial profiles and different dependences on the spin parameter a, so their combined signal is in general not degenerate with either alone.

All results follow from the single identification: the CL condition is the statement that galactic vacuum flows organise themselves on isorapidity surfaces of the composed Lorentz factor in hyperbolic two-rapidity space. That surface encodes the flat rotation curve, the sigma-channel residual, the galactic geodetic precession, and — in its strong-field limit at the galactic core — a fully relativistic, parameter-free geodetic precession of the accretion disk with five distinct and falsifiable observational consequences.

8 Conclusion

This paper established the gravitational rapidity field χ_r as the primary scalar object of a vacuum-flow description of gravity, and derived its consequences across three scales: the ontological hierarchy of approximations, the constant-Lagrangian galactic flow, and the strong-field accretion regime of galactic cores. The central results are collected here.

The rapidity ontology and its approximation hierarchy

The Newtonian potential $\Phi = -GM/r$, the Painlevé–Gullstrand river velocity $v_r = \sqrt{2GM/r - H(t)r}$ (Painlevé 1921; Gullstrand 1922; Hamilton and Lisle 2008), and the Einsteinian metric $g_{\mu\nu}$ are not three independent descriptions of gravity but three successive approximations of a single primary object: the gravitational rapidity field $\chi_r(r,t) = \psi_N(r) - \psi_H(r,t)$. The chain of approximations is

$$\chi_r \xrightarrow{\cosh} \gamma_r^{\mathrm{FR}} \xrightarrow{\chi_r^2/2} 1 - \Phi_{\mathrm{eff}}/c^2 \xrightarrow{H\to 0} 1 + GM/(rc^2), \tag{92}$$

and the metric arises by two further irreversible steps: a slow-flow approximation ($\cosh\chi_r \approx 1$) that discards the Lorentz factor, followed by a coframe quadratic that discards the bivector phase encoding the direction of flow. Each step loses information that cannot be recovered from $g_{\mu\nu}$ alone. General Relativity is therefore acceleration-relativistic but not flow-relativistic: it constrains the curvature of v_r^2 but carries no information about $\cosh\chi_r$, $\mathrm{sgn}(\chi_r)$, or the absolute rapidity χ_r itself (de Haas 2025b). The physical velocity $v_r^{\mathrm{FR}} = c\tanh\chi_r$ is bounded by construction at both horizons of the combined Newton–Hubble field, without any coordinate singularity or superluminal disclaimer.

The constant-Lagrangian galactic flow

The constant-Lagrangian (CL) condition $D\Gamma/Dt = 0$, where $\Gamma = \cosh\chi_r \cosh\chi_\phi$, was re-read in rapidity language and shown to have three equivalent formulations at decreasing levels of exactness (Table 4): the isorapidity condition $D\psi_\Gamma/Dt = 0$ (exact), a velocity-weighted rapidity balance (slow-flow), and the kinetic energy conservation $D(v_r^2 + v_\phi^2)/Dt = 0$ (non-relativistic). At the rapidity level two features are visible that are invisible at the lower levels.

First, every CL streamline traces a circle of constant hyperbolic radius ψ_Γ on the Poincaré rapidity disk in the two-rapidity plane (χ_r, χ_ϕ). The flat rotation curve is the projection of this circle onto the χ_ϕ axis at $r = r_c$, where $\chi_r(r_c) = 0$ and $\cosh\chi_\phi(r_c) = \Gamma_0$ fixes the flat rotation speed through $v_\infty = c\tanh\chi_\phi(r_c) = c\sqrt{1-\Gamma_0^{-2}}$.

Second, the isorapidity condition $\partial_r\Gamma = 0$ is precisely the condition under which all Newtonian, centrifugal, and Hubble cross-terms cancel identically in the radial sigma channel of the field equation $\mathbf{H}_g = \partial^T U_g$, leaving the purely cosmological residual $c\Pi_{g,r} = \dot{H}(t)\,r$ (de Haas 2026d,c). The seven pairwise cancellations of the velocity description collapse to one geometric statement in rapidity space. The Baryonic Tully–Fisher relation has its unique origin not in the geodetic structure but in the CL boundary condition at the bulge radius R combined with the identification $R \sim r_c = (2GM/H^2)^{1/3}$ (de Haas 2025a, 2026a).

