Matter-First Dynamics in LQG: Fermion Determinants, Spacetime-Algebra Tetrads, and an Induced Discrete Field Equation

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Abstract

Loop quantum gravity (LQG) is usually formulated as a quantisation of geometry, while operational measurements of length, time, and orientation are made through relations among matter fields. Penrose's combinatorial spacetime and Altaisky's recent locally Lorentzian matter-spin-network construction suggest a matter-first route: interaction events form the vertices of a graph, fermion-line segments form its edges, and geometry is reconstructed from the relations carried by those lines. We develop this kinematical proposal into an induced dynamical model with a computable discrete field equation. In the spacetime algebra $\mathrm{Cl}(1,3)$, the $\mathrm{SL}(2,\mathbb{C})$ interaction vertices become versors; their sandwich action transports the matter-built tetrad, agrees with both the Hermitian-matrix and Ruehl Lorentz maps, and sends elementary areas to simple grade-two blades. Loop curvature is the bivector logarithm of a holonomy and is classified by the scalar and pseudoscalar invariants of B^2; the generic loxodromic case is handled by a closed-form commuting elliptic–hyperbolic split. Because the discrete Grassmann matter action is bilinear, its path integral is the exact fermion determinant $Z = \det D$. On an oriented graph Dirac operator, gauge invariance reduces Z to a functional of cycle-space holonomies, while loops that share interaction events fail to factorise. The connected non-factorising determinant defines the interaction part of the effective action. Varying $S_{\text{eff}} = -\log|\det D|$ gives a local, gauge-covariant stationarity condition relating each independent cycle connection to the fermion propagator. Flat geometry solves the homogeneous equation as a Lorentzian saddle, and fixed boundary holonomies source curvature in the remaining cycles. We demonstrate the construction on the tetrahedral S^2 toy universe and on the K_5 four-simplex 1-skeleton. The resulting equation is induced, Sakharov-style, and no continuum limit to Einstein gravity is claimed.

1 Introduction

The quantisation of gravity inherits a conceptual asymmetry. Quantum mechanics elevates the Poisson brackets of classical mechanics to operator commutators and keeps time as a privileged parameter; general relativity is invariant under the diffeomorphisms $x'^{\mu} = f^{\mu}(x)$ and grants no coordinate any preference. Loop quantum gravity (LQG) sharpens the asymmetry to a paradox: the canonical programme begins from the Einstein–Hilbert functional of a *matterless* spacetime, $T_{\mu\nu} \equiv 0$, and quantises that, yet a length or a time can be measured only by a material clock or rod [1, 2, 3]. If the metric is, operationally, a record of relations between matter fields, a theory that excises matter at the outset has quantised the wrong object.

Penrose proposed the cure half a century ago [4]. Direction and angle are not given by a background manifold; they are read off from the recoupling of blocks of spin-$\frac{1}{2}$ matter, with the probability that two large blocks are parallel fixing the angle through $P(\theta) = \frac{1}{2}(1 + \cos\theta)$. The carrier of geometry is then a graph—a *spin network*—whose edges are labelled by representations and whose vertices are intertwiners. Ponzano and Regge [5], and later the Ashtekar reformulation [6] and the loop representation [7], brought spin networks into gravity, but along the way the matter content was dropped: the half-integers became edge lengths and the network became the dual of a triangulation rather than a history of particles. Recently Altaisky returned to Penrose's original intent and gave a locally Lorentz-invariant construction of combinatorial spacetime directly from matter [8]: the events of spacetime are the interaction vertices of fermion loops, the edges are the matter lines, a tetrad is built from the $\mathrm{SL}(2,\mathbb{C})$ spinor on each edge, and all curvature is concentrated, Regge-fashion, at the vertices of the triangulation graph Γ rather than spread over a smooth connection.

That construction is kinematic. It assigns a geometry to a matter configuration but does not yet say which configurations are dynamically preferred, nor whether the emergent geometry obeys anything resembling a field equation. The present paper supplies the missing dynamical layer and, with it, four results.

1. **A spacetime-algebra account of the Lorentz sector** (Sec. 3). The $\mathrm{SL}(2,\mathbb{C})$ vertex matrix is a versor in $\mathrm{Cl}(1,3)$; the tetrad transport is its sandwich action; three independent expressions for the Lorentz matrix coincide; and the elementary areas are simple grade-two blades.

