

Octonionic Projection, Vacuum Closure, and Fundamental Constants

Ruediger Giesel^{a,*}

^a*Independent Researcher, Stuttgart, Germany, 20-06-2026*

Abstract

We formulate a compact effective-field-theory version of an octonionic projection model for the origin and calibration of fundamental constants. The model treats the constants not as isolated inputs, but as effective quantities arising from emergent spacetime projection, quantum phase normalization, and dynamical vacuum selection inside the non-associative octonion algebra. The non-associative sector is governed by an octonionic field triplet and its associator energy. Within this framework, the speed of light is interpreted as the conversion between projected time and space units, the quantum of action as the conversion between action normalization and quantum phase, Newton's constant as the inverse stiffness of the projected vacuum, and the cosmological constant as residual vacuum curvature. Gauge couplings are represented as eigenvalues of a vacuum-dependent kinetic tensor rather than fixed bare group numbers. An explicit Fano-plane calculation supplies algebraic normalization references and shows why a strongly non-associative vacuum would generate a Planck-scale curvature contribution, whereas the observed smallness of the cosmological constant requires a nearly associative physical vacuum. The report provides a numerical-closure step based on a controlled one-parameter vacuum deformation and an exact calibration condition for Newton's constant. It does not claim that all observed constants follow from the Fano plane alone; instead, it identifies the vacuum, projection, spectral and renormalization data required for parameter-free prediction.

Keywords: octonions, non-associative geometry, physical constants, vacuum selection

*Corresponding author.

Email address: rdigergiesel@yahoo.com (Ruediger Giesel)

1. Introduction

The numerical values of dimensional physical constants cannot be derived from a mathematical structure alone without a convention for units; the relevant measured input values and recommended constants are conventionally fixed by precision data compilations [17, 13]. A theory may nevertheless explain why constants enter in particular combinations, why certain dimensionless ratios are small or large, and how measured constants emerge from a deeper set of normalization, projection, and vacuum data. This distinction is essential for any attempt to connect fundamental constants to a pregeometric algebraic framework.

This article develops such a formulation for an octonionic projection model. The octonions $\mathbb{O}$ form the largest real normed division algebra. Their automorphism group is G_2 , and their multiplication is non-associative but alternative. These properties have long motivated applications to particle physics, exceptional structures, and non-associative formulations of quantum theory [1, 2, 3, 4]. In the present framework, the physical four-dimensional world is not identified with the entire seven-dimensional imaginary octonionic sector. Instead, the observed Lorentzian, quantum, and gauge structures are effective projections of a deeper octonionic field system.

The guiding question is the following. Can the constants

$$c, \quad \hbar, \quad G, \quad \Lambda, \quad e, \quad g_2, \quad g_3, \quad m_f, \tag{1}$$

be understood as consequences of an octonionic projection mechanism? The answer proposed here is affirmative in a structural sense: these quantities can be assigned definite origins as projection, normalization, coupling, and vacuum-eigenvalue data. However, their measured SI values or low-energy values are not obtained from the algebra alone. The numerical values require a dynamical vacuum solution, canonical normalization, and renormalization-group evolution to the scale at which the constants are measured.

This distinction leads to a conservative and testable formulation. The algebra fixes exact associator identities and constrains possible projection sectors. The effective action fixes dimensions and stability conditions. The vacuum equations select the physical associative or nearly associative subspace. The fluctuation spectrum supplies thresholds. Finally, the renormalization-group flow maps the high-scale projection couplings to low-energy observables.

1.1. Relation to other octonionic approaches

Several octonionic approaches to fundamental physics have been developed. Furey's programme derives Standard-Model-like particle and charge structures from division-algebraic ideals [5], while Dray and Manogue emphasize Lorentzian and spinorial structures over the octonions [4]. Singh's programme is closer to the present work, since it also aims at deriving dimensionless constants and particle mass ratios from octonionic or exceptional-algebraic structures [6, 7]. The mechanism proposed here is different. Fundamental constants are not obtained from the characteristic equation of the exceptional Jordan algebra or from trace dynamics, but from a vacuum-projection mechanism. A dynamically selected nearly associative octonionic vacuum fixes the gravitational scale; the residual associator norm contributes to the cosmological constant; and the eigenvalues of a vacuum-dependent gauge-kinetic tensor determine the bare gauge couplings. Renormalization-group flow then connects these projection-scale quantities to low-energy observables. The advantage of this formulation is that the derivation of constants is cast as an effective-field-theory problem. Algebraic projection, vacuum selection, residual non-associativity and empirical calibration remain separated. This makes the framework modular, transparent and testable: Newton's constant measures vacuum stiffness, the cosmological constant measures residual non-associativity, and gauge couplings arise from the projected kinetic tensor.

2. Octonionic algebra and associator sector

Let

$$\mathbb{O} = \mathbb{R} \oplus \text{Im } \mathbb{O}, \quad \text{Im } \mathbb{O} = \text{span}\{e_1, \dots, e_7\}. \quad (2)$$

We use the multiplication convention

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k, \quad (3)$$

where f_{ijk} are completely antisymmetric octonionic structure constants. The oriented Fano-plane convention used in this work is

$$(123), (145), (176), (246), (257), (347), (365), \quad (4)$$

with $f_{ijk} = +1$ for the listed oriented triples and their cyclic permutations.

The associator is

$$[x, y, z] = (xy)z - x(yz), \quad (5)$$

and vanishes whenever x, y, z lie in a common associative quaternionic subalgebra. The model uses a triplet of octonionic fields,

$$\Psi_I(x) \in \mathbb{O}, \quad I = 1, 2, 3, \quad (6)$$

rather than a single field, because alternativity implies $[\Psi, \Psi, \Psi] = 0$. A non-trivial local associator energy therefore requires three generally independent octonionic directions.

The basic non-associative invariant is

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) = [\Psi_1, \Psi_2, \Psi_3], \quad V_A = \lambda \|\mathcal{A}\|^2. \quad (7)$$

The sign condition

$$\lambda \geq 0 \quad (8)$$

is required for stability of the associator potential in the absence of compensating higher-order terms. In four spacetime dimensions, if the projected scalar field dimension is $[\Psi_I] = 1$, then $[\mathcal{A}] = 3$, $[\|\mathcal{A}\|^2] = 6$, and

$$[\lambda] = -2, \quad \lambda = \frac{\eta_A}{M_*^2}, \quad (9)$$

where η_A is dimensionless and M_* is the non-associative excitation scale.

Definition 2.1 (Real component representation). *Write*

$$\Psi_I = \psi_{I0} + \sum_{a=1}^7 \psi_{Ia} e_a, \quad \psi_{IA} \in \mathbb{R}, \quad A = 0, \dots, 7. \quad (10)$$

The real inner product is defined by

$$\langle x, y \rangle = \text{Re}(x\bar{y}), \quad \|x\|^2 = \langle x, x \rangle. \quad (11)$$

All functional derivatives below are understood as ordinary real derivatives with respect to the components ψ_{IA} . This avoids ambiguities associated with differentiating non-associative products.

Lemma 2.2 (First variation of the associator energy). *Let*

$$A = [\Psi_1, \Psi_2, \Psi_3]. \quad (12)$$

Then the first variation of $\|A\|^2$ is

$$\delta\|A\|^2 = 2\langle A, [\delta\Psi_1, \Psi_2, \Psi_3] \rangle + 2\langle A, [\Psi_1, \delta\Psi_2, \Psi_3] \rangle + 2\langle A, [\Psi_1, \Psi_2, \delta\Psi_3] \rangle. \quad (13)$$

Thus the associator contribution to the Euler–Lagrange equation is a real-linear adjoint of the trilinear maps

$$L_1(\delta X) = [\delta X, \Psi_2, \Psi_3], \quad L_2(\delta X) = [\Psi_1, \delta X, \Psi_3], \quad L_3(\delta X) = [\Psi_1, \Psi_2, \delta X]. \quad (14)$$

Proof. The associator is real trilinear, hence δA is the sum of the three displayed variations. Since the octonionic norm is induced by the real inner product, $\delta\|A\|^2 = 2\langle A, \delta A \rangle$. $\square$

Proposition 2.3 (Necessity of triplet fields). *A single octonionic field cannot generate a nonzero local associator self-energy of the form $\|[\Psi, \Psi, \Psi]\|^2$.*

Proof. The algebra $\mathbb{O}$ is alternative, hence the subalgebra generated by any two elements is associative and in particular $[x, x, x] = 0$ for all $x \in \mathbb{O}$. Therefore $[\Psi, \Psi, \Psi] = 0$ pointwise. Three generally independent fields are the minimal field content for a generic nonzero associator invariant. $\square$

3. Effective action and projection chain

The effective projected action used in this section follows from the minimal invariance, locality, norm-positivity, and non-associativity requirements summarized in Appendix A. With those assumptions, the projected low-energy action is taken to be

$$S_{\text{eff}} = S_0 \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa_{\mathbb{O}}} (R - 2\Lambda_0) - \sum_{I=1}^3 \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle - V_{\text{loc}}(\Psi_I) - \lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 - \xi R \sum_I \langle \Psi_I, \Psi_I \rangle \right]. \quad (15)$$

Here S_0 is the fundamental action normalization, $\kappa_{\mathbb{O}}$ is the dimensionless or model-normalized gravitational stiffness parameter, α_I are kinetic normalizations, and ξ is a curvature-coupling parameter. The projection chain is schematically

$$\mathbb{O} \longrightarrow \mathbb{H}_{\text{phys}} \longrightarrow \mathbb{C} \longrightarrow (M_4, g_{\mu\nu}). \quad (16)$$

The first step selects a preferred associative quaternionic sector; the second selects the complex phase used in quantum mechanics; the third identifies the Lorentzian spacetime projection. The geometric language of bundles, projections, and symmetry reduction follows the standard differential-geometric setting [12, 8].

Definition 3.1 (Projection datum). *A projection datum for the low-energy theory is the tuple*

$$\Pi = (P_{\mathbb{H}}, P_{\mathbb{C}}, u, \Theta, \mu), \quad (17)$$

where $P_{\mathbb{H}} : \mathbb{O} \rightarrow \mathbb{H}_{\text{phys}}$ is an idempotent real-linear projector onto an associative quaternionic subalgebra, $P_{\mathbb{C}} : \mathbb{H}_{\text{phys}} \rightarrow \mathbb{C}$ selects a complex unit, u is the emergent clock direction, Θ is the phase map, and μ is the renormalization scale. Physical constants are functions of Π and of the selected vacuum Ψ_0 .

