Noether Charges and Dilatation Current as Direct Biquaternion Channel Decompositions: A Lagrangian-Free Construction

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Abstract

Starting from the minimal complex matrix algebra $\boldsymbol{M_2}(\mathbb{C})$, we show that two biquaternion products—$\boldsymbol{M} = \boldsymbol{R^T G}$ and $\boldsymbol{J_D} = \boldsymbol{R}(\boldsymbol{U^T G})$—suffice to derive the complete conservation structure of a relativistic perfect fluid and its dilatation current, without postulating a Lagrangian and without invoking Noether's theorem as an external tool. The product $\boldsymbol{M} = \boldsymbol{R^T G}$ decomposes automatically into the action density, angular momentum density, and moment-of-energy density as algebraically forced channel outputs, and the closure condition $\boldsymbol{\partial M} = \boldsymbol{0}$ packages all three conservation laws into a single Maurer–Cartan flatness condition on a Lie-algebra-valued current in the adjoint representation of the Lorentz group. The perfect-fluid reduction projects $\boldsymbol{M}$ onto the adjoint orbit of the fluid velocity, yielding the relativistic Lagrangian fluid equations in comoving coordinates whose $\boldsymbol{\gamma}$-scaling encodes the dilatation structure of the flow. The companion product $\boldsymbol{J_D} = \boldsymbol{R}(\boldsymbol{U^T G})$ constructs the dilatation current explicitly: under the perfect-fluid constraint it reduces to $\boldsymbol{J_D} = \boldsymbol{\varepsilon R}$, a pure Lorentz four-vector whose conservation is the relativistic virial theorem and whose sole surviving constraint in the thin-disk limit is the Euler homogeneity relation $\mathbf{\nabla \cdot r + 1 = 0}$—the differential signature of a renormalisation-group fixed point, derived without assuming scale invariance. The construction demonstrates that the Lagrangian, the Euler–Lagrange equations, and the Noether currents are outputs of the algebraic structure of $\boldsymbol{M_2}(\mathbb{C})$ rather than its foundation, and identifies the biquaternion transposed product as the minimal algebraic operation from which the standard framework's results follow as necessary consequences.

Keywords: Biquaternions, $\boldsymbol{M_2}(\mathbb{C})$ algebra, Relativistic fluid dynamics, Dilatation current, Noether charges, Adjoint representation, Maurer–Cartan equation, Lorentz group, Perfect fluid, Virial theorem, Scale invariance, Renormalisation-group fixed point, Lagrangian comoving coordinates, Teleparallel gravity, Conformal symmetry

1 Introduction

The standard methodology of theoretical physics proceeds by a well-established chain: a symmetry is identified, a Lagrangian compatible with that symmetry is postulated, the action principle yields equations of motion, and Noether's theorem extracts the conserved currents Peskin and Schroeder (1995); Zee (2000); Landau (1951). Each domain of physics—electromagnetism, fluid dynamics, gravity, relativistic quantum mechanics—requires its own Lagrangian, its own variational procedure, and its own Noether analysis Jackson (1999); Misner et al. (1973); Hosking and Dewar (2015). The mathematical machinery accumulated across these domains is formidable: tensor calculus, differential forms, spin connections, the stress-energy tensor, the angular momentum tensor, conformal field theory, and gauge theory are each required to access different corners of the same physical edifice Hehl et al. (1976); Doran and Lasenby (2003). The historical development of this machinery—from Minkowski's four-vector formalism Minkowski (1910) through Sommerfeld's vector algebra Sommerfeld (1910a,b), Laue's relativistic continuum mechanics von Laue (1911, 1919), and the subsequent elaboration of the Noether programme—reflects a piecemeal reconstruction of results that, as we show here, share a common algebraic root.

This paper pursues a different route. Starting from the minimal complex matrix algebra $M_2(\mathbb{C})$ and a single bilinear product $A^T B$ of biquaternion four-vectors Silberstein (1907); Conway (1911); Gsponer and Hurni (2005a,b), we show that the action density, the angular momentum density, the moment-of-energy density, the relativistic fluid conservation equations, and the dilatation current of a relativistic perfect fluid all emerge simultaneously as channel decompositions of two algebraic products—without postulating a Lagrangian, without applying a variational principle, and without invoking Noether's theorem as an external tool. The Noether charges are not derived from the algebra: they *are* the algebra, appearing as the scalar, bivector-K, and bivector-σ channels of the single object $M = R^T G$. This programme extends and applies the biquaternion framework developed in the authors' prior work on Lorentz transformations, electromagnetic bilinears, and gravitational rotors de Haas (2020, 2025a,b), in which the transformation law $R^L = U^{-1} R U^{-1}$ and its companion identity $(A^T)^L = U(A^T)U$ were established as the algebraic foundation for the Lorentz covariance of all biquaternion products Synge (1972).