2. **A closed-form Lorentz logarithm** (Sec. 4). The curvature of a loop is a bivector classified by the two invariants of its square; through the isomorphism of the centre of the even subalgebra with $\mathbb{C}$, its logarithm splits into commuting rotation and boost parts even in the generic loxodromic case.

3. **The matter path integral as a fermion determinant** (Secs. 5–6). The discrete Grassmann action is bilinear, so $Z = \det D$ exactly; gauge invariance reduces Z to a function of the cycle-space holonomies, and the determinant fails to factorise exactly when loops share events.

4. **An induced discrete field equation** (Sec. 7). The effective action $S_{\text{eff}} = -\log|\det D|$ has a stationarity condition that is local on each independent cycle and gauge covariant; flat geometry is a Lorentzian saddle solving the vacuum equation, and boundary matter data source curvature in the bulk.

We are explicit about scope. The field equation is of the induced, Sakharov type [14]: it is what the matter determinant dictates for the geometry it feels, not a discretisation of the Einstein equations, and we do not claim a general-relativistic continuum limit. What the construction does show is that local gauge covariance, a Lorentzian tetrad geometry, an exact partition function, and a nontrivial field equation can all be carried by a finite combinatorial substrate with no differentiable manifold underneath.

2 Matter spin networks and the comparator

2.1 Events, lines, and the gauge group on edges

A matter spin network is an oriented graph $\Gamma = (V, E)$. The vertices V are interaction events; the edges E are fermion-line segments running from one event to the next. In the language of category theory the edges are objects and the vertices are morphisms [8]. Each edge e carries an internal frame F_e valued in a gauge group G—for the Lorentz sector $G = \mathrm{SL}(2,\mathbb{C})$, the double cover of the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$, acting on the two-component spinor of the line.

In ordinary gauge theory a local transformation rotates the field independently at each *point*, $\psi(x) \mapsto U(g(x))\psi(x)$, and the covariant derivative $D_\mu = \partial_\mu + A_\mu$ requires a differentiable manifold to compare neighbouring points. On Γ there are no points and no derivative. The local transformation instead rotates each fermion line independently,

$$F_e \;\longmapsto\; h_e\, F_e, \qquad h_e \in G \text{ chosen per edge,} \tag{1}$$

inverting the usual lattice dictionary in which the gauge parameter lives at sites. To compare the frames of two edges e, e' meeting at a shared event one needs a *comparator*, the discrete analogue of the connection. Because the edges already carry their own frames, the comparator is not an independent field but a function of the matter,

$$g_{e'e} \;=\; F_{e'} F_e^{-1} \;\in\; G, \tag{2}$$

and it transforms as a link variable, $g_{e'e} \mapsto h_{e'} g_{e'e} h_e^{-1}$. The "gauge field" is thereby bookkeeping about matter rather than a new substance, and a Wilson loop is a trace over a cycle of comparators in which the edge rotations telescope away.

2.2 Curvature is a vertex effect

The comparator (2) built purely from frames is, however, *flat.*

Proposition 1 (Flatness of the bare comparator)**.** *For any closed cycle of edges $e_0 \to e_1 \to \cdots \to e_{n-1} \to e_0$, the ordered product of comparators is the identity,*

$$g_{e_0 e_{n-1}} \cdots g_{e_2 e_1}\, g_{e_1 e_0} = F_{e_0} F_{e_{n-1}}^{-1} \cdots F_{e_2} F_{e_1}^{-1}\, F_{e_1} F_{e_0}^{-1} = \mathbb{1}. \tag{3}$$

Hence the holonomy of the bare comparator vanishes on every loop, and the corresponding Wilson trace, though gauge invariant, carries no curvature.

The proof is the telescoping of adjacent factors. Proposition 1 is the combinatorial counterpart of the statement that a pure-gauge connection is flat, and it forces a choice: curvature must be carried by the morphisms at the events themselves. One therefore equips each event with a vertex transport $K_v \in G$, so that the transport across the event from e to e' is

$$U_{e'e}(v) \;=\; F_{e'}\, K_v\, F_e^{-1}. \tag{4}$$

The holonomy of a loop is then conjugate to the ordered product $\prod_v K_v$ of the vertex transports it threads; curvature lives at the interaction events and nowhere else. This is precisely the Regge philosophy [9], in which the simplices are flat and the deficit is concentrated at the hinges, transported here from the dual lattice to the matter graph itself (Fig. 1).