Proposition 3.2 (Dimensional normalization). *If S_0 is the action normalization, $\ell_{\mathbb{O}}$ and $\tau_{\mathbb{O}}$ are the projected length and time units, and Φ_0 is the vacuum norm, then all dimensionful constants may be written as products of these three primitive scales and dimensionless projection factors. Consequently, only dimensionless combinations such as α_i , m_f/m_P , and $\Lambda \ell_P^2$ are invariant predictions of the model.*

The variation of Eq. (15) gives an octonionic field equation of the form

$$\alpha_I \square \Psi_I - \frac{\partial V_{\text{loc}}}{\partial \overline{\Psi}_I} - \lambda \frac{\partial}{\partial \overline{\Psi}_I} \|[\Psi_1, \Psi_2, \Psi_3]\|^2 - 2\xi R \Psi_I = 0, \quad (18)$$

where

$$\frac{\partial}{\partial \overline{\Psi}_I} \|[\Psi_1, \Psi_2, \Psi_3]\|^2 \quad (19)$$

is shorthand for the real gradient defined by Eq. (13). The metric variation gives the Einstein equation with additional stress-energy contributions from kinetic, local, associator, and nonminimal-curvature sectors. The purely algebraic contribution from the associator potential contains

$$T_{\mu\nu}^{(A)} = -g_{\mu\nu} \lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 \equiv -g_{\mu\nu} V_A \quad (20)$$

when no derivative dependence is present.

4. Dimensional constants as projection data

4.1. Speed of light

The speed of light arises as the conversion factor between the time and space units selected by the Lorentzian projection. Let u^μ denote the preferred clock direction and P_3 the spatial associative sector. The emergent line element is written as

$$ds^2 = -c^2 dt^2 + h_{ij} dx^i dx^j. \quad (21)$$

If $\ell_\mathbb{O}$ and $\tau_\mathbb{O}$ are the fundamental projected length and time units, then

$$c = \frac{\ell_\mathbb{O}}{\tau_\mathbb{O}}. \quad (22)$$

Thus c is not a bare algebraic number in the model. It is the conversion rate imposed by the emergent Lorentzian projection and the chosen units of length and time.

4.2. Planck constant

The quantum sector is obtained after selecting a complex subalgebra. A projected state has phase

$$\psi = |\psi| e^{i\theta}, \quad \theta = \frac{S_{\text{phys}}}{\hbar}. \quad (23)$$

The Planck constant is therefore the conversion factor between projected physical action and complex phase. With S_0 the fundamental octonionic action scale,

$$\hbar = S_0 Z_\hbar, \quad (24)$$

where $Z_\hbar$ is the dimensionless phase-projection normalization. The Born rule is then associated with the norm projection

$$\|\psi\|^2 = \sum_n |c_n|^2, \quad P_n = |c_n|^2. \quad (25)$$

4.3. Newton constant

The gravitational sector of Eq. (15) is compared with the Einstein-Hilbert action in SI normalization,

$$S_{\text{EH}} = \int d^4x \sqrt{-g} \frac{c^4}{16\pi G} (R - 2\Lambda). \quad (26)$$

Comparison with the first term in Eq. (15) gives

$$\frac{S_0}{2\kappa_{\mathbb{O}}} = \frac{c^4}{16\pi G}, \quad G = \frac{\kappa_{\mathbb{O}} c^4}{8\pi S_0}. \quad (27)$$

If the octonionic vacuum norm is

$$\Phi_0^2 = \sum_{I=1}^3 \langle \Psi_{I,0}, \Psi_{I,0} \rangle, \quad (28)$$

and $\kappa_{\mathbb{O}} \propto \Phi_0^{-2}$, then G^{-1} measures the stiffness of the projected vacuum. A large vacuum norm corresponds to a stiff spacetime projection and therefore a weak gravitational coupling.

4.4. Numerical closure condition for Newton's constant

The previous subsection identifies Newton's constant as the inverse stiffness of the projected vacuum. This statement becomes numerically exact only after the coefficient of the Ricci scalar has been fixed by the octonionic vacuum itself. In this subsection we work first in natural units, $c = \hbar = 1$, and restore SI normalization only for comparison with measured values.

Let the curvature-coupling functional multiplying the Ricci scalar be denoted by $F(\Psi)$. A dimensionally consistent minimal form is

$$F(\Psi) = \xi \sum_{I=1}^3 \langle \Psi_I, \Psi_I \rangle + \zeta \frac{\|[\Psi_1, \Psi_2, \Psi_3]\|^2}{M_*^4} + \dots \quad (29)$$

Here $[\Psi_I] = 1$, so the associator norm squared has mass dimension six and the denominator must be M_*^4 in order for $F(\Psi)$ to have mass dimension two. The effective gravitational action contains

$$S_R = \int d^4x \sqrt{-g} F(\Psi) R. \quad (30)$$

Matching to the Einstein-Hilbert normalization in natural units gives the exact closure condition

$$\boxed{\frac{1}{16\pi G} = F(\Psi_0) = \xi\Phi_0^2 + \zeta\frac{\mathcal{A}_0^2}{M_*^4} + \dots} \quad (31)$$

where

$$\Phi_0^2 = \sum_{I=1}^3 \langle \Psi_{I,0}, \Psi_{I,0} \rangle, \quad \mathcal{A}_0^2 = \|[\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}]\|^2. \quad (32)$$

Thus the model predicts G if and only if the vacuum norm, the nonminimal coupling, the associator correction, and the threshold scale are determined internally by the vacuum equations. If these quantities are instead fixed by imposing the observed value of G , then Eq. (31) is a calibration condition rather than a prediction.

In a nearly associative vacuum the associator term is subleading. Then

$$\boxed{G \simeq \frac{1}{16\pi\xi\Phi_0^2}} \quad (33)$$

up to corrections of order $\zeta\mathcal{A}_0^2/(\xi\Phi_0^2M_*^4)$. Restoring SI units gives

$$\frac{c^4}{16\pi G} = F_{\text{SI}}(\Psi_0), \quad F_{\text{SI}}(\Psi_0) = c^4 F(\Psi_0) \quad (34)$$

for the normalization used in Eq. (26). The section therefore separates the two logically different possibilities:

$$\text{internal solution for } (\Phi_0, \xi, \zeta, M_*, \mathcal{A}_0) \Rightarrow \text{prediction of } G, \quad (35)$$

whereas

$$G \text{ inserted to reconstruct } \Phi_0 \Rightarrow \text{calibration of the vacuum scale.} \quad (36)$$

Proposition 4.1 (Prediction versus calibration of Newton's constant). *Within the octonionic projection model, Newton's constant is numerically predicted precisely when the coefficient $F(\Psi_0)$ of the emergent Ricci scalar is computed from the same octonionic vacuum equations that determine the projection sector. If $F(\Psi_0)$ is adjusted to match the measured value of G , the construction remains internally consistent but does not constitute a parameter-free derivation of Newton's constant.*

4.5. Cosmological constant

At a homogeneous vacuum $\Psi_I = \Psi_{I,0}$, the effective vacuum energy density is

$$\rho_{\text{vac}} = V_{\text{loc}}(\Psi_0) + \lambda \|\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}\|^2. \quad (37)$$

The resulting curvature contribution is

$$\Lambda = \frac{8\pi G}{c^4} \rho_{\text{vac}} \quad (38)$$

in the convention where ρ_{vac} is an energy density. In natural units this is $\Lambda = \kappa \rho_{\text{vac}}$ with $\kappa = 8\pi G$. Thus Λ is the residual curvature of the projected octonionic vacuum. The associator term provides a possible non-associative vacuum contribution, but the observed smallness of $\Lambda \ell_P^2$ strongly constrains it.

5. Exact Fano-plane calculation and vacuum implication

Using the Fano convention (4), one finds

$$[e_1, e_2, e_3] = 0, \quad (39)$$

because $\{1, e_1, e_2, e_3\}$ is a quaternionic subalgebra. For the non-quaternionic triple (e_1, e_2, e_4) ,

$$(e_1 e_2) e_4 = e_3 e_4 = e_7, \quad e_1 (e_2 e_4) = e_1 e_6 = -e_7, \quad (40)$$

and therefore

$$[e_1, e_2, e_4] = 2e_7, \quad \|[e_1, e_2, e_4]\|^2 = 4. \quad (41)$$

Theorem 5.1 (Elementary associator dichotomy). *For three distinct imaginary basis units e_i, e_j, e_k in the convention (4), the associator vanishes if the triple lies on one oriented Fano line, and otherwise has norm 2 up to sign and basis direction. Hence*

$$\|[e_i, e_j, e_k]\|^2 \in \{0, 4\}. \quad (42)$$

Proof. If e_i, e_j, e_k lie on a common Fano line, they generate a quaternionic subalgebra and associativity gives zero. If they do not lie on a common line, the two bracketings follow two different Fano products and differ by two opposite imaginary units. Direct inspection of the seven oriented triples in Eq. (4), together with total antisymmetry, gives $[e_i, e_j, e_k] = \pm 2e_\ell$ for the complementary basis direction e_ℓ . $\square$

This is an exact algebraic result. It shows that the elementary basis associators take discrete algebraic values: the quaternionic basis triple is exactly associative, while the non-quaternionic basis triple has squared associator norm four. For general real linear combinations of octonionic directions the associator amplitude is continuous, but Eq. (41) provides an exact normalization reference for the non-associative sector.

For the vacuum it is useful to separate amplitudes from unit octonionic directions. We write

$$\Psi_{I,0} = \phi_I n_I, \quad \langle n_I, n_I \rangle = 1, \quad \Phi_0^2 = \sum_{I=1}^3 \phi_I^2. \quad (43)$$

The vacuum associator is then

$$A_0 = [\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}] = \phi_1 \phi_2 \phi_3 [n_1, n_2, n_3], \quad (44)$$

and the dimensionless associator amplitude is defined by

$$\epsilon_A^2 = \frac{\|A_0\|^2}{\Phi_0^6}. \quad (45)$$

This definition is invariant under an overall rescaling of the vacuum amplitudes. For equal amplitudes, $\phi_1 = \phi_2 = \phi_3 = \phi$ and $\Phi_0^2 = 3\phi^2$, the elementary non-associative Fano triple gives

$$\epsilon_A^2 = \frac{\phi^6 \|[e_1, e_2, e_4]\|^2}{(3\phi^2)^3} = \frac{4}{27}. \quad (46)$$

If instead one uses the unnormalized directional reference $n_1 = e_1$, $n_2 = e_2$, $n_3 = e_4$ without dividing by the total triplet norm, the corresponding basis associator norm is the exact value 4 in Eq. (41). The distinction is purely a convention and must be kept fixed when comparing dimensionless vacuum amplitudes.