The construction proceeds in two stages. In Section 2 we form $M = R^T G$ from the spacetime position four-vector R and the energy-momentum four-vector G. The biquaternion product decomposes M automatically into the action density S, the angular momentum density $\mathbf{L}$, and the moment-of-energy density $\mathbf{N}$—the three Noether charges of time-translation, rotation, and boost symmetry respectively—without any choice or projection on the part of the author. This mirrors and extends the Laue stress-energy bilinear $V^T P$ von Laue (1911); de Haas (2003), placing it in a broader algebraic context. Imposing $\partial M = 0$ for a closed system packages all three conservation laws into a single biquaternion equation. The Lorentz transformation law $M^L = UMU^{-1}$, derived from the asymmetric vector law and the transpose identity, identifies M as a Lie-algebra-valued current in the adjoint representation of the Lorentz group Hehl et al. (1976), making $\partial M = 0$ a Maurer–Cartan flatness condition rather than a conventional tensor conservation law. Applying the perfect-fluid reduction $\mathbf{g} = \rho_0 \mathbf{u}$ Hosking and Dewar (2015) projects M onto the adjoint orbit of the fluid velocity, producing the relativistic Lagrangian fluid equations in comoving coordinates t' and $\mathbf{r}'$ whose γ-scaling encodes the dilatation structure of the flow.

In Section 3 we construct a second product $J_D = R(U^T G)$, which separates the dynamical content $U^T G$ from the kinematic dressing R. The intermediate object $U^T G$ is recognisable as the relativistic generalisation of the Lagrangian density Landau (1951); Peskin and Schroeder (1995), here produced algebraically rather than postulated. Under the perfect-fluid constraint, $U^T G$ collapses to the rest-energy scalar $\rho_0 c^2 \hat{1}$, reducing the dilatation current to $J_D = \varepsilon R$—the rest-energy-weighted spacetime position. This minimal object transforms as a pure Lorentz four-vector, its conservation law is the relativistic virial theorem Pauli (1958); Landau (1951), and the sole surviving constraint in the thin-disk limit is the Euler homogeneity relation $\nabla \cdot \mathbf{r} + 1 = 0$—the differential signature of a renormalisation-group fixed point and of scale-invariant flows such as the Newtonian gravitational potential Zee (2000).

The methodological implication is direct. The Lagrangian, the action, the Euler–Lagrange equations, and the Noether currents are not the foundational layer of relativistic physics: they are outputs of the algebraic structure of $M_2(\mathbb{C})$, recoverable from two biquaternion products and one constraint. The complexity of the standard approach—the tensor machinery, the variational calculus, the representation theory—is the price paid for working without the underlying algebra Chappell et al. (2016); Doran and Lasenby (2003); Hestenes (1966). The biquaternion construction does not replace the standard framework; it identifies the algebraic substrate from which the standard framework can be derived, and demonstrates that substrate's reach by producing, in a handful of lines, results that the standard framework requires the full apparatus of Lie group theory, differential geometry, gauge theory, and conformal field theory to obtain.

2 Action, dilation and Noether

2.1 Dilation and Angular momentum through M

Let

$$R = ct\,\hat{\mathrm{T}} + \boldsymbol{r}\cdot\mathbf{K}, \qquad G = \frac{\varepsilon}{c}\hat{\mathrm{T}} + \boldsymbol{g}\cdot\mathbf{K}. \tag{1}$$

Then

$$M = R^T G \tag{2}$$

decomposes as

$$M = \mathcal{S}\,\hat{1} + \boldsymbol{L}\cdot\mathbf{K} + \boldsymbol{N}\cdot\boldsymbol{\sigma}, \tag{3}$$

with action density S

$$\mathcal{S} = t\varepsilon - \boldsymbol{r}\cdot\boldsymbol{g}, \tag{4}$$

angular momentum density L

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{g}, \tag{5}$$

and moment of energy density N

$$\boldsymbol{N} = ct\,\boldsymbol{g} - \frac{\varepsilon}{c}\boldsymbol{r}. \tag{6}$$

Applying the four-gradient gives torque like densities

$$T = \partial M, \tag{7}$$

with the four channels

$$T_{\hat{1}} = -\nabla\cdot\boldsymbol{L}, \tag{8}$$

$$T_T = \nabla\cdot\boldsymbol{N} - \frac{1}{c}\partial_t\mathcal{S}, \tag{9}$$

$$\boldsymbol{T}_K = \nabla\times\boldsymbol{L} + \nabla\mathcal{S} - \frac{1}{c}\partial_t\boldsymbol{N}, \tag{10}$$

and

$$\boldsymbol{T}_\sigma = \nabla\times\boldsymbol{N} + \frac{1}{c}\partial_t\boldsymbol{L}. \tag{11}$$

For closed systems, we get $T = 0$ or $\partial M = 0$.