3 Spacetime-algebra representation of the Lorentz sector

3.1 The vertex as a versor

At an event the matter field is rotated by an $\mathrm{SL}(2,\mathbb{C})$ transformation, which in the Pauli basis reads [8]

$$\hat{A} \;=\; a_0 \mathbb{1} + \sum_{k=1}^{3} a_k\, \sigma_k, \qquad \det \hat{A} = a_0^2 - \sum_k a_k^2 = 1, \qquad a_\mu \in \mathbb{C}. \tag{5}$$

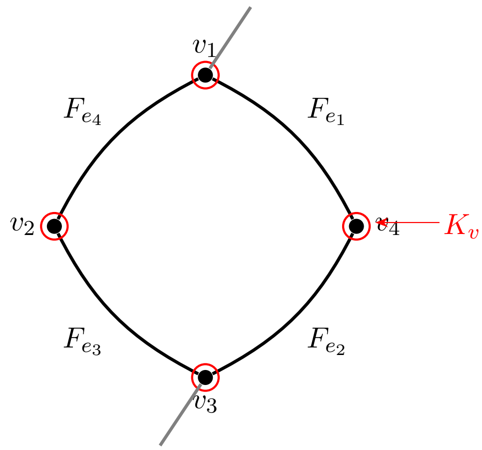

Figure 1: A fragment of a matter spin network. Filled dots are interaction events (morphisms); thick lines are fermion-line segments carrying frames F_e. The bare frame comparators $F_{e'}F_e^{-1}$ telescope to the identity around the loop (Proposition 1); the curvature is carried by the vertex transports K_v (red), so the loop holonomy is conjugate to $\prod_v K_v$.

We pass to the spacetime algebra $\mathrm{Cl}(1,3)$, the real Clifford algebra of Minkowski space with $\gamma_0^2 = +1$, $\gamma_i^2 = -1$ [10, 11]. Its even subalgebra $\mathrm{Cl}^+(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$, with the relative (boost) bivectors $\hat\sigma_k = \gamma_k\gamma_0$ ($\hat\sigma_k^2 = +1$) playing the role of the Pauli matrices and the volume element $I_3 = \hat\sigma_1\hat\sigma_2\hat\sigma_3$, equal to the pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$, playing the role of the imaginary unit ($I^2 = -1$, I central in $\mathrm{Cl}^+(1,3)$). The matrix (5) corresponds to the even multivector

$$R \;=\; \Re(a_0) + \Im(a_0)\, I + \sum_k \Re(a_k)\, \hat\sigma_k + \sum_k \Im(a_k)\, I\hat\sigma_k, \tag{6}$$

a sum of scalar, bivector and pseudoscalar parts.

Proposition 2 (Versor normalisation)**.** *R in (6) is a versor of $\mathrm{Spin}^+(1,3)$, that is $R\widetilde{R} = 1$, if and only if $\det \hat A = 1$. The unit-determinant condition of (5) is the versor norm of geometric algebra.*

3.2 Tetrad transport: three expressions, one map

Following [8] one attaches a tetrad leg e^0 to the first fermion of a loop and transports it by the vertex matrix, $u^2 = \hat A u^1$, so that the next leg is $e^1_\rho = e^0_\mu \Lambda^\mu{}_\rho$, with the Lorentz matrix $\Lambda(a)$ given in closed form by Rühl [12]:

$$\begin{aligned}
&\Lambda^0{}_0 = |a_0|^2 + \textstyle\sum_k |a_k|^2, && \Lambda^k{}_0 = a_0\bar a_k + \bar a_0 a_k + \imath\,\epsilon_{klm} a_l \bar a_m,\\
&\Lambda^0{}_k = a_0\bar a_k + \bar a_0 a_k - \imath\,\epsilon_{klm} a_l \bar a_m, && \Lambda^l{}_k = \delta^l_k\Big(|a_0|^2 - \textstyle\sum_j |a_j|^2\Big) + a_k\bar a_l + \bar a_k a_l + \imath\,\epsilon_{klm}(\bar a_0 a_m - a_0 \bar a_m).
\end{aligned} \tag{7}$$

In the spacetime algebra the same transport is the versor sandwich $e^1 = R\, e^0\, \widetilde{R}$. A third route reads Λ off the Hermitian-matrix map $X \mapsto \hat A X \hat A^\dagger$ on $X = x^\mu\sigma_\mu$.