The dimensionless vacuum-curvature relation may be written as

$$\Lambda \ell_P^2 = C_\Lambda \left(v_0 + \eta_A \epsilon_A^2 \right), \quad (47)$$

where v_0 denotes the dimensionless local-potential contribution, η_A the dimensionless associator strength, and C_Λ specifies the Planck-density convention. In the reduced Planck-density convention used for the estimates below,

$C_\Lambda = 1/(8\pi)$. Other Planck-unit conventions simply rescale v_0 and η_A . If the vacuum were strongly non-associative, Eq. (41) or Eq. (46) would give an order-unity associator contribution, and for $\eta_A \sim 1$ one would obtain a Planck-scale curvature contribution. Since cosmological observations imply an extremely small value of $\Lambda\ell_P^2$ [18], the physical vacuum must be nearly associative,

$$\epsilon_A \ll 1, \quad v_0 + \eta_A \epsilon_A^2 \ll 1. \quad (48)$$

If the local contribution is cancelled or symmetry-suppressed, the estimate becomes

$$\epsilon_A \sim \eta_A^{-1/2} \left(\frac{\Lambda\ell_P^2}{C_\Lambda} \right)^{1/2}. \quad (49)$$

For the reduced Planck-density convention $C_\Lambda = 1/(8\pi)$ this is $\epsilon_A \sim \eta_A^{-1/2} (8\pi\Lambda\ell_P^2)^{1/2}$. For the observed dark-energy scale one has, order-of-magnitude, $\Lambda\ell_P^2 \sim 10^{-122} - 10^{-123}$. Therefore, if v_0 is absent, cancelled, or protected by a symmetry, the nearly associative deviation must satisfy

$$\epsilon_A \sim 10^{-61} \eta_A^{-1/2}. \quad (50)$$

This is an extremely small number, but it is technically natural in the t'Hooft sense if the limit $\epsilon_A \rightarrow 0$ restores exact quaternionic associativity and an enhanced projection symmetry [19]. Thus the smallness of the cosmological constant is not removed by the present model; rather, it is translated into the dynamical requirement that the physical vacuum be aligned to an associative sector with extraordinary precision.

This converts the cosmological-constant problem into a vacuum-alignment problem: the selected physical vacuum must lie extremely close to an associative quaternionic sector.

Corollary 5.2 (Vacuum-alignment bound). *Assume $v_0 = 0$ and $C_\Lambda = 1/(8\pi)$. Then*

$$\Lambda\ell_P^2 \leq 10^{-122} \implies \epsilon_A \leq \sqrt{8\pi} 10^{-61} \eta_A^{-1/2}. \quad (51)$$

Thus a natural value $\eta_A = \mathcal{O}(1)$ forces the normalized associator of the vacuum to be smaller than the elementary basis value by roughly sixty-one decimal orders.

6. Numerical closure from a nearly associative vacuum

The preceding estimate can be sharpened by choosing a concrete one-parameter deformation away from an associative quaternionic vacuum. This subsection is the simplest place where the model gains numerical force: the small observed value of $\Lambda\ell_P^2$ is converted into a directly calculable distance from the associative subspace.

Choose the unit vacuum directions

$$n_1 = e_1, \quad n_2 = e_2, \quad n_3(\varepsilon) = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4, \quad 0 \leq \varepsilon \ll 1. \quad (52)$$

The parameter ε measures the component of the third vacuum direction outside the quaternionic subalgebra

$$\mathbb{H}_{123} = \text{span}\{1, e_1, e_2, e_3\}. \quad (53)$$

By trilinearity of the associator and by Eqs. (39)–(41),

$$[n_1, n_2, n_3(\varepsilon)] = \sqrt{1 - \varepsilon^2} [e_1, e_2, e_3] + \varepsilon [e_1, e_2, e_4] \quad (54)$$

$$= 2\varepsilon e_7. \quad (55)$$

Therefore

$$\|[n_1, n_2, n_3(\varepsilon)]\|^2 = 4\varepsilon^2. \quad (56)$$

This result is exact for the ansatz (52); it is not a small- ε approximation.

In the directional normalization of Eq. (56), the dimensionless vacuum-curvature relation becomes

$$\Lambda\ell_P^2 = \frac{1}{8\pi} (v_0 + 4\eta_A \varepsilon^2). \quad (57)$$

If the local contribution is absent, symmetry-cancelled, or absorbed into a zero-curvature reference vacuum, $v_0 = 0$, then

$$\varepsilon = \sqrt{\frac{2\pi}{\eta_A} \Lambda\ell_P^2} \quad (58)$$

so that the observed dark-energy scale implies

$$\varepsilon \simeq \sqrt{\frac{2\pi}{\eta_A}} 10^{-61} \quad (59)$$

up to the conventional order-of-magnitude range in $\Lambda\ell_P^2$. Equivalently, if the deformation is generated dynamically by the ratio of a symmetry-breaking misalignment parameter ζ to an alignment stiffness ϵ ,

$$\varepsilon = \frac{\zeta}{\epsilon} + O\left(\frac{\zeta^2}{\epsilon^2}\right), \quad (60)$$

then Eq. (57) gives

$$\Lambda\ell_P^2 = \frac{\eta_A}{2\pi} \left(\frac{\zeta}{\epsilon}\right)^2 + O\left(\frac{\zeta^3}{\epsilon^3}\right) \quad (61)$$

for $v_0 = 0$. Thus the numerical smallness of the cosmological constant is equivalent, in this minimal deformation sector, to the hierarchy

$$\frac{\zeta}{\epsilon} \simeq \sqrt{\frac{2\pi}{\eta_A}} 10^{-61}. \quad (62)$$

Proposition 6.1 (Numerical vacuum-closure criterion). *For the nearly associative vacuum family (52), the model predicts the exact relation (57). Consequently, once η_A and v_0 are fixed by the vacuum potential, $\Lambda\ell_P^2$ is no longer a free parameter. In the special protected case $v_0 = 0$ and $\eta_A = O(1)$, the observed value of $\Lambda\ell_P^2$ is equivalent to a vacuum misalignment of order 10^{-61} from the associative quaternionic sector.*

This section increases the numerical content of the model in a minimal and falsifiable way. The algebraic input is fixed by the Fano-plane calculation, the deformation ansatz is explicit, and the output is the dimensionless number ε required by the observed vacuum curvature. The remaining dynamical problem is to derive this value of ε from the parameters of V_{eff} rather than to impose it.

7. Gauge couplings as vacuum-dependent kinetic eigenvalues

A purely group-theoretic embedding cannot by itself determine the observed low-energy values of e , g_2 , and g_3 . In the present formulation, gauge couplings are eigenvalues of a vacuum-dependent kinetic tensor. Let T_a be generators acting on the projected octonionic field space. Define

$$K_{ab}(\Psi_0) = \langle T_a \Psi_0, T_b \Psi_0 \rangle. \quad (63)$$

Quantity	Exact expression in the deformation model	Numerical implication
Vacuum direction	$n_3 = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4$	one-parameter misalignment
Associator	$[n_1, n_2, n_3] = 2\varepsilon e_7$	exact
Associator norm	$\ [n_1, n_2, n_3]\ ^2 = 4\varepsilon^2$	exact
Vacuum curvature	$\Lambda \ell_P^2 = (v_0 + 4\eta_A \varepsilon^2)/(8\pi)$	model output after v_0, η_A
Protected case	$\varepsilon = \sqrt{2\pi \Lambda \ell_P^2 / \eta_A}$	$\varepsilon \sim 10^{-61} \eta_A^{-1/2}$

Table 1: Numerical closure from a nearly associative vacuum. The table makes explicit which number is calculated by the associator algebra and which remaining quantities must be supplied by the dynamical vacuum potential.

The gauge kinetic term then has the form

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} K_{ab}(\Psi_0) F_{\mu\nu}^a F^{b\mu\nu}. \quad (64)$$

After diagonalization and canonical normalization,

$$K v_i = k_i v_i, \quad g_i^{-2}(\mu_0) = Z_i k_i, \quad \alpha_i(\mu_0) = \frac{g_i^2(\mu_0)}{4\pi}. \quad (65)$$

Here Z_i encodes the chosen generator and field normalization; in a convention with canonically normalized generators one may set $Z_i = 1$. The earlier projection norms $\mathcal{N}_i$ are therefore not arbitrary constants but normalized eigenvalues of $K_{ab}(\Psi_0)$:

$$\mathcal{N}_i(\Psi_0) = Z_i k_i. \quad (66)$$

Proposition 7.1 (Canonical coupling extraction). *Let $K(\Psi_0)$ be positive definite on a projected simple gauge factor G_i and let T_a be normalized by $\text{tr}(T_a T_b) = \delta_{ab}/2$. After the field redefinition $A_\mu^a = (K^{-1/2})^a_b \hat{A}_\mu^b$, the inverse squared coupling multiplying the canonically normalized curvature is the corresponding eigenvalue of K . Therefore distinct low-energy couplings can arise from a single octonionic kinetic form only if the vacuum breaks the kinetic tensor into nondegenerate projected blocks.*

For the electromagnetic coupling,

$$e = g_{\mathbb{O}} \|P_{\mathbb{C}} T_{U(1)}\|, \quad \alpha = \frac{e^2}{4\pi\hbar c} \quad (67)$$

Quantity	Bare minimal projection	Low-energy target	Interpretation
$\ [e_1, e_2, e_3] \ ^2$	0	associative sector	exact
$\ [e_1, e_2, e_4] \ ^2$	4	non-associative sector	exact
$\sin^2 \theta_W$	$3/8 = 0.375$	≈ 0.231 at M_Z	running required
α_s/α	$8/3 \approx 2.67$	≈ 15.1 at M_Z	thresholds/RG
Λ_P^2	$\mathcal{O}(10^{-1})$ for strong associator	$\sim 10^{-122}$	nearly associative vacuum

Table 2: Numerical diagnostic consequences of the minimal octonionic projection before dynamical vacuum renormalization. The first two rows are exact algebraic results. The remaining rows show that bare projection norms cannot be identified with measured electroweak-scale quantities without renormalization-group evolution and threshold corrections.

in unit conventions without explicit ϵ_0 . For the weak and strong sectors,

$$g_2 = g_{\mathbb{O}} \|P_{\mathbb{H}} T_{SU(2)}\|, \quad g_3 = g_{\mathbb{O}} \|P_{SU(3)} T_{G_2}\|. \quad (68)$$

Here $SU(3)$ should be understood as a stabilizer subgroup of a selected imaginary octonionic direction, while the full Standard Model group is not a single direct subgroup of one G_2 because G_2 has rank 2 whereas $SU(3)_c \times SU(2)_L \times U(1)_Y$ has rank 4 [8, 9, 10]. The correct interpretation is therefore that the Standard Model gauge sectors arise as compatible projected sectors, not as one simultaneous direct embedding inside a single G_2 .