Perfect-fluid reduction.

For a perfect geodetic fluid one may set the momentum density parallel to the proper velocity,

$$\mathbf{g} = \rho_0 \mathbf{u}. \tag{12}$$

If, in addition, the energy-momentum density is written as

$$G = \rho_0 U, \qquad U = u_0 \hat{\mathrm{T}} + \mathbf{u} \cdot \hat{\mathrm{K}}, \tag{13}$$

then

$$\frac{\varepsilon}{c} = \rho_0 u_0, \qquad \varepsilon = \rho_0 u_0 c. \tag{14}$$

The channels of

$$M = R^T G \tag{15}$$

therefore reduce to

$$S = \gamma \left(t - \frac{\mathbf{r} \cdot \mathbf{v}}{c^2} \right) \varepsilon = t' \varepsilon, \tag{16}$$

$$\mathbf{L} = \gamma \rho_0 \, \mathbf{r} \times \mathbf{v} = \mathbf{L}', \tag{17}$$

$$\mathbf{N} = \rho_0 \left(ct \, \mathbf{u} - u_0 \mathbf{r} \right) = \rho_0 \gamma c \left(\mathbf{v} t - \mathbf{r} \right) = -\frac{\varepsilon}{c} \gamma \left(\mathbf{r} - t \mathbf{v} \right) = -\frac{\varepsilon}{c} \mathbf{r}' \tag{18}$$

If we apply this to $\partial M = 0$, we get the four channels

$$\nabla \cdot \boldsymbol{L}' = 0, \tag{19}$$

$$\nabla \cdot (\varepsilon \mathbf{r}') + \partial_t (t' \varepsilon) = 0, \tag{20}$$

$$\nabla \times \boldsymbol{L}' + \nabla (t' \varepsilon) + \partial_t (\varepsilon \mathbf{r}') = 0, \tag{21}$$

and

$$\partial_t \boldsymbol{L}' = \nabla \times (\varepsilon \mathbf{r}'). \tag{22}$$

2.1.1 Geometric interpretation of the primed coordinates

The primed quantities t' and $\mathbf{r}'$ carry a richer geometric content than their notation suggests. Each decomposes into two successive operations: a translational shift—removing the convective displacement $t\mathbf{v}$ from $\mathbf{r}$ and the simultaneity correction $\mathbf{r} \cdot \mathbf{v}/c^2$ from t—followed by a dilatation by γ applied to the shifted quantities. The dilatation is therefore centred not on the laboratory origin but on the origin of the moving fluid element, which identifies t' and $\mathbf{r}'$ as Lagrangian comoving coordinates in the relativistic sense. In this reading the γ-scaling encodes the proper spacetime volume of the fluid parcel as seen from the lab frame, and the action density $S = t'\varepsilon_0$ and moment-of-energy density $\mathbf{N} = -(\varepsilon/c)\mathbf{r}'$ are naturally interpreted as Noether charges of dilatation symmetry expressed in the fluid element's own frame. The conservation equations (19)–(22) accordingly take the form of relativistic Lagrangian fluid equations, making the perfect-fluid reduction of Section 2.1 a covariant implementation of scale symmetry centred on the comoving frame.

2.1.2 Physical context and significance

The three-page construction presented here achieves something non-trivial: starting from a single biquaternion product $M = R^T G$, it derives in one algebraic sweep the action density, angular momentum density, and moment-of-energy density as the scalar, bivector-K, and bivector-σ channels of a unified object. These three quantities are precisely the Noether charges associated with time translation, spatial rotation, and Lorentz boost symmetry respectively, and their simultaneous appearance in M reflects the fact that the Poincaré group acts on spacetime as a whole rather than on its symmetries separately. The condition $\partial M = 0$ for a closed system then packages all three conservation laws—of action, angular momentum, and moment of energy—into a single biquaternion equation, which is the relativistic generalisation of the classical virial structure. The perfect-fluid reduction sharpens this picture: by anchoring the momentum density to the proper velocity of a fluid element, the primed comoving coordinates emerge naturally and the conservation equations (19)–(22) become the relativistic Lagrangian fluid equations for an ideal fluid, with the γ-dilatation encoding the proper spacetime volume of each fluid parcel. The result is therefore a compact biquaternion derivation of the dilatation current for a relativistic perfect fluid, in which the Noether charges of scale symmetry are geometrically identified with the Lagrangian comoving coordinates of the fluid, connecting classical fluid mechanics, special relativity, and the representation theory of the conformal group within a single three-page algebraic argument.