Proposition 3 (Coincidence of the three transports)**.** *For every $\hat A \in \mathrm{SL}(2,\mathbb{C})$ with associated versor R and parameters a, the three Lorentz matrices—the trace form $\Lambda^\mu{}_\nu = \frac{1}{2}\operatorname{Tr}(\sigma_\mu \hat A \sigma_\nu \hat A^\dagger)$, the Rühl form (7), and the sandwich form $R\gamma_\nu\widetilde{R} = \Lambda^\mu{}_\nu\,\gamma_\mu$—are equal and lie in $\mathrm{SO}^+(1,3)$.*

Remark 1 (A parity pitfall). The trace form requires the *same* basis $\sigma_\mu = (\nVdash, +\sigma_i)$ both to embed $X = x^\mu \sigma_\mu$ and to extract $x'^\mu = \frac{1}{2}\operatorname{Tr}(\sigma_\mu X')$. The choice $(\nVdash, -\sigma_i)$ in the extraction yields an *improper* transform with $\det \Lambda = -1$ that nonetheless preserves the metric, $\Lambda^\top \eta \Lambda = \eta$. Metric preservation alone does not certify a proper Lorentz transformation; one must also verify $\det \Lambda = +1$.

3.3 Areas as simple bivectors

With tetrad legs in hand, the elementary area associated with two consecutive edges of a loop is the bivector [13, 8]

$$F^{[ij]} \;=\; \tfrac{1}{2}\, e^{[i,j]} \wedge e^{[i+1,j]}, \qquad e_r^{[i+1,j]} = e_m^{[i,j]} \Lambda^m{}_r. \tag{8}$$

As a wedge of two vectors $F^{[ij]}$ is a *simple* grade-two blade, $F \wedge F = 0$, and its magnitude $|\langle F\widetilde{F}\rangle_0|^{1/2}$ is a Lorentz scalar, invariant under the sandwich $F \mapsto RFR^{-1}$. The areas are honest oriented plane elements of $\mathrm{Cl}(1,3)$, the natural carriers of the discrete geometry.

4 Curvature as a bivector and the Lorentz logarithm

By (4) the holonomy of a loop is a versor R; its curvature is the bivector $B = \log R$, the deficit concentrated at the loop. A Lorentz bivector is characterised by the two invariants in the scalar and pseudoscalar parts of its square,

$$B^2 \;=\; s + p\, I, \qquad s = \langle B^2 \rangle_0, \quad p\, I = \langle B^2 \rangle_4, \tag{9}$$

which are the discrete analogues of the two relativistic invariants of the field strength. The classification is

$$\begin{aligned} &p = 0 : \text{simple: } s < 0 \text{ elliptic (rotation)},\ s > 0 \text{ hyperbolic (boost)},\ s = 0 \text{ parabolic (null)};\\ &p \neq 0 : \text{loxodromic: a commuting rotation} + \text{boost.} \end{aligned} \tag{10}$$

A simple closed form for the exponential and logarithm follows from the structure of the centre.

Proposition 4 (Lorentz logarithm via the centre)**.** *The centre of* $\mathrm{Cl}^+(1,3)$ *is* $\{1, I\} \cong \mathbb{C}$ *under* $I \leftrightarrow \imath$*, with* I *central among even elements. Every Lorentz bivector is a single complex multiple of one unit blade,*

$$B = \zeta\, \hat{F}, \qquad \zeta = a + bI \in \{1, I\}, \quad \hat{F}^2 = -1, \tag{11}$$

and consequently

$$\exp(B) = \cos\zeta + \hat{F}\sin\zeta, \qquad \cos\zeta = \langle R\rangle_0 + \langle R\rangle_4, \quad \sin\zeta\, \hat{F} = \langle R\rangle_2, \tag{12}$$

the trigonometric functions being evaluated as ordinary complex functions of ζ*. The logarithm therefore exists in closed form for the generic loxodromic rotor, the branch point being* $R = -1$ *(a* 2π *rotation), where it is multivalued.*

The canonical commuting split is read directly from ζ,

$$B = B_{\mathrm{e}} + B_{\mathrm{h}}, \qquad B_{\mathrm{e}} = \Re(\zeta)\, \hat{F} \ (\text{elliptic},\ B_{\mathrm{e}}^2 \le 0), \quad B_{\mathrm{h}} = \Im(\zeta)\, I\hat{F} \ (\text{hyperbolic},\ B_{\mathrm{h}}^2 \ge 0), \tag{13}$$

with $[B_{\mathrm{e}}, B_{\mathrm{h}}] = 0$. The invariants (s, p) are conjugation invariant, so $2|B_{\mathrm{e}}|$ and $2|B_{\mathrm{h}}|$ recover the deficit angle and the boost rapidity of the loop. Figure 2 shows the classification in the (s, p) plane.