8. Bare projection diagnostics for gauge couplings

The preceding construction also clarifies what the minimal algebraic projection does *not* predict. Introduce bare projection normalizations $\mathcal{N}_i$ by

$$\alpha_i^{(0)} = \frac{g_{\mathbb{O}}^2}{4\pi\mathcal{N}_i}. \quad (69)$$

A grand-unification-like high-scale normalization may be written as

$$\mathcal{N}_1 : \mathcal{N}_2 : \mathcal{N}_3 = \frac{5}{3} : 1 : 1. \quad (70)$$

With the usual electroweak relation

$$\sin^2 \theta_W^{(0)} = \frac{\mathcal{N}_2}{\mathcal{N}_1 + \mathcal{N}_2}, \quad (71)$$

Eq. (70) gives

$$\sin^2 \theta_W^{(0)} = \frac{3}{8} = 0.375. \quad (72)$$

This is the familiar high-scale value in simple unified normalizations, not the measured electroweak-scale value [9, 10]. Conversely, using the approximate measured value $\sin^2 \theta_W(M_Z) \simeq 0.23122$ [13] gives the effective low-energy ratio

$$\frac{\mathcal{N}_1(M_Z)}{\mathcal{N}_2(M_Z)} = \frac{1 - \sin^2 \theta_W(M_Z)}{\sin^2 \theta_W(M_Z)} \simeq 3.325. \quad (73)$$

Thus a bare high-scale ratio $\mathcal{N}_1 : \mathcal{N}_2 = 5/3 : 1$ is not the same object as the renormalized electroweak-scale ratio.

The same point appears in the strong sector. With $\mathcal{N}_3 = 1$ and Eq. (70), the minimal bare ratio is

$$\frac{\alpha_s^{(0)}}{\alpha^{(0)}} = \frac{\mathcal{N}_1 + \mathcal{N}_2}{\mathcal{N}_3} = \frac{8}{3}. \quad (74)$$

At the Z scale, however, $\alpha(M_Z)^{-1} \simeq 127.95$ and $\alpha_s(M_Z) \simeq 0.118$ [13, 17], giving

$$\frac{\alpha_s(M_Z)}{\alpha(M_Z)} \simeq 15.1. \quad (75)$$

Combining this with Eq. (73), and setting $\mathcal{N}_2(M_Z) = 1$ for comparison, gives the approximate effective norm pattern

$$\mathcal{N}_1(M_Z) : \mathcal{N}_2(M_Z) : \mathcal{N}_3(M_Z) \simeq 3.325 : 1 : 0.287. \quad (76)$$

This numerical diagnostic is useful because it identifies precisely where the model must do dynamical work: the effective projection norms must be eigenvalues of $K_{ab}(\Psi_0)$ and must be evolved from the octonionic scale to M_Z .

9. Renormalization-group flow and thresholds

The measured coupling constants are low-energy quantities, and their comparison with high-scale projection data requires the standard renormalization-group framework of quantum field theory [11, 10]. If $M_{\mathbb{O}}$ is the octonionic

projection scale and μ the experimental scale, then

$$\mu \frac{dg_i}{d\mu} = \beta_i(g_1, g_2, g_3, \lambda, y_f, \dots). \quad (77)$$

At one loop, in terms of inverse fine-structure constants,

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M_{\mathbb{O}}) + \frac{b_i}{2\pi} \ln \frac{M_{\mathbb{O}}}{\mu} + \sum_r \frac{\Delta b_i^{(r)}}{2\pi} \ln \frac{M_{\mathbb{O}}}{M_r}. \quad (78)$$

The threshold masses M_r are not additional arbitrary parameters once the vacuum potential is fixed; the use of such threshold corrections follows the usual decoupling logic for heavy states [20]. More generally, when kinetic mixing is present one evolves the full matrix $K_{ab}(\mu)$,

$$\mu \frac{dK_{ab}}{d\mu} = \mathcal{B}_{ab}(K, Y, \lambda, \Psi_0), \quad (79)$$

and only afterwards diagonalizes the projected blocks. Equation (78) is the diagonal one-loop limit of this matrix equation. The threshold masses are eigenvalues of the Hessian of the effective potential,

$$(M^2)_{Ia,Jb} = \left. \frac{\partial^2 V_{\text{eff}}}{\partial \Psi_{Ia} \partial \Psi_{Jb}} \right|_{\Psi=\Psi_0}, \quad M^2 w_r = M_r^2 w_r. \quad (80)$$

Thus the renormalization structure is dynamically linked to the selected octonionic vacuum.

The Standard Model one-loop coefficients alone do not generally produce exact unification of the three observed low-energy couplings [11, 10]. A useful diagnostic is obtained from the one-loop coefficients

$$b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7, \quad (81)$$

with $SU(5)$ -normalized hypercharge. Using approximate M_Z inputs, the α_1 - α_2 crossing occurs around

$$M_{12} \sim 10^{13} \text{ GeV}, \quad \alpha_{12}^{-1} \sim 42, \quad (82)$$

whereas the corresponding one-loop extrapolation of the strong coupling does not meet the same point. Equivalently, running back to M_Z from a single

common coupling would yield a strong coupling of order $\alpha_s(M_Z) \sim 0.07$, rather than the observed value near 0.118. This is not a failure of the projection idea; rather, it identifies the need for threshold states, non-universal projection renormalization, or additional sectors. In the present framework, those corrections must be computed from Eq. (80) and the representations of the resulting fluctuation modes under the projected gauge sectors.

10. Fermion masses as vacuum and associator eigenvalues

Fermion masses are represented as eigenvalues of vacuum-dependent Yukawa operators. Let χ_f denote projected fermionic modes. Define

$$Y_{fg}(\Psi_0) = \langle \chi_f, Y(\Psi_0) \chi_g \rangle. \quad (83)$$

Then

$$M_{fg} = \frac{v_{\text{eff}}}{\sqrt{2}} Y_{fg}(\Psi_0), \quad U_L^\dagger M U_R = \text{diag}(m_1, m_2, m_3). \quad (84)$$

In an associator-eigenmode picture one introduces the linearized operator

$$\mathcal{A}_{\text{lin}} \chi_n = a_n \chi_n, \quad n = 1, 2, 3. \quad (85)$$

The generation structure may then be represented as a spectrum of associator modes. This statement is a hypothesis until the operator $\mathcal{A}_{\text{lin}}$ is derived from a fully specified vacuum potential and shown to yield the observed hierarchies. A phenomenological reconstruction of the a_n from known masses is not a prediction. A prediction requires computing a_n from Eq. (80) or an equivalent linearized associator operator.

As a diagnostic, one may parameterize a charged-lepton hierarchy by

$$\frac{m_n}{m_1} = \exp[\beta_\ell(a_n - a_1)]. \quad (86)$$

For $m_e = 0.511 \text{ MeV}$, $m_\mu = 105.66 \text{ MeV}$, and $m_\tau = 1776.86 \text{ MeV}$ [13], one obtains

$$\ln \frac{m_\mu}{m_e} \simeq 5.331, \quad \ln \frac{m_\tau}{m_e} \simeq 8.154. \quad (87)$$

If $a_1 = 0$ and $a_2 = 1$ are used as a normalization convention, the third eigenvalue reconstructed from the charged-lepton hierarchy is

$$a_3 \simeq \frac{8.154}{5.331} \simeq 1.53. \quad (88)$$

This reconstruction is not a derivation of the lepton masses; it is a target spectrum for the linearized associator operator. The model becomes predictive only if $\mathcal{A}_{\text{lin}}$ yields such a hierarchy without fitting it to the observed masses.

11. Dynamical vacuum selection and renormalization structure

The preceding sections show that numerical predictions require a dynamical prescription for Ψ_0 rather than a chosen basis triplet. A mathematically closed model is specified by the quadruple

$$\mathfrak{M} = (\mathbb{O}, f_{ijk}, V_{\text{eff}}, \Pi), \quad (89)$$

where f_{ijk} fixes the Fano convention, V_{eff} fixes vacuum selection, and Π fixes the low-energy projection datum. The vacuum is the solution set

$$\mathcal{V} = \{\Psi_0 \in \mathbb{O}^3 \mid dV_{\text{eff}}(\Psi_0) = 0, \ d^2V_{\text{eff}}(\Psi_0) \geq 0\}. \quad (90)$$

A physical vacuum is an element of $\mathcal{V}$ that also satisfies the near-associativity and phenomenological coupling constraints.

Definition 11.1 (Predictive closure). *The model is predictively closed at one loop if the following data are computable from V_{eff} and Π without fitting low-energy constants:*

$$\Psi_0, \quad K_{ab}(\Psi_0), \quad \text{Spec } \mathcal{M}^2(\Psi_0), \quad \Delta b_i^{(r)}, \quad Y_{fg}(\Psi_0). \quad (91)$$

Then $\alpha_i(M_Z)$, m_f/m_P , and Λ_P^2 are outputs rather than inputs.

Theorem 11.2 (Minimal conditions for numerical predictivity). *A bare octonionic projection model determines numerical low-energy constants only if:*

1. *the vacuum equations determine Ψ_0 up to gauge-equivalent degeneracies;*
2. *the kinetic tensor $K_{ab}(\Psi_0)$ has identifiable $U(1)$, $SU(2)$, and $SU(3)$ projected blocks;*
3. *the Hessian spectrum supplies all threshold masses and representations;*
4. *the renormalization-group equations are integrated from M_0 to the measurement scale;*
5. *the local and associator vacuum energies determine Λ_P^2 in a fixed Planck convention.*

Without these ingredients, the theory provides structural origins of the constants but not numerical derivations.

Proof. The low-energy constants are functions of canonically normalized kinetic terms, vacuum eigenvalues, and scale-dependent effective couplings. Items (1)–(2) fix the high-scale boundary conditions, item (3) fixes the decoupling thresholds, item (4) maps the boundary conditions to the experimental scale, and item (5) fixes the dimensionless curvature. If any item is absent, at least one measured constant remains an adjustable input. $\square$

12. Concrete calculational program

A quantitative version of the model is determined by the following steps.

1. Fix the Fano-plane convention, for example Eq. (4), so that all products $e_i e_j$ and all associators are unambiguous.
2. Choose a stable effective potential. A minimal alignment model is

$$V_{\text{eff}} = \frac{g}{4} \left(\sum_I \|\Psi_I\|^2 - \Phi_0^2 \right)^2 + \lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 + \epsilon \sum_I \|(1 - P_{\mathbb{H}})\Psi_I\|^2 + \delta V_{\text{br}}. \quad (92)$$

Here $P_{\mathbb{H}}$ projects onto a preferred quaternionic subspace and δV_{br} introduces controlled departures from exact associativity.

3. Solve the vacuum equations

$$\frac{\partial V_{\text{eff}}}{\partial \Psi_{Ia}} = 0, \quad I = 1, 2, 3, \quad a = 0, \dots, 7, \quad (93)$$

for the 24 real vacuum components $\Psi_{Ia,0}$.

4. Compute the associator vacuum amplitude using the fixed normalization convention,

$$A_0 = [\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}], \quad \epsilon_A^2 = \frac{\|A_0\|^2}{\Phi_0^6}, \quad \Phi_0^2 = \sum_I \|\Psi_{I,0}\|^2. \quad (94)$$

For the equal-amplitude basis reference this convention gives $\epsilon_A^2 = 4/27$, while the raw elementary basis associator norm remains 4.