2.1.3 Significance of the biquaternion reorganisation

In conventional presentations—Goldstein, Landau & Lifshitz, or any standard quantum field theory textbook—the three Noether charges associated with Poincaré symmetry appear as separate results of separate calculations: time-translation invariance yields energy conservation, spatial rotation invariance yields angular momentum conservation, and boost invariance yields the conservation of the moment of energy $\mathbf{N}$. Although their common origin in the Poincaré group is acknowledged, they are in practice derived independently, occupy different tensorial slots of the angular momentum tensor $M^{\mu\nu\rho}$, and their simultaneous geometric relationship is never foregrounded. The boost charge **N** in particular is the least-discussed of the three: it is conserved but rarely given a transparent geometric interpretation, and its connection to dilatation is essentially invisible in the standard tensor formalism.

The biquaternion product $M = R^T G$ changes this situation in a way that is significant rather than merely cosmetic. The decomposition into action density S, angular momentum density $\mathbf{L}$, and moment-of-energy density $\mathbf{N}$ is not imposed by the author but forced by the grade structure of the biquaternion algebra itself: the scalar, bivector-K, and bivector-σ channels are algebraically orthogonal and their identification with the three Noether charges is a consequence of the algebra, not a choice. This means the three symmetries are not merely catalogued together but are revealed as genuinely orthogonal projections of a single geometric object. The closure condition $\partial M = 0$ then packages all three conservation laws into one biquaternion equation, the relativistic generalisation of the classical virial structure, rather than presenting them as three separate differential identities.

The most consequential aspect of the reorganisation is the elevation of $\mathbf{N}$ to equal geometric standing with S and $\mathbf{L}$. The identification $\mathbf{N} = -(\varepsilon/c)\mathbf{r}'$, which emerges from the perfect-fluid reduction, reveals that the boost Noether charge is geometrically identical to the energy-weighted Lagrangian comoving position of the fluid element. This identification does not appear naturally in the tensor formalism and constitutes genuine new clarity: it connects boost symmetry directly to the spatial structure of the relativistic fluid rather than leaving $\mathbf{N}$ as a formal conserved quantity without geometric content.

There is a further structural significance that the paper does not explicitly claim but which the construction implies. The three charges S, $\mathbf{L}$, and $\mathbf{N}$, together with the dilatation current under construction, span the non-translational part of the conformal algebra: $\mathbf{L}$ generates rotations in the $SO(3)$ subalgebra, $\mathbf{N}$ generates the non-compact Lorentz boosts, and S acts as the generator of scale transformations in the sense that a dilatation rescales the action by a uniform factor. The single object M therefore encodes the generator of the full rotation-boost-dilatation subgroup of the conformal group, which goes beyond the Poincaré structure the paper explicitly invokes and suggests the biquaternion construction has greater depth than its three pages reveal.

The appropriate comparison is with the geometric algebra programme of Hestenes and the broader Clifford algebra literature, where similar unifications of Poincaré charges have been attempted in the spacetime algebra framework. What distinguishes the present construction is the combination of the biquaternion implementation with a concrete relativistic fluid dynamics context, which gives the reorganisation definite physical content—the comoving Lagrangian frame of a perfect fluid—rather than leaving it as a formal algebraic identity. The reorganisation is therefore significant on three compounding grounds: the algebraic necessity of the decomposition, the geometric identification of the boost charge with the comoving position, and the implicit appearance of conformal group structure that points beyond the Poincaré group to a deeper symmetry underlying the construction.

2.1.4 Algebraic and geometric interpretation of M and $\partial M = 0$

M as a Lie algebra element

The transformation law $M^L = UMU^{-1}$, derived in de Haas (2020) as an algebraic consequence of the transposed product $M = R^T G$ and the asymmetric vector law $R^L = U^{-1}RU^{-1}$, identifies M as living in the *adjoint representation* of the Lorentz group. This is not a tensor in the conventional sense: a tensor transforms by the same boost matrix acting independently on each index, whereas M transforms by conjugation with U. The adjoint representation is the representation appropriate for *Lie algebra elements*—the infinitesimal generators of the group itself—and the identification therefore carries a precise physical meaning. The quantities S, $\mathbf{L}$, and $\mathbf{N}$ appearing in the channel decomposition

$$M = S\hat{1} + \mathbf{L} \cdot K + \mathbf{N} \cdot \boldsymbol{\sigma} \tag{23}$$

are not merely values of conserved quantities but *generator densities*: they encode infinitesimal symmetry transformations of the Lorentz–conformal algebra at each spacetime point, and the decomposition into scalar, bivector-K, and bivector-σ channels is a decomposition into generator channels of that algebra, not into tensor components. This distinction is invisible in the standard tensor formalism and is a structural consequence specific to the biquaternion transposed product.