The galactic geodetic precession

The isorapidity circle $\cosh\chi_r \cosh\chi_\phi = \Gamma_0$ is structurally identical to the rapidity circle of de Haas (2014): the same perpendicular-boost composition rule, the same Poincaré-disk area theorem first stated by Borel (1913) and given its modern graphical form by Rhodes and Semon (2004). The Poincaré-disk area $(\Gamma_0 - 1)\,2\pi$

is the Thomas–Wigner rotation angle accumulated by any spin vector parallel-transported along a CL streamline — the galactic geodetic precession, the direct structural counterpart of the de Sitter–Fokker precession (de Sitter 1916; Fokker 1921) confirmed by Gravity Probe B (Everitt et al. 2011). At galactic scales both rapidities are small and $\Gamma_0 - 1 \approx v_\infty^2/2c^2 \sim 10^{-6}$: the precession is negligible throughout cosmic history, and there is no cosmic era in which it becomes non-negligible.

Five parameter-free predictions from galactic cores

The transition from the weak-field galactic tail ($\Gamma_0 - 1 \sim 10^{-6}$) to the strong-field accretion regime occurs as r decreases into the SMBH sphere of influence. In galactic cores the CL vacuum flow gives way to the direct GP-B scenario of de Haas (2014): pure radial vacuum inflow composing with pure azimuthal matter orbiting, both rapidities of order unity. The geodetic precession angle per orbit is then

$$\vartheta_G^{\mathrm{acc}}(r) = \left(\frac{1}{\sqrt{(1-\alpha)(1-\alpha/2)}} - 1 \right) 2\pi, \qquad \alpha = r_s/r, \tag{93}$$

exact and without approximation. This yields five distinct, parameter-free predictions (Section 7.8), each distinguishable from Lense–Thirring precession (Stella and Vietri 1998) and requiring no black-hole spin:

1. **Universal ISCO frequency ratio.** $\nu_G/\nu_{\mathrm{orb}}|_{\mathrm{ISCO}} = 0.374$, independent of black hole mass, distance, inclination, or spin — the only parameter-free, mass-independent QPO prediction of this kind.
2. **M87* jet precession period.** $T_G \approx 26$ yr at $r = 10\,r_s$ for M87* ($M_{\mathrm{BH}} \approx 6.5 \times 10^9\, M_\odot$), directly testable with continued VLBI monitoring.
3. **QPO floor in low-spin systems.** Low-frequency QPOs at $\nu_G \approx 0.374\,\nu_{\mathrm{orb}}$ at the ISCO, present even when Lense–Thirring precession is negligible.
4. **Distinct radial QPO frequency profile.** The profile $\nu_G(r) \propto [(1-\alpha)(1-\alpha/2)]^{-1/2} - 1$ differs from both nodal and periastron precession, discriminable across a black-hole state transition.
5. **Spinless disk warp.** A disk warp driven without black-hole spin, with a radial profile distinct from the Bardeen–Petterson shape (Bardeen and Petterson 1975), testable in VLBI water-maser AGN systems.

In spinning black holes the geodetic and Lense–Thirring contributions are additive with different radial profiles and different dependences on the spin parameter a, so their combined signal is in general not degenerate with either mechanism alone. The five predictions are most cleanly testable in systems with known or constrained low spin, and most precisely quantified for M87* through prediction 2.

Position of the framework

The BQ vacuum-flow hierarchy positions Newtonian gravity, the Hubble flow, the constant-Lagrangian galactic disk, and the strong-field accretion disk as four regimes of the same first-order field equation $\mathbf{H}_g = \partial^T U_g$, distinguished by the rapidity content of the rotor G_g and the approximation level applied, rather than as separate theories connected only through limiting procedures (de Haas 2026d,c). The metric and the Einstein tensor are secondary: they arise by projecting U_g through two information-reducing steps, whereas U_g itself retains the full rapidity content of G_g exactly and invertibly. In this sense the present framework is a first-order precursor to General Relativity: GR describes the second-order curvature consequences of the rapidity field; the BQ vacuum-flow hierarchy describes the field itself.

The central result is that a single geometric object — the isorapidity surface $\cosh\chi_r \cosh\chi_\phi = \Gamma_0$ in hyperbolic two-rapidity space — simultaneously encodes the flat galactic rotation curve, the exact sigma-channel cancellation and its cosmological residual, the galactic geodetic precession, and the fully relativistic geodetic precession of black hole accretion disks with five falsifiable observational consequences. That a single surface does all of this is not a coincidence of the framework; it is the framework.

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