5. Compute the gauge kinetic tensor $K_{ab}(\Psi_0)$, diagonalize it, and extract the high-scale couplings $g_i(M_0)$.

6. Compute the fluctuation Hessian (80); classify the eigenmodes under the projected gauge sectors; compute their threshold coefficients $\Delta b_i^{(r)}$.
7. Integrate the renormalization-group equations to the measurement scale and compare with α , $\sin^2 \theta_W$, α_s , fermion masses, and cosmological constraints.

This program converts the structural octonionic model into a falsifiable effective theory. It also identifies which inputs are algebraic, which are dynamical, and which are phenomenological.

13. Complete calibration and numerical closure

The preceding sections define the dynamical route from octonionic algebra to observable constants. For a numerical statement one must also specify the complete calibration data at a reference scale. This section gives the combined formulation: it keeps the algebraic and dynamical derivation explicit, but it also states the finite data set that fixes the numerical constants of the effective four-dimensional theory.

Definition 13.1 (Complete octonionic calibration configuration). *A complete calibration configuration is the ordered tuple*

$$\mathcal{C}_{\text{full}} = (\mathbb{O}, \varphi, \mathbb{H}_{\text{grav}}, \mathbb{C}_{\text{qm}}, e_7, \Phi_0, \chi, \lambda, \xi, V_{\text{loc}}, \mathcal{P}_Y, \mathcal{P}_Q, \mathcal{A}, \mu_\star, \mathcal{R}_{\mu_\star}). \quad (95)$$

Here φ is the G_2 three-form determined by the Fano-plane orientation, $\mathbb{H}_{\text{grav}}$ is the associative quaternionic subalgebra used in the space-time projection, $\mathbb{C}_{\text{qm}}$ is the complex quantum-phase subalgebra, e_7 fixes the $SU(3)$ stabilizer sector, Φ_0 is the vacuum norm, χ is the Einstein-Hilbert projection coefficient, $\mathcal{P}_Y$ and $\mathcal{P}_Q$ are the hypercharge and electric-charge projectors, $\mathcal{A}$ is the associator Hessian in the matter sector, $\mu_\star$ is the reference scale, and $\mathcal{R}_{\mu_\star}$ denotes the renormalization conditions at that scale.

Theorem 13.2 (Determinacy inside the effective model). *Given $\mathcal{C}_{\text{full}}$ and the effective action (15), every constant appearing in the canonically normalized four-dimensional effective Lagrangian is fixed at the scale $\mu_\star$.*

Proof. The multiplication of $\mathbb{O}$ is fixed by φ , hence by the structure constants f_{ijk} . The Lorentzian, quantum and internal projections are fixed by $\mathbb{H}_{\text{grav}}$,

$\mathbb{C}_{\text{qm}}$ and the stabilizer direction e_7 . The coefficient of the Einstein-Hilbert term is fixed by the pair (Φ_0, χ) through

$$\frac{1}{2\kappa} = \frac{\chi}{2}\Phi_0^2, \quad G = \frac{1}{8\pi\chi\Phi_0^2} \quad (c = \hbar = 1). \quad (96)$$

The gauge kinetic terms are fixed by the vacuum-dependent kinetic tensor $K_{ab}(\Psi_0)$ and by the renormalization conditions contained in $\mathcal{R}_{\mu\star}$. Fermion masses are fixed by the Yukawa map built from the associator Hessian $\mathcal{A}$ and the corresponding matter projectors. The cosmological constant is fixed by the vacuum value of $V_{\text{loc}} + \lambda\|\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}\|^2$. Therefore no further numerical constant remains undetermined once the complete configuration has been specified. $\square$

Remark 13.3 (Calibration versus parameter-free prediction). The theorem does not claim that the Fano plane alone predicts all constants. It states a weaker but mathematically controlled result: a complete octonionic vacuum, together with its projector norms and renormalization conditions, fixes the constants. A parameter-free derivation would require additional dynamical theorems that determine Φ_0 , V_{loc} , the projector norms, the associator spectrum and the threshold spectrum from the algebra and the variational principle alone.

13.1. Numerical calibration layer

In SI units the constants c , $\hbar$, e , k_B and N_A are exact definitions of the unit system. They should therefore not be presented as independent predictions of the octonionic algebra. In the present framework they enter as exact conversion normalizations after the projection to the measured four-dimensional description. By contrast, G , α , α_s , electroweak couplings, fermion masses and Λ are empirical or scale-dependent quantities that must be matched by the vacuum and renormalization structure.

The Planck scales are derived once c , $\hbar$ and G are fixed:

$$\ell_{\text{P}} = \sqrt{\frac{\hbar G}{c^3}}, \quad t_{\text{P}} = \sqrt{\frac{\hbar G}{c^5}}, \quad m_{\text{P}} = \sqrt{\frac{\hbar c}{G}}. \quad (97)$$

For the canonical Einstein-Hilbert normalization $\chi = 1$, Eq. (96) gives

$$\Phi_0 = \overline{M}_{\text{P}} = \frac{1}{\sqrt{8\pi G}} \quad (98)$$

in natural units. Thus Newton’s constant is not generated by the Fano-plane signs alone; it is the inverse stiffness scale of the projected vacuum.

Table 3: Numerical constants in the combined formulation. “SI exact” means exact by unit definition; “derived” means obtained from other constants; “adjusted” or “scale-fixed” means empirical or renormalized at the specified scale.

Quantity	Octonionic role	Representative numerical value	Status
c	Lorentzian projection conversion	$299\,792\,458\,\text{m s}^{-1}$	SI exact
h	quantum phase normalization	$6.626\,070\,15 \times 10^{-34}\,\text{J s}$	SI exact
$\hbar$	$\hbar/(2\pi)$	$1.054\,571\,817 \times 10^{-34}\,\text{J s}$	derived exact
e	electric-charge normalization after projection	$1.602\,176\,634 \times 10^{-19}\,\text{C}$	SI exact
k_B	energy-temperature conversion for projected states	$1.380\,649 \times 10^{-23}\,\text{J K}^{-1}$	SI exact
N_A	mole normalization	$6.022\,140\,76 \times 10^{23}\,\text{mol}^{-1}$	SI exact
G	inverse Einstein-Hilbert vacuum stiffness $G = (8\pi\chi\Phi_0^2)^{-1}$	$6.674\,30(15) \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$	adjusted
$\alpha(0)$	electromagnetic projector normalization	$7.297\,352\,5643 \times 10^{-3}$	adjusted
$\alpha^{-1}(0)$	inverse electromagnetic coupling	$137.035\,999\,1776$	adjusted
ℓ_P	$\sqrt{\hbar G/c^3}$	$1.616\,255\,02 \times 10^{-35}\,\text{m}$	derived
t_P	$\sqrt{\hbar G/c^5}$	$5.391\,246\,45 \times 10^{-44}\,\text{s}$	derived
m_P	$\sqrt{\hbar c/G}$	$2.176\,434\,34 \times 10^{-8}\,\text{kg}$	derived
$\overline{M}_P c^2$	reduced vacuum-stiffness scale	$2.435\,323\,46 \times 10^{18}\,\text{GeV}$	derived
$g_Y(M_Z)$	hypercharge projector norm at M_Z	0.35741705	scale-fixed
$g_2(M_Z)$	weak quaternionic projector norm at M_Z	0.65172383	scale-fixed
$g_3(M_Z)$	color/stabilizer projector norm at M_Z	1.21719969	scale-fixed
v	electroweak vacuum normalization	$246.219\,650\,8\,\text{GeV}$	adjusted

Table 4: Structural origin of physical constants in the octonionic projection model.

Constant	Origin	Representative relation
c	Lorentzian space-time projection	$c = \ell_{\mathbb{O}}/\tau_{\mathbb{O}}$
$\hbar$	Action-to-phase normalization	$\hbar = S_0 Z_{\hbar}$
G	Inverse vacuum stiffness	$G = \kappa_{\mathbb{O}} c^4 / (8\pi S_0)$
Λ	Local plus associator vacuum curvature	$\Lambda = 8\pi G \rho_{\text{vac}} / c^4$
e	Complex $U(1)$ projection	$e = g_{\mathbb{O}} \ P_{\mathbb{C}} T_{U(1)}\ $
g_2	Quaternionic $SU(2)$ projection	$g_2 = g_{\mathbb{O}} \ P_{\mathbb{H}} T_{SU(2)}\ $
g_3	$SU(3)$ color projection/stabilizer sector	$g_3 = g_{\mathbb{O}} \ P_{SU(3)} T_{G_2}\ $
m_f	Vacuum-Yukawa eigenvalue	$M_{fg} = v_{\text{eff}} Y_{fg} / \sqrt{2}$
ℓ_P	Combined stiffness/action scale	$\ell_P = \sqrt{\hbar G / c^3}$
λ	Non-associative coupling	$\lambda = \eta_A / M_*^2$

Quantity	Octonionic role	Representative numerical value	Status
Λ	residual local plus associator vacuum curvature	$\sim 1.09 \times 10^{-52} \text{ m}^{-2}$	cosmological

This table is not a claim that all numerical values have already been derived parameter-free. It is a closure table: it shows where each number enters the effective theory and which part of the octonionic configuration must determine it in a fully predictive version.

14. Summary of constants

The central message of Table 4 is that the constants are not inserted at random. They correspond to definite projection or vacuum structures. The dimensionless measured quantities, such as α , α_s , m_f/m_P , and $\Lambda \ell_P^2$, are the true targets for numerical prediction.

15. Attempt at parameter-free numerical closure

This section makes explicit how far the present formulation can be pushed toward a parameter-free numerical derivation. The purpose is not to hide the

remaining assumptions, but to convert them into a finite set of calculable closure equations. A fully predictive version of the theory is obtained only if these equations determine all vacuum, projection, and running data without using the measured constants as inputs.

15.1. Minimal closure ansatz

We impose the following minimal closure assumptions.

- (i) The gravitational nonminimal coupling is fixed at the conformal infrared value,

$$\xi = \frac{1}{6}. \quad (99)$$

- (ii) The associator potential is canonically normalized at the projection scale,

$$\eta_A = 1, \quad \lambda = M_*^{-2}. \quad (100)$$

- (iii) The local vacuum contribution is protected by the associative limit,

$$v_0 = 0. \quad (101)$$

- (iv) The normalized vacuum is the smallest deformation of an associative quaternionic frame,

$$n_1 = e_1, \quad n_2 = e_2, \quad n_3 = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4. \quad (102)$$

- (v) The remaining dimensionless deformation ε must be obtained from the stationary equation of the same effective potential that fixes the vacuum.

With the Fano convention used in the main text, this gives exactly

$$[n_1, n_2, n_3] = 2\varepsilon e_7, \quad \|[n_1, n_2, n_3]\|^2 = 4\varepsilon^2. \quad (103)$$

Thus the protected-vacuum prediction for the dimensionless cosmological constant becomes

$$\Lambda \ell_P^2 = \frac{\varepsilon^2}{2\pi}. \quad (104)$$

Equation (104) is parameter-free once ε has been computed from the vacuum equations. Conversely, if ε is reconstructed from the observed value of $\Lambda \ell_P^2$, the equation is a calibration relation rather than a prediction.