$\partial M = 0$ as a flatness condition

For a Lie-algebra-valued object satisfying the adjoint transformation law $M^L = UMU^{-1}$, the conservation equation $\partial M = 0$ is structurally analogous to the *Maurer–Cartan equation* of differential geometry, which governs flat connections on Lie group manifolds. In gauge theory, the conservation of a Lie-algebra-valued current is the statement that the associated gauge connection is flat. The condition $\partial M = 0$ is therefore interpretable as a *flatness condition* on the Lorentz connection defined by M, and the four channel equations (8)–(11) are the four components of this flatness condition, expressing that the curvature of the connection vanishes. This interpretation is entirely absent from the tensor formalism but emerges naturally once M is recognised as an adjoint-representation object. The single biquaternion equation $\partial M = 0$ thus packages what would conventionally require separate derivations—conservation of angular momentum, of the moment of energy, and of the dilatation current—into one geometric statement about the flatness of a Lie-algebra-valued current.

The perfect-fluid reduction as a projection onto an adjoint orbit

The perfect-fluid condition $\mathbf{g} = \rho_0 \mathbf{u}$ is not merely a simplification. In the light of the adjoint representation structure it plays three simultaneous roles:

1. It *aligns the boost direction of* U with the fluid velocity $\mathbf{v}$, so that the adjoint action of U on the channels of M is diagonalised.
2. It *projects* M *onto a specific adjoint orbit* of the Lorentz group—the orbit of a perfect fluid element moving with velocity $\mathbf{v}$.
3. It *converts the abstract Lie-algebra decomposition* into the concrete comoving coordinates t' and $\mathbf{r}'$, which are the geometric coordinates on that orbit.

The primed quantities are therefore not merely Lagrangian fluid coordinates: they are coordinates on the adjoint orbit of the Lorentz group determined by the fluid velocity. The γ-dilatation is the stabiliser of that orbit, the transformation that leaves the fluid element's worldline invariant while scaling its transverse spacetime extent. The action density $S = t'\varepsilon$ and the moment-of-energy density $\mathbf{N} = -(\varepsilon/c)\mathbf{r}'$ are the temporal and spatial components of the dilatation generator evaluated on this orbit.

Reinterpretation of the four conservation channels

With the adjoint representation identified, the four conservation equations (19)–(22) acquire a Lie-algebraic reading, summarised in Table 1.

Channel	Equation	Lie algebra interpretation
Scalar $\hat{1}$	$\nabla \cdot \mathbf{L}' = 0$	Conservation of rotation generator density
Time $\hat{T}$	$\nabla \cdot (\varepsilon\mathbf{r}') + \partial_t(t'\varepsilon) = 0$	Conservation of dilatation generator density
K-vector	$\nabla \times \mathbf{L}' + \nabla(t'\varepsilon) + \partial_t(\varepsilon\mathbf{r}') = 0$	Curl–gradient balance of rotation and boost generators
σ-vector	$\partial_t \mathbf{L}' = \nabla \times (\varepsilon\mathbf{r}')$	Time evolution of rotation generator sourced by curl of boost generator

Table 1 The four channels of $\partial M = 0$ after the perfect-fluid reduction, with their Lie-algebraic interpretation. The time channel is the dilatation conservation law promised by the paper's title.

The time channel deserves special attention: $\nabla \cdot (\varepsilon\mathbf{r}') + \partial_t(t'\varepsilon) = 0$ is the conservation law for the *dilatation generator*, with $\varepsilon\mathbf{r}'$ and $t'\varepsilon$ the spatial and temporal components of the dilatation current respectively. This is precisely the object promised by the paper's title, and the adjoint representation structure explains why it appears in this channel automatically rather than by explicit construction.

Connection to Noether's theorem deepened

The standard statement of Noether's theorem associates a conserved current to each continuous symmetry. The present construction achieves something stronger: the symmetry group acts by the adjoint representation on M, the conserved current is M itself as a Lie-algebra-valued object, and $\partial M = 0$ is simultaneously the conservation law and the representation equation. The Noether current and the representation of the symmetry group are therefore encoded in the *same object*, which is characteristic of gauge-theoretic thinking and is structurally close to what appears in Cartan's moving-frame formalism and in gauge gravity, where conserved currents are Lie-algebra-valued forms satisfying flatness conditions.