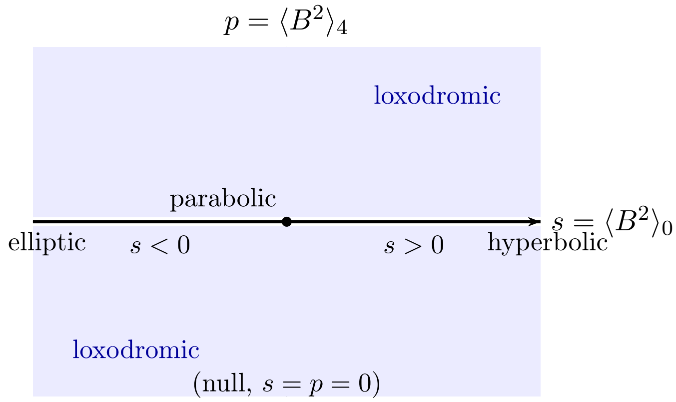

Figure 2: Classification of a loop's curvature bivector B by the two invariants of $B^2 = s + pI$. On the axis $p = 0$ the bivector is simple—a pure rotation ($s < 0$), a pure boost ($s > 0$), or null ($s = p = 0$). Off the axis it is loxodromic, a commuting rotation and boost, the generic case resolved by the split (13).

5 The matter path integral

5.1 A bilinear action and an exact determinant

On a fermion loop with N interaction vertices the matter Lagrangian is discretised by replacing the loop derivative with a finite difference [8]. Taking the fields at successive events as independent Grassmann variables one may eliminate the explicit connections and write the action as

$$S \;=\; \sum_k \Big[\tfrac{1}{2}\big(\bar{u}^k u^{k+1} - \bar{u}^{k+1} u^k\big) + \imath M\, \bar{u}^k u^k \Big] \;=\; \bar{u}^k D_{kl}\, u^l, \tag{14}$$

a bilinear form in the anticommuting fields with kernel the discrete Dirac operator D. The Berezin integral of a bilinear Grassmann action is the determinant of its kernel; hence the partition function is exact and finite,

$$Z \;=\; \int \prod_k \mathrm{d}\bar{u}^k\, \mathrm{d}u^k\; e^{\,\bar{u} D u} \;=\; \det D. \tag{15}$$

For a single loop, $D = \imath M \nVdash + \frac{1}{2}(S - S^{-1})$ with S the cyclic shift, and the eigenvalues are organised by the loop's Fourier modes $u(s) = \sum_m C_m e^{\imath m s}$, the integer m being the conserved charge of the U(1) loop-translation symmetry [8]:

$$\det D \;=\; \prod_{m=0}^{N-1} \Big(\imath M + \imath \sin \tfrac{2\pi m}{N} \Big). \tag{16}$$

The loop has N modes and, by the Bohr–Sommerfeld condition $\oint p\, \mathrm{d}x = 2\pi\hbar N$, a length $\lambda = N\ell_0$ and a Compton mass $M = h/(\lambda c)$: more interaction events make a longer, lighter mode.

5.2 Integrating over matter is integrating over geometry

The covariant version of (14) carries a comparator U_e on each edge, and the determinant is gauge invariant. Its value therefore depends only on the gauge-invariant content of the geometry—the holonomies—a statement we make precise in the next section. This is the dynamical face of Altaisky's claim that, by virtue of $u^{k+1} = \hat{A}^k u^k$, summing over all matter configurations is summing over all geometries [8]: the fermion determinant (15) is a functional of the connection alone.