15.2. Vacuum equation for the misalignment

To make the preceding statement calculational, introduce the one-dimensional effective potential obtained by inserting the above nearly associative ansatz into the full vacuum functional,

$$U(\varepsilon) = V_{\text{eff}}(n_1, n_2, n_3(\varepsilon)). \quad (105)$$

The misalignment is then fixed by

$$\frac{dU}{d\varepsilon} = 0, \quad \frac{d^2U}{d\varepsilon^2} > 0. \quad (106)$$

For example, a symmetry-protected expansion near the associative point has the form

$$U(\varepsilon) = U_0 + 2\eta_A \varepsilon^2 + u_4 \varepsilon^4 - s\varepsilon + O(\varepsilon^6), \quad (107)$$

where the linear coefficient s parametrizes a small explicit breaking of the associative vacuum alignment. Equation (106) gives, to leading order,

$$\varepsilon = \frac{s}{4\eta_A} + O(s^3). \quad (108)$$

Consequently,

$$\Lambda \ell_P^2 = \frac{s^2}{32\pi\eta_A^2} + O(s^4). \quad (109)$$

This is the simplest explicit route to a numerical prediction: the theory must compute the small symmetry-breaking coefficient s . If s is inserted by hand, the smallness of Λ is not explained; if s follows from a fixed-point, instanton, anomaly, or spectral condition of the octonionic vacuum, Eq. (109) becomes a genuine prediction.

15.3. Newton constant in the minimal closure

With the same closure assumptions, the Newton coupling reduces to a vacuum-stiffness relation. In natural units,

$$\frac{1}{16\pi G} = \xi \Phi_0^2 + \frac{\zeta \mathcal{A}_0^2}{M_*^4}, \quad (110)$$

where $\mathcal{A}_0^2 = \|[\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}]\|^2$. For the nearly associative vacuum, the associator correction is suppressed by ε^2 . If $\xi = 1/6$ and the correction term is negligible, one obtains

$$\Phi_0^2 = \frac{3}{8\pi G}, \quad \Phi_0 = \sqrt{6} \bar{M}_P, \quad (111)$$

where $\overline{M}_P = (8\pi G)^{-1/2}$ is the reduced Planck mass in natural units. Equation (111) is not yet a prediction of G ; it shows the unique vacuum scale required once the conformal curvature coupling is adopted. It becomes predictive only if Φ_0 is independently obtained from the minimum of the octonionic potential.

15.4. Gauge-coupling closure equations

The corresponding parameter-free target for the gauge sector is the eigenvalue problem

$$K_{ab}(\Psi_0)v_i^b = \kappa_i v_i^a, \quad \frac{1}{g_i^2(M_\odot)} = \kappa_i. \quad (112)$$

Given a vacuum solution Ψ_0 , Eq. (112) contains no adjustable gauge couplings. The observed electroweak- and strong-scale constants are then obtained by integrating

$$\alpha_i^{-1}(M_Z) = \alpha_i^{-1}(M_\odot) + \frac{b_i}{2\pi} \ln \frac{M_\odot}{M_Z} + \sum_r \frac{\Delta b_i^{(r)}}{2\pi} \ln \frac{M_\odot}{M_r}. \quad (113)$$

The masses M_r must be the eigenvalues of the Hessian of the same vacuum potential,

$$\mathcal{M}^2 w_r = M_r^2 w_r. \quad (114)$$

Thus a complete numerical derivation of α , $\sin^2 \theta_W$, and α_s is equivalent to the coupled spectral problem

$$V_{\text{eff}} \longrightarrow \Psi_0, \quad K_{ab}(\Psi_0), \quad \mathcal{M}^2(\Psi_0) \longrightarrow \alpha_i(M_Z). \quad (115)$$

15.5. Obstruction and remaining calculational task

The above construction also gives a precise obstruction theorem.

Proposition 15.1 (No constants from Fano signs alone). *The finite Fano-plane multiplication table cannot by itself determine the continuous observed values of G , $\Lambda \ell_P^2$, α , $\sin^2 \theta_W$, α_s , or m_f/m_P . It can determine exact algebraic normalizations, such as $\| [e_1, e_2, e_4] \|^2 = 4$, but the numerical constants require continuous vacuum data, kinetic eigenvalues, and renormalization-scale information.*

Proof. The Fano-plane data form a finite discrete multiplication table. They can produce discrete algebraic constants and selection rules. The measured

dimensionless constants, however, form nontrivial real numbers depending on vacuum amplitudes, eigenvalues of real kinetic operators, and logarithmic scale evolution. Therefore the Fano table alone cannot supply the required continuous data. A numerical prediction requires the stationary solution of a real-valued effective potential and the spectra of the associated Hessian and kinetic tensors. $\square$

Therefore the present attempt reduces the problem to a finite, well-defined computation. The constants become parameter-free precisely when the same master action determines

$$\Psi_0, \quad \varepsilon, \quad \Phi_0, \quad K_{ab}(\Psi_0), \quad \mathcal{M}^2(\Psi_0), \quad \mathcal{A}_{\text{lin}}(\Psi_0) \quad (116)$$

without using the measured constants as boundary conditions. At the current stage, the cosmological-constant relation is the most explicit numerical closure, while the gauge and fermion sectors are reduced to concrete spectral problems.

16. Explicit computational implementation

The preceding section reduces the numerical problem to a finite set of closure equations. We now state the corresponding calculation in a form that can be implemented directly. This section does not introduce new physical input; it rewrites the vacuum, kinetic, Hessian, and renormalization problems as explicit finite-dimensional operations on the real components of three octonionic fields.

16.1. Real component representation

Write each octonionic field as

$$\Psi_I = \sum_{a=0}^7 \psi_{Ia} e_a, \quad I = 1, 2, 3, \quad (117)$$

where $e_0 = 1$ and e_a , $a = 1, \dots, 7$, obey the fixed Fano-plane multiplication table. The multiplication constants are defined by

$$e_a e_b = C_{ab}^c e_c, \quad (118)$$

with $C_{00}^0 = 1$, $C_{0a}^a = C_{a0}^a = 1$, $C_{ab}^0 = -\delta_{ab}$, and $C_{ab}^c = f_{abc}$ for $a, b, c \geq 1$. The associator components are therefore the explicit cubic polynomials

$$A^d(\Psi_1, \Psi_2, \Psi_3) = \psi_{1a} \psi_{2b} \psi_{3c} \left(C_{ab}^m C_{mc}^d - C_{bc}^m C_{am}^d \right), \quad (119)$$

where repeated indices are summed. Hence

$$\|[\Psi_1, \Psi_2, \Psi_3]\|^2 = \sum_{d=0}^7 A^d A^d \quad (120)$$

is an explicit sixth-order polynomial in the 24 real variables ψ_{Ia} . This is the algebraic core of the implementation.

16.2. Vacuum solver

Choose the minimal vacuum functional

$$\begin{aligned} V_{\min} = & \frac{g}{4} \left(\sum_{I=1}^3 \|\Psi_I\|^2 - \Phi_0^2 \right)^2 + \lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 \\ & + \mu_H^2 \sum_{I=1}^3 \|(1 - P_{\mathbb{H}})\Psi_I\|^2 - s \Phi_0^4 B(\Psi_1, \Psi_2, \Psi_3), \end{aligned} \quad (121)$$

where $P_{\mathbb{H}}$ projects onto the selected associative quaternionic subalgebra and B is a dimensionless alignment-breaking functional. The vacuum is the solution of the 24 real equations

$$\frac{\partial V_{\min}}{\partial \psi_{Ia}} = 0, \quad I = 1, 2, 3, \quad a = 0, \dots, 7. \quad (122)$$

For the nearly associative branch one may consistently reduce the system to

$$n_1 = e_1, \quad n_2 = e_2, \quad n_3 = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4. \quad (123)$$

Using $[e_1, e_2, e_3] = 0$ and $[e_1, e_2, e_4] = 2e_7$, the reduced potential has the form

$$U(\varepsilon) = U_0 + (2\eta_A + \bar{\mu}_H^2) \varepsilon^2 + u_4 \varepsilon^4 - s \varepsilon + O(\varepsilon^6), \quad (124)$$

where $\bar{\mu}_H^2$ denotes the dimensionless alignment stiffness after normalization by the vacuum scale. The stationary point is

$$\varepsilon = \frac{s}{2(2\eta_A + \bar{\mu}_H^2)} + O(s^3), \quad (125)$$

and therefore

$$\Lambda \ell_P^2 = \frac{\eta_A}{2\pi} \left[\frac{s}{2(2\eta_A + \bar{\mu}_H^2)} \right]^2 + O(s^4). \quad (126)$$

Equation (126) is the first explicit numerical implementation target: once the symmetry-breaking coefficient s and the alignment stiffness $\bar{\mu}_H^2$ are fixed by the same vacuum functional, the dimensionless cosmological constant is no longer an independent input.

16.3. Gauge-kinetic eigenvalue computation

Let T_a be a chosen real matrix representation of the projected internal generators acting on the eight-dimensional real vector of octonionic components. For the vacuum solution Ψ_0 , construct

$$K_{ab}(\Psi_0) = \sum_{I=1}^3 \langle T_a \Psi_{I,0}, T_b \Psi_{I,0} \rangle. \quad (127)$$

Then solve

$$K_{ab} v_i^b = \kappa_i v_i^a. \quad (128)$$

The bare projection-scale couplings are

$$g_i^{-2}(M_\mathbb{O}) = \kappa_i, \quad \alpha_i^{-1}(M_\mathbb{O}) = 4\pi\kappa_i. \quad (129)$$

For the nearly associative branch, the eigenvalues admit an expansion

$$\kappa_i = \kappa_i^{(0)} + \kappa_i^{(2)} \varepsilon^2 + O(\varepsilon^4), \quad (130)$$

because the sign of the small misalignment is physically unobservable in the quadratic kinetic form unless the breaking functional violates this parity. Thus the gauge-coupling calculation becomes a finite matrix diagonalization once the generator basis and the vacuum solution are specified.

16.4. Hessian spectrum and threshold masses

The threshold spectrum is obtained from the Hessian

$$\mathcal{M}_{Ia,Jb}^2 = \left. \frac{\partial^2 V_{\min}}{\partial \psi_{Ia} \partial \psi_{Jb}} \right|_{\Psi=\Psi_0}. \quad (131)$$

This is a real symmetric 24×24 matrix. Its eigenvalue equation is

$$\mathcal{M}^2 w_r = M_r^2 w_r. \quad (132)$$

The masses M_r are the threshold scales entering the running couplings. The implementation should classify the eigenvectors into radial, associative, non-associative, and alignment-breaking modes. Schematically,

$$\begin{aligned} \text{radial mode} & : M_\rho^2 \sim g\Phi_0^2, \\ \text{quaternionic orientation modes} & : M_H^2 \sim \mu_H^2, \\ \text{non-associative modes} & : M_A^2 \sim \lambda\Phi_0^4 + \mu_H^2, \\ \text{breaking mode} & : M_s^2 = M_A^2 + O(s). \end{aligned} \quad (133)$$

The precise coefficients are fixed by the component Hessian (131); they should not be treated as adjustable threshold parameters.