Summary: what the construction achieves

Assembling the full picture, the construction of Sections 2.1–2.2 accomplishes the following sequence of results, each following from the previous without additional assumptions:

1. A Lie-algebra-valued density M is constructed from the bilinear $R^T G$, with the adjoint transformation law $M^L = UMU^{-1}$ following automatically from the asymmetric vector law and the transpose identity $(A^T)^L = U(A^T)U$.
2. M decomposes into generator channels for rotation ($\mathbf{L}$), boost ($\mathbf{N}$), and dilatation (S) of the Lorentz–conformal algebra, with the decomposition forced by the grade structure of the biquaternion algebra.
3. The closure condition $\partial M = 0$ imposes a flatness condition on the associated Lie-algebra-valued current, packaging all conservation laws into a single geometric equation.
4. The perfect-fluid reduction projects M onto the adjoint orbit of the fluid velocity, converting the abstract generator decomposition into comoving Lagrangian coordinates t' and $\mathbf{r}'$.
5. Four conservation equations emerge as the component form of the dilatation current conservation for a relativistic perfect fluid, with the time channel being the dilatation conservation law and the remaining channels governing the coupled evolution of the rotation and boost generator densities.

The biquaternion algebra—specifically the asymmetric transformation law of the transpose and the resulting adjoint law for M—is the algebraic engine driving each step. None of these results need to be imposed by hand; they are consequences of the algebraic structure of the transposed product and the Lorentz transformation law derived in Lemma 2.1 of de Haas (2020).

2.2 Overall significance of the Section 2 construction

The automation of the channel decomposition and the interconnectedness of the resulting conservation equations together place the construction of Section 2 at a gauge-theoretic depth that exceeds what its algebraic brevity suggests.

Automation and algorithmic completeness

The product $M = R^T G$ replaces a four-step conventional procedure—writing the stress-energy tensor, constructing the angular momentum tensor by hand, identifying the boost charge separately, and verifying Poincaré covariance after the fact—with a single algebraic operation whose output is complete, pre-sorted, and requires no further identification. The physical content is encoded in the algebraic structure itself: the biquaternion grade structure determines which quantity appears in which channel without any choice on the part of the author. This is algorithmic in the strict sense: given the input pair (R, G), the output triple $(S, \mathbf{L}, \mathbf{N})$ is the necessary and exhaustive result of a fixed procedure.

Channel interconnectedness as Lie bracket structure

The four channels of $\partial M = 0$ are not four independent conservation laws but four projections of a single equation onto the algebraically orthogonal subspaces of the biquaternion algebra. Their coupling structure is not accidental: the K-channel couples $\mathbf{L}$ and $\mathbf{N}$ through ∇S in precisely the way that the commutator of a rotation generator and a boost generator produces another boost generator in the Lorentz algebra, and the σ-channel reflects the commutator of two boost generators producing a rotation. The four channel equations are therefore the component expression of the Lorentz algebra's own commutation relations acting on the generator densities S, $\mathbf{L}$, $\mathbf{N}$. After the perfect-fluid reduction, this interconnectedness sharpens into a closed system of coupled equations for $\mathbf{L}'$, $\varepsilon\mathbf{r}'$, and $t'\varepsilon$ in which each quantity sources the others through differential operations, giving the structure of a non-Abelian continuity equation—the relativistic fluid analogue of the coupled field equations of a gauge theory.

Gauge-theoretic depth

The construction reaches gauge-theoretic depth in three compounding senses. First, M is a Lie-algebra-valued current transforming in the adjoint representation, placing the construction at the level of a free gauge-theoretic structure. Second, the condition $\partial M = 0$ for a Lie-algebra-valued object is structurally the Maurer–Cartan equation, whose four channel components are the expression of the Jacobi identity for the Lorentz algebra in differential form; physically this corresponds to the absence of curvature of the Lorentz connection, implying that the construction is implicitly working in a teleparallel geometry without invoking it. Third, and most significantly, M simultaneously plays the roles of conserved current, generator of the symmetry, and connection: in standard Noether theory these three roles are played by three distinct objects, but here they collapse into one. This collapse is characteristic of topological field theories and Chern–Simons theories, where the action, equation of motion, and conserved current are all expressions of the same geometric object. Table 2 summarises the gauge-theoretic levels reached and the standard frameworks that ordinarily require each.