6 The graph Dirac operator and the interaction action

6.1 Definition and the cycle-space reduction

We promote (14) from a single loop to the whole triangulation graph. The gauge-covariant graph Dirac operator acts on $\bigoplus_{v\in V}\mathbb{C}^n$ by

$$(Du)_v \;=\; \imath M\,u_v + \tfrac{1}{2}\sum_{e=(v\to w)} U_e\,u_w - \tfrac{1}{2}\sum_{e=(w\to v)} U_e^{-1}u_w, \tag{17}$$

which reduces to the loop operator on a single cycle. Under $u_v \mapsto g_v u_v$, $U_e \mapsto g_a U_e g_b^{-1}$ one has $D \mapsto GDG^{-1}$ with $G = \bigoplus_v g_v$, so $\det D$ is gauge invariant.

Proposition 5 (Cycle-space dependence)**.** *$\det D$ depends on the link variables only through the holonomies of a basis of the cycle space of Γ, of dimension $b_1 = |E| - |V| + 1$. In particular, on a tree ($b_1 = 0$) every link is pure gauge and $\det D$ is independent of the U_e; gauge-fixing a spanning tree to the identity turns the b_1 chord links into the fundamental-cycle holonomies.*

6.2 Non-factorisation as interaction

When two loops are disjoint, D is block diagonal and the determinant factorises, $Z = Z_1Z_2$: the loops do not interact. When they share an interaction event, the shared vertex couples their hopping terms and the determinant ceases to factorise. The connected part,

$$S_{\text{int}} \;=\; -\log\frac{Z_{\text{coupled}}}{Z_1\,Z_2}, \tag{18}$$

is the gauge-invariant interaction action carried by the shared event—the discrete counterpart of the statement that geometry couples matter loops. It vanishes for disjoint loops and grows monotonically with the strength of the shared transport.

6.3 The toy universe

The smallest closed example is the tetrahedron K_4: four events, six fermion-line segments, the boundary of a 3-simplex, topologically S^2 [8]. It has $b_1 = 3$ independent cycles, a well-defined partition function $Z = \det D$ and effective action $S_{\text{eff}} = -\log|Z|$, and its curvature is distributed over the three fundamental cycles. For generic link data each fundamental holonomy is loxodromic, so the closed-form logarithm of Proposition 4 is precisely what reads the curvature of the toy universe.

7 The induced discrete field equation

7.1 Variation of the effective action

The matter determinant induces an effective action for the geometry,

$$S_{\text{eff}} \;=\; -\log|\det D|, \tag{19}$$

in the spirit of Sakharov's induced gravity [14]. We ask whether its stationarity is a field equation. Varying a link by a Lie-algebra rotation $\delta U_c = U_c\,\varepsilon$, $\varepsilon \in \mathfrak{sl}(2,\mathbb{C})$, and using $\delta\log\det D = \operatorname{tr}(D^{-1}\delta D)$ together with the block structure $D_{ab} = \tfrac{1}{2}U_c$, $D_{ba} = -\tfrac{1}{2}U_c^{-1}$ on the edge $c = (a \to b)$, one finds

$$\delta\log\det D \;=\; \operatorname{tr}\big[\varepsilon\,J_c\big], \qquad J_c = \tfrac{1}{2}\Big((D^{-1})_{ba}\,U_c + U_c^{-1}\,(D^{-1})_{ab}\Big). \tag{20}$$

Stationarity for all ε is the vanishing of the trace-free part of the current.

Proposition 6 (The discrete field equation)**.** *The stationarity condition $\delta S_{\rm eff} = 0$ is, on each independent cycle c,*

$$\big[J_c\big]_{\rm traceless} \;=\; 0, \qquad J_c = \tfrac{1}{2}\Big((D^{-1})_{ba}\,U_c + U_c^{-1}\,(D^{-1})_{ab}\Big), \tag{21}$$

a determined system of $6\,b_1$ real equations for the $6\,b_1$ real parameters of the cycle holonomies. It is built from the fermion propagator D^{-1}—the matter two-point function—so the connection on each cycle is fixed by the matter current it carries. Under a gauge transformation the current conjugates, $J_c \mapsto g_b\,J_c\,g_b^{-1}$, so the equation is gauge covariant and its solution set is gauge invariant.

7.2 The vacuum and its signature

Proposition 7 (Flat vacuum)**.** *Flat geometry—all holonomies trivial—solves the homogeneous field equation* (21) *identically: it is the discrete Minkowski vacuum. Every pure-gauge configuration (trivial holonomy, nontrivial links) solves it as well.*

The character of the vacuum is read from the Hessian of $S_{\rm eff}$ at the flat point, equivalently the Jacobian of (21), which is symmetric. For the tetrahedron its 18×18 spectrum has signature $(9, 9)$; for the four-simplex K_5 (five events, ten lines, $b_1 = 6$, the first genuinely four-dimensional vertex) the 36×36 spectrum has signature $(18, 18)$. In each case the equal split of positive and negative directions is the boost–rotation balance of the Lorentzian geometry.