16.5. RG integration as final numerical map

The low-energy observables are obtained from

$$\alpha_i^{-1}(M_Z) = 4\pi\kappa_i + \frac{b_i}{2\pi} \ln \frac{M_{\mathbb{O}}}{M_Z} + \sum_r \frac{\Delta b_i^{(r)}}{2\pi} \ln \frac{M_{\mathbb{O}}}{M_r}. \quad (134)$$

This equation is no longer a formal statement if κ_i and M_r have been computed from Eqs. (128) and (132). The numerical comparison is then made with the three dimensionless targets

$$\alpha^{-1}(M_Z), \quad \sin^2 \theta_W(M_Z), \quad \alpha_s(M_Z). \quad (135)$$

A successful implementation must reproduce all three from the same vacuum and threshold spectrum. A failure would identify which of the assumptions in (121) must be changed.

16.6. Implementation checklist

The calculation can be carried out in the following finite sequence.

1. Fix the Fano structure constants C_{ab}^c .
2. Build the component associator polynomial from Eq. (119).
3. Specify $P_{\mathbb{H}}$, the alignment-breaking functional B , and the minimal potential (121).
4. Solve the 24 equations (122) for Ψ_0 , or solve the reduced branch (125) when the nearly associative ansatz is imposed.
5. Construct and diagonalize $K_{ab}(\Psi_0)$.
6. Construct and diagonalize the Hessian $\mathcal{M}^2(\Psi_0)$.
7. Insert the resulting κ_i and M_r into Eq. (134).
8. Compare the resulting dimensionless constants with experiment without refitting κ_i , M_r , or ε .

The report is fully parameter-free only if every item in this list is fixed by the master action and not by the observed constants themselves.

17. Predictive status and limitations

The present framework should be read as a closed determination scheme rather than as a completed parameter-free derivation of all constants. Its predictive content depends on which quantities are fixed internally by the octonionic dynamics and which are imposed as reference-scale calibration data. This distinction is summarized in Table 5.

Table 5: Predictive status of the constants in the present formulation.

Quantity	Status in the model	Required step for parameter-free prediction
c and $\hbar$	Projection and normalization scales for Lorentzian geometry and quantum phase	Internal normalization of the clock, length, and action units
G	Exact closure condition through vacuum stiffness and nonminimal curvature coupling	Internal determination of Φ_0 , ξ , ζ , M_* , and the vacuum associator amplitude
$\Lambda \ell_P^2$	Concrete numerical closure through near-associativity of the vacuum	Derivation of the small misalignment parameter from the vacuum potential
α , α_2 , α_s	Eigenvalues of the vacuum-dependent kinetic tensor followed by RG flow	Explicit computation and diagonalization of $K_{ab}(\Psi_0)$, thresholds, and running
m_f/m_P	Spectrum of vacuum-dependent Yukawa and associator operators	Explicit solution of the associator eigenvalue problem and flavor projectors

Thus the most concrete numerical result in the present version is the cosmological-constant closure from a nearly associative vacuum. Newton's constant is given by an exact calibration condition, while gauge couplings and fermion masses require further computation of the vacuum kinetic tensor, Hessian spectrum, threshold corrections, and associator eigenmodes. A future parameter-free version of the model must compute these objects from the same effective potential rather than reconstructing them from measured constants.

17.1. Submission-level scope of the result

The central claim of the paper is therefore deliberately limited. The work does not present a completed numerical derivation of all observed constants. It presents a closed computational framework in which the constants become fixed once the vacuum, projection tensors, associator spectrum, Hessian thresholds and renormalization map are determined internally. This restricted claim is important for two reasons. First, it prevents dimensional constants from being treated as pure algebraic numbers independent of unit conventions. Second, it makes the remaining tests precise: a successful parameter-free version must compute the vacuum solution, the kinetic eigenvalues, the threshold spectrum and the low-energy running without refitting them to the measured constants.

18. Discussion and limitations

The combined report gives exact algebraic information, such as Eq. (41), a consistent effective-field-theory route from an octonionic vacuum to low-energy constants, and a closed numerical calibration layer summarized in Table 3. It does not yet give a complete parameter-free numerical derivation of all constants. The missing ingredients are not conceptual placeholders but specific calculational objects: the full vacuum potential, the solution of the 24-component vacuum equations, the Hessian spectrum, the representation content of new modes, the projector norms, and the renormalization-group flow including thresholds.

This limitation is important. For example, a bare algebraic normalization may yield a high-scale value such as $\sin^2 \theta_W = 3/8$ in grand-unification-like constructions, but the observed electroweak-scale value is a renormalized quantity. Similarly, the strong coupling at the Z scale cannot be inferred from a bare G_2 projection alone; the effective low-energy norm pattern (76) shows explicitly how far the renormalized quantities are from the simplest bare normalization. Finally, the cosmological constant requires a nearly associative vacuum or a cancellation mechanism in $v_0 + \eta_A \epsilon_A^2$, with the limiting estimate (50) indicating the required alignment precision if no local contribution remains.

The proposed framework is therefore best understood as a constrained effective theory of projection and vacuum selection. Its value lies in making the required steps explicit and testable: compute the vacuum, compute its

associator amplitude, compute kinetic eigenvalues, compute thresholds, and compare the resulting low-energy constants with experiment.

19. Conclusions

We have presented how the principal physical constants can arise in an octonionic projection model, including both the dynamical vacuum-selection route and an explicit numerical calibration layer. The constants c and $\hbar$ originate as projection and normalization factors for Lorentzian geometry and complex quantum phase. Newton's constant measures the inverse stiffness of the projected octonionic vacuum. The cosmological constant is the residual curvature produced by local and associator vacuum energy. Gauge couplings are eigenvalues of a vacuum-dependent kinetic tensor, and fermion masses are eigenvalues of vacuum-dependent Yukawa and associator operators.

The exact Fano-plane computation

$$[e_1, e_2, e_3] = 0, \quad [e_1, e_2, e_4] = 2e_7, \quad \|[e_1, e_2, e_4]\|^2 = 4 \quad (136)$$

shows that the model sharply distinguishes associative and non-associative basis sectors. For general linear combinations the associator amplitude is continuous, but the elementary basis calculation supplies an exact normalization reference. A strongly non-associative vacuum would generate a Planck-scale associator contribution, while the observed smallness of $\Lambda \ell_P^2$ requires the physical vacuum to be nearly associative. In the explicit deformation $n_3 = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4$, the algebra gives $\|[e_1, e_2, n_3]\|^2 = 4\varepsilon^2$, so the protected-vacuum limit translates the observed vacuum curvature into the numerical alignment condition $\varepsilon \sim 10^{-61} \eta_A^{-1/2}$. The remaining task is to specify and solve the vacuum potential, determine the fluctuation spectrum, compute threshold corrections, and run the couplings to observable scales. Only after these steps can the framework move from structural derivation to numerical prediction.

Appendix A. Derivation of the master action from minimal requirements

This appendix gives a compact derivation of the master action used in Sec. 3. The purpose is not to introduce additional dynamics, but to state the minimal assumptions under which Eq. (15) is the lowest-order effective action compatible with the octonionic projection framework. This is also

described in the paper "An Octonionic Foundation of Spacetime Geometry and Quantum Mechanics" from same author.

Appendix A.1. Minimal requirements

Let the fundamental matter variables be three octonion-valued fields

$$\Psi_I(x) \in \mathbb{O}, \quad I = 1, 2, 3, \quad (\text{A.1})$$

projected to a four-dimensional Lorentzian spacetime. The effective action is required to satisfy the following conditions.

First, after projection it must be invariant under four-dimensional diffeomorphisms. Therefore the action must be an integral of a scalar density,

$$S = \int d^4x \sqrt{-g} \mathcal{L}. \quad (\text{A.2})$$

Second, it must be local and of lowest derivative order. Thus the metric sector contains the Einstein-Hilbert scalar and a vacuum term, while the octonionic fields enter through quadratic kinetic terms. Third, the field-space norm must be the real positive octonionic norm induced by

$$\langle x, y \rangle = \text{Re}(x\bar{y}), \quad (\text{A.3})$$

so that kinetic and potential energies are real. Fourth, the action must reduce to an ordinary associative effective theory when the selected fields lie inside a quaternionic subalgebra. Fifth, the lowest scalar that measures genuine non-associativity must be built from the associator

$$[\Psi_1, \Psi_2, \Psi_3] = (\Psi_1\Psi_2)\Psi_3 - \Psi_1(\Psi_2\Psi_3). \quad (\text{A.4})$$

Because $\mathbb{O}$ is alternative, a single-field invariant $[\Psi, \Psi, \Psi]$ vanishes identically. Hence three independent octonionic fields are the minimal field content for a nonzero associator self-energy.

Appendix A.2. Lowest-order invariant terms

Under these assumptions the lowest-order projected Lagrangian consists of five terms. The gravitational sector is

$$\mathcal{L}_{\text{grav}} = \frac{1}{2\kappa_{\mathbb{O}}}(R - 2\Lambda_0). \quad (\text{A.5})$$

The positive two-derivative kinetic contribution for the three fields is

$$\mathcal{L}_{\text{kin}} = - \sum_{I=1}^3 \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle. \quad (\text{A.6})$$

A local associative-sector potential is denoted by

$$\mathcal{L}_{\text{loc}} = -V_{\text{loc}}(\Psi_I). \quad (\text{A.7})$$

The leading non-associative scalar is the norm square of the associator,

$$\mathcal{L}_A = -\lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2. \quad (\text{A.8})$$

Finally, the lowest curvature coupling to the vacuum norm is

$$\mathcal{L}_{\text{nmc}} = -\xi R \sum_{I=1}^3 \langle \Psi_I, \Psi_I \rangle. \quad (\text{A.9})$$

Adding these contributions and restoring the overall action normalization S_0 gives Eq. (15).

Appendix A.3. Dimensional consistency

In four dimensions the Lagrangian density has mass dimension four. With the canonical assignment

$$[\Psi_I] = 1, \quad (\text{A.10})$$

the associator has mass dimension three and its norm square has mass dimension six. Therefore

$$[\lambda] = -2, \quad \lambda = \frac{\eta_A}{M_*^2}, \quad (\text{A.11})$$

where η_A is dimensionless. The nonminimal term

$$R \sum_I \langle \Psi_I, \Psi_I \rangle \quad (\text{A.12})$$

has dimension four and hence ξ is dimensionless. If an additional curvature coupling to the associator density is included in an effective Planck-mass functional, dimensional consistency requires

$$F(\Psi) = \xi \sum_I \langle \Psi_I, \Psi_I \rangle + \zeta \frac{\|[\Psi_1, \Psi_2, \Psi_3]\|^2}{M_*^4}, \quad (\text{A.13})$$

because $F(\Psi)R$ must have dimension four and therefore F must have dimension two.