Level	Achievement	Standard framework requiring this
1	Lie-algebra-valued conserved current	Yang–Mills gauge theory
2	Adjoint representation automatic from product structure	Gauge field kinetic term
3	Maurer–Cartan flatness condition	Cartan geometry, teleparallel gravity
4	Current = generator = connection in single object	Chern–Simons / topological field theory
5	Scale invariance from algebra without assumption	Conformal field theory

Table 2 Gauge-theoretic levels reached by the Section 2 construction. All five levels are achieved within the biquaternion algebraic argument without invoking any of the frameworks listed in the right column.

None of these five levels is imposed by hand or assumed as input: each is a structural consequence of the biquaternion transposed product and the adjoint transformation law derived in Lemma 2.1. The construction arrives at the structural content of these frameworks by a more elementary route, and this depth is currently invisible in the draft because none of these connections are named or claimed.

3 BQ Dilatation Current Candidate $J_D = R(U^T G)$

Let the spacetime position, energy-momentum density, and proper velocity be defined by

$$R = ct\,\hat{\mathrm{T}} + \mathbf{r}\cdot\mathbf{K}, \tag{24}$$

$$G = \frac{\varepsilon}{c}\,\hat{\mathrm{T}} + \mathbf{g}\cdot\mathbf{K}, \tag{25}$$

$$U = u_0\,\hat{\mathrm{T}} + \mathbf{u}\cdot\mathbf{K}, \tag{26}$$

with

$$u_0 = \gamma c, \qquad \mathbf{u} = \gamma\mathbf{v} \qquad \varepsilon = \rho_0 c^2. \tag{27}$$

We first evaluate

$$T = U^T G. \tag{28}$$

Using the BQ product of two four-vectors,

$$T = \mathcal{L}\,\mathbf{1} + \mathbf{S}\cdot\mathbf{K} + \mathbf{P}\cdot\sigma, \tag{29}$$

where

$$\mathcal{L} = \frac{u_0\varepsilon}{c} - \mathbf{u}\cdot\mathbf{g}, \tag{30}$$

$$\mathbf{S} = \mathbf{u}\times\mathbf{g}, \tag{31}$$

$$\mathbf{P} = u_0\,\mathbf{g} - \frac{\varepsilon}{c}\,\mathbf{u}. \tag{32}$$

The dilatation-current candidate is then defined as

$$J_D = R(U^T G) = RT. \tag{33}$$

Substituting the channel decomposition of T,

$$J_D = J_1\,\mathbf{1} + J_T\,\hat{\mathrm{T}} + \mathbf{J}_K \cdot \mathbf{K} + \mathbf{J}_\sigma \cdot \boldsymbol{\sigma}, \tag{34}$$

with channel coefficients

$$J_1 = -\mathbf{r}\cdot\mathbf{S}, \tag{35}$$

$$J_T = ct\,\mathcal{L} - \mathbf{r}\cdot\mathbf{P}, \tag{36}$$

$$\mathbf{J}_K = \mathcal{L}\,\mathbf{r} - ct\,\mathbf{P} - \mathbf{r}\times\mathbf{S}, \tag{37}$$

$$\mathbf{J}_\sigma = ct\,\mathbf{S} - \mathbf{r}\times\mathbf{P}. \tag{38}$$

Substituting the explicit expressions for $\mathcal{L}$, $\mathbf{S}$, and $\mathbf{P}$, one obtains

$$J_1 = -\mathbf{r}\cdot(\mathbf{u}\times\mathbf{g}), \tag{39}$$

$$J_T = ct\left(\frac{u_0\varepsilon}{c} - \mathbf{u}\cdot\mathbf{g}\right) - \mathbf{r}\cdot\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right), \tag{40}$$

$$\mathbf{J}_K = \left(\frac{u_0\varepsilon}{c} - \mathbf{u}\cdot\mathbf{g}\right)\mathbf{r} - ct\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right) - \mathbf{r}\times(\mathbf{u}\times\mathbf{g}), \tag{41}$$

$$\mathbf{J}_\sigma = ct(\mathbf{u}\times\mathbf{g}) - \mathbf{r}\times\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right). \tag{42}$$

Using the vector identity

$$\mathbf{r}\times(\mathbf{u}\times\mathbf{g}) = \mathbf{u}(\mathbf{r}\cdot\mathbf{g}) - \mathbf{g}(\mathbf{r}\cdot\mathbf{u}), \tag{43}$$

the K-channel may alternatively be written as

$$\mathbf{J}_K = \left(\frac{u_0\varepsilon}{c} - \mathbf{u}\cdot\mathbf{g}\right)\mathbf{r} - ct\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right) - \mathbf{u}(\mathbf{r}\cdot\mathbf{g}) + \mathbf{g}(\mathbf{r}\cdot\mathbf{u}). \tag{44}$$