Proposition 8 (Lorentzian saddle)**.** *The flat vacuum is a non-degenerate saddle of $S_{\rm eff}$, with Hessian of split signature equal to the number of boost and rotation directions of the cycle space. It is a critical point but not an extremum; the negative directions are the boosts.*

7.3 Matter sources curvature

A vertex mass is internally isotropic and does not deform the vacuum; genuine sources are anisotropic in the internal space. Fixing the holonomies of some cycles as boundary data and leaving the remainder dynamical turns (21) into a boundary-value problem. With nontrivial boundary data the flat bulk is no longer a solution, and the field equation has a curved solution in which the dynamical cycles acquire curvature in response to the boundary (Fig. 3). This holds on both the tetrahedron and the four-simplex, and is the field-equation form of Altaisky's thesis that geometry is a function of matter: given the matter configuration on part of the network, the induced field equation determines the geometry of the rest.

8 Discussion

We have turned the kinematic matter-spin-network reformulation of loop quantum gravity into a theory with dynamics. The Lorentz sector is the versor geometry of $\mathrm{Cl}(1, 3)$; the curvature of a loop is a bivector with a closed-form logarithm valid in the generic loxodromic case; the matter path integral is the exact fermion determinant $Z = \det D$, a functional of the cycle-space holonomies; loops that share interaction events fail to factorise, and that failure is the interaction action; and the induced effective action carries a local, gauge-covariant field equation whose vacuum is a Lorentzian saddle and whose boundary-value problem lets matter source curvature.

Three boundaries deserve emphasis. First, the field equation is of induced type: it expresses what the matter determinant demands of the geometry it feels, in the manner of Sakharov [14], and

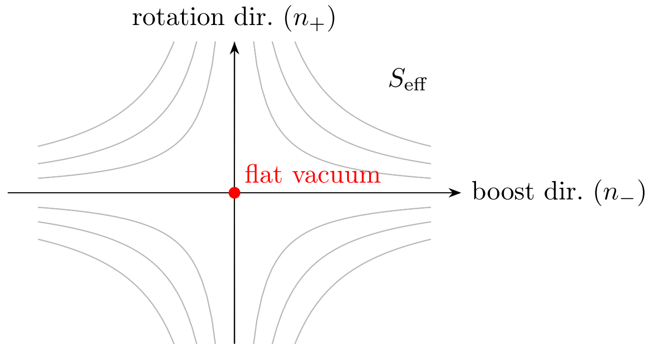

Figure 3: Schematic of the effective action near the flat vacuum. The Hessian is indefinite with equal numbers of positive (rotation) and negative (boost) directions, so flat is a Lorentzian saddle (Proposition 8). Fixing boundary holonomies displaces the critical point along the negative directions, sourcing curvature in the bulk.

we do not claim it reduces to the Einstein equations in any limit; establishing or refuting a general-relativistic continuum limit is the central open problem. Second, the construction is a quantum geometry but not yet a quantised spin foam: it has a partition function and a propagator, but no Hilbert space of spin-network states and no Engle–Pereira–Rovelli–Livine vertex amplitude [2]. The natural bridge is the relation between the bivector areas (8), which here are unconstrained, and the simplicity constraints that select gravitational from topological bivectors in spin-foam models [13]. Third, the saddle character of the vacuum (Proposition 8) raises the question Altaisky leaves open: whether a refinement or coarse-graining of the network, with the number of degrees of freedom playing the role of an evolution parameter, defines a renormalisation-group flow of S_{eff} and selects the geometry dynamically. The negative directions of the Hessian are the natural candidates to drive such a flow.

What the construction establishes is deliberately modest but nontrivial: local gauge covariance, a Lorentzian tetrad geometry, an exact partition function, and a nontrivial field equation can be carried, without contradiction and without approximation, by a finite combinatorial substrate of interacting matter loops, with no differentiable manifold beneath it. The most basic assumption of field theory—that there is a smooth place for things to happen—turns out to be dispensable, with the relations among matter taken as primary and the geometry reconstructed from them.

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