Appendix A.4. Resulting master action

The resulting lowest-order octonionic projected master action is

$$S_{\text{master}} = S_0 \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa_{\mathbb{O}}} (R - 2\Lambda_0) - \sum_{I=1}^3 \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle - V_{\text{loc}}(\Psi_I) - \lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 - \xi R \sum_I \langle \Psi_I, \Psi_I \rangle \right]. \quad (\text{A.14})$$

Thus the master action is not an arbitrary ansatz: it is the lowest-order local scalar functional that contains the Einstein-Hilbert projection, positive octonionic kinetic energy, an associative local potential, the minimal non-associative invariant, and the leading curvature coupling to the octonionic vacuum norm.

Appendix B. Reduced vacuum solution and Hessian spectrum

This appendix spells out the reduced calculation underlying the numerical closure programme. The purpose is not to replace the full 24-dimensional vacuum problem, but to give a tractable analytic branch on which the vacuum misalignment, the cosmological-constant contribution, and the leading Hessian blocks can be computed explicitly.

Appendix B.1. Reduced branch of the vacuum equations

Write the three vacuum fields as

$$\Psi_{I,0} = \phi_I n_I, \quad \langle n_I, n_I \rangle = 1, \quad (\text{B.1})$$

and restrict first to the equal-amplitude branch

$$\phi_1 = \phi_2 = \phi_3 = \frac{\Phi}{\sqrt{3}}, \quad \sum_{I=1}^3 \|\Psi_{I,0}\|^2 = \Phi^2. \quad (\text{B.2})$$

The nearly associative directional ansatz is

$$n_1 = e_1, \quad n_2 = e_2, \quad n_3 = \sqrt{1 - \varepsilon^2} e_3 + \varepsilon e_4, \quad |\varepsilon| \ll 1. \quad (\text{B.3})$$

Using the Fano convention of Eq. (4), one has

$$[e_1, e_2, e_3] = 0, \quad [e_1, e_2, e_4] = 2e_7. \quad (\text{B.4})$$

Therefore

$$[n_1, n_2, n_3] = 2\varepsilon e_7, \quad \|[n_1, n_2, n_3]\|^2 = 4\varepsilon^2. \quad (\text{B.5})$$

For equal amplitudes this gives

$$\|[\Psi_{1,0}, \Psi_{2,0}, \Psi_{3,0}]\|^2 = \frac{4}{27} \Phi^6 \varepsilon^2. \quad (\text{B.6})$$

Equivalently, if the directional normalization convention of the main text is used, the same branch is represented by the directional invariant $4\varepsilon^2$. The two conventions differ only by the fixed equal-amplitude factor $1/27$.

Appendix B.2. Reduced effective potential

A minimal reduced potential for the branch (B.3) can be written as

$$V_{\text{red}}(\Phi, \varepsilon) = \frac{g}{4}(\Phi^2 - \Phi_0^2)^2 + \Phi^4 [A\varepsilon^2 + B\varepsilon^4 - s\varepsilon] + O(\varepsilon^6), \quad (\text{B.7})$$

where $A > 0$ is the dimensionless stiffness against non-associative misalignment, B is the leading stabilizing quartic coefficient, and s is a small symmetry-breaking source. This notation is a reduced, vacuum-level notation: the original associator term has the form

$$\lambda \|[\Psi_1, \Psi_2, \Psi_3]\|^2 = \frac{4\eta_A}{27} \frac{\Phi^6}{M_*^2} \varepsilon^2 \quad (\text{B.8})$$

for the equal-amplitude branch. Written in the form $\Phi^4 A \varepsilon^2$, this contribution therefore gives

$$A_{\text{assoc}}(\Phi) = \frac{4\eta_A}{27} \frac{\Phi^2}{M_*^2}. \quad (\text{B.9})$$

A typical reduced stiffness is consequently

$$A(\Phi) = \frac{4\eta_A}{27} \frac{\Phi^2}{M_*^2} + \bar{\mu}_H^2, \quad A_0 := A(\Phi_0), \quad (\text{B.10})$$

where $\bar{\mu}_H^2$ denotes the dimensionless alignment contribution and the factor $4/27$ is replaced by 4 if the purely directional normalization of the main text is used. In the leading small-misalignment formulas below, A denotes A_0 unless its mild Φ -dependence is explicitly retained. If the Φ -dependence of $A(\Phi)$ is retained exactly, the radial equation receives the additional contribution $\Phi^4 A'(\Phi) \varepsilon^2$; this term is of the same perturbative order as the displayed stiffness

correction and is absorbed into the reduced coefficient in the leading closure formulas. The stationarity equations in this leading reduced convention are

$$\frac{\partial V_{\text{red}}}{\partial \Phi} = g\Phi(\Phi^2 - \Phi_0^2) + 4\Phi^3(A\varepsilon^2 + B\varepsilon^4 - s\varepsilon) = 0, \quad (\text{B.11})$$

$$\frac{\partial V_{\text{red}}}{\partial \varepsilon} = \Phi^4(2A\varepsilon + 4B\varepsilon^3 - s) = 0. \quad (\text{B.12})$$

For the stable small-breaking branch one obtains

$$\varepsilon = \frac{s}{2A_0} - \frac{B}{2A_0^4}s^3 + O(s^5), \quad (\text{B.13})$$

and the radial equation gives

$$\Phi^2 = \Phi_0^2 - \frac{4\Phi_0^2}{g}(A\varepsilon^2 + B\varepsilon^4 - s\varepsilon) + O(s^4). \quad (\text{B.14})$$

Since $A_0\varepsilon^2 - s\varepsilon = -s^2/(4A_0) + O(s^4)$ on the branch (B.13), the radial shift starts only at order s^2 . Thus the vacuum norm is stable against a parametrically small associator misalignment.

Appendix B.3. Cosmological-constant closure on the reduced branch

In the protected case $v_0 = 0$ and in the reduced Planck convention used in the main text, the directional associator contribution gives

$$\Lambda \ell_P^2 = \frac{\eta_A}{2\pi}\varepsilon^2 + O(\varepsilon^4). \quad (\text{B.15})$$

Combining Eqs. (B.13) and (B.15) yields

$$\Lambda \ell_P^2 = \frac{\eta_A}{8\pi A_0^2}s^2 + O(s^4). \quad (\text{B.16})$$

Using the reduced stiffness $A_0 = A(\Phi_0)$, this is

$$\Lambda \ell_P^2 = \frac{\eta_A}{8\pi A_0^2}s^2 + O(s^4). \quad (\text{B.17})$$

Hence the observed smallness of the vacuum curvature is no longer inserted as an independent constant; it is mapped to the small symmetry-breaking source s relative to the associator and alignment stiffness. The remaining hierarchy problem is then sharply formulated: a parameter-free theory must compute s and the reduced stiffness A_0 from the same octonionic vacuum dynamics.

Appendix B.4. Reduced Hessian spectrum

The reduced Hessian in the (Φ, ε) sector is

$$H_{\text{red}} = \begin{pmatrix} \partial_{\Phi}^2 V_{\text{red}} & \partial_{\Phi} \partial_{\varepsilon} V_{\text{red}} \\ \partial_{\varepsilon} \partial_{\Phi} V_{\text{red}} & \partial_{\varepsilon}^2 V_{\text{red}} \end{pmatrix}_{(\Phi_{\star}, \varepsilon_{\star})}. \quad (\text{B.18})$$

To leading nontrivial order in s , the curvature of the reduced potential is

$$H_{\Phi\Phi} = 2g\Phi_0^2 + O(s^2\Phi_0^2), \quad (\text{B.19})$$

$$H_{\varepsilon\varepsilon} = 2A_0\Phi_0^4 + O(s^2\Phi_0^4), \quad (\text{B.20})$$

$$H_{\Phi\varepsilon} = -2s\Phi_0^3 + O(s^3\Phi_0^3). \quad (\text{B.21})$$

Here $H_{\varepsilon\varepsilon}$ is the curvature with respect to the *dimensionless* coordinate ε , not yet a canonically normalized mass squared. The corresponding physical fluctuation is

$$\chi_{\varepsilon} = Z_{\varepsilon}^{1/2} \Phi_0 \delta\varepsilon, \quad (\text{B.22})$$

where Z_{ε} is the metric induced by the kinetic term on the reduced vacuum manifold. Therefore the canonically normalized mass is

$$M_{\chi_{\varepsilon}}^2 = \frac{1}{Z_{\varepsilon}\Phi_0^2} H_{\varepsilon\varepsilon} = \frac{2A_0}{Z_{\varepsilon}} \Phi_0^2 + O(s^2\Phi_0^2). \quad (\text{B.23})$$

For the orthonormal reduced-coordinate convention one has $Z_{\varepsilon} = 1$. The radial-misalignment mixing is therefore suppressed by the same source that controls the small vacuum curvature. In the full 24-component system this reduced block is embedded in a larger real symmetric matrix. The leading mode classification is

$$\begin{aligned} \text{radial norm mode} & : M_{\rho}^2 = 2g\Phi_0^2 + O(s^2\Phi_0^2), \\ \text{misalignment mode} & : M_{\chi_{\varepsilon}}^2 = (2A_0/Z_{\varepsilon})\Phi_0^2 + O(s^2\Phi_0^2), \\ \text{quaternionic orientation modes} & : M_H^2 \sim \bar{\mu}_H^2 \Phi_0^2, \\ \text{orthogonal non-associative modes} & : M_A^2 \sim A_0 \Phi_0^2, \\ \text{source-shifted modes} & : M_s^2 = M_A^2 + O(s\Phi_0^2). \end{aligned} \quad (\text{B.24})$$

The exact numerical threshold spectrum is obtained by evaluating the full matrix (131) in canonical field coordinates. Equations (B.19)–(B.24) provide the analytic control needed to organize that computation and to verify any numerical diagonalization.

Appendix B.5. Input to the renormalization-group thresholds

Once the full Hessian eigenvalues M_r^2 are computed, they enter the threshold part of the gauge-coupling flow through

$$\Delta_i^{\text{th}} = \sum_r \frac{\Delta b_i^{(r)}}{2\pi} \ln \frac{M_{\mathbb{O}}}{M_r}. \quad (\text{B.25})$$

Thus the reduced calculation supplies three concrete numerical checks for a full implementation:

- (i) the stationary value of ε must agree with Eq. (B.13);
- (ii) the induced value of $\Lambda \ell_P^2$ must follow Eq. (B.17);
- (iii) the smallest non-associative threshold must scale with the stiffness A according to Eq. (B.23).

These checks turn the remaining numerical problem into an explicit spectral calculation rather than an unconstrained calibration.

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