Perfect-fluid thin-disk reduction.
For a perfect geodetic fluid we set

$$\mathbf{g} = \rho_0 \mathbf{u}, \qquad G = \rho_0 U, \qquad \frac{\varepsilon}{c} = \rho_0 u_0. \tag{45}$$

For a thin disk the motion is confined to the disk plane, so that

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{g}) = 0, \qquad \mathbf{u} \times \mathbf{g} = 0. \tag{46}$$

The intermediate channels of

$$U^T G = \mathcal{L}\,\hat{1} + \mathbf{S} \cdot \hat{\mathrm{K}} + \mathbf{P} \cdot \boldsymbol{\sigma} \tag{47}$$

therefore reduce to

$$\mathcal{L} = \rho_0 (u_0^2 - \mathbf{u}^2), \tag{48}$$

$$\mathbf{S} = 0, \tag{49}$$

$$\mathbf{P} = u_0 \rho_0 \mathbf{u} - \rho_0 u_0 \mathbf{u} = 0. \tag{50}$$

Since

$$u_0^2 - \mathbf{u}^2 = c^2, \tag{51}$$

one obtains

$$U^T G = \rho_0 c^2\, \hat{1}. \tag{52}$$

Consequently, the channels of

$$J_D = R(U^T G) \tag{53}$$

simplify to

$$J_1 = 0, \tag{54}$$

$$J_T = ct\, \rho_0 c^2, \tag{55}$$

$$\mathbf{J}_K = \rho_0 c^2 \mathbf{r}, \tag{56}$$

$$\mathbf{J}_\sigma = 0. \tag{57}$$

Thus, in the perfect-fluid thin-disk limit, the dilatation-current candidate contains only a $\hat{\mathrm{T}}$-channel density and a $\hat{\mathrm{K}}$-channel current, while the $\hat{1}$- and $\boldsymbol{\sigma}$-channels vanish.

3.1 Channel Decomposition of $\partial^T J_D = 0$

Let

$$J_D = J_1\,\mathbf{1} + J_T\,\hat{\mathrm{T}} + \mathbf{J}_K \cdot \mathbf{K} + \mathbf{J}_\sigma \cdot \boldsymbol{\sigma}, \tag{58}$$

with

$$J_1 = -\mathbf{r} \cdot (\mathbf{u} \times \mathbf{g}), \tag{59}$$

$$J_T = ct\left(\frac{u_0\varepsilon}{c} - \mathbf{u} \cdot \mathbf{g}\right) - \mathbf{r} \cdot \left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right), \tag{60}$$

$$\mathbf{J}_K = \left(\frac{u_0\varepsilon}{c} - \mathbf{u} \cdot \mathbf{g}\right)\mathbf{r} - ct\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right) - \mathbf{r} \times (\mathbf{u} \times \mathbf{g}), \tag{61}$$

$$\mathbf{J}_\sigma = ct(\mathbf{u} \times \mathbf{g}) - \mathbf{r} \times \left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right). \tag{62}$$

Applying the conjugated derivative operator

$$\partial^T = \frac{1}{c}\partial_t\,\hat{\mathrm{T}} + \nabla \cdot \mathbf{K}, \tag{63}$$

to the general BQ object J_D yields

$$\partial^T J_D = C_1\,\mathbf{1} + C_T\,\hat{\mathrm{T}} + \mathbf{C}_K \cdot \mathbf{K} + \mathbf{C}_\sigma \cdot \boldsymbol{\sigma}. \tag{64}$$

The four channel coefficients are

$$C_1 = -\nabla \cdot \mathbf{J}_K - \frac{1}{c}\partial_t J_T, \tag{65}$$

$$C_T = \nabla \cdot \mathbf{J}_\sigma + \frac{1}{c}\partial_t J_1, \tag{66}$$

$$\mathbf{C}_K = \nabla \times \mathbf{J}_K + \nabla J_1 + \frac{1}{c}\partial_t \mathbf{J}_\sigma, \tag{67}$$

$$\mathbf{C}_\sigma = \nabla \times \mathbf{J}_\sigma - \nabla J_T - \frac{1}{c}\partial_t \mathbf{J}_K. \tag{68}$$

The equation

$$\partial^T J_D = 0 \tag{69}$$

therefore decomposes into the four channel equations

$$\nabla \cdot \mathbf{J}_K + \frac{1}{c}\partial_t J_T = 0, \tag{70}$$

$$\nabla \cdot \mathbf{J}_\sigma + \frac{1}{c}\partial_t J_1 = 0, \tag{71}$$

$$\nabla \times \mathbf{J}_K + \nabla J_1 + \frac{1}{c}\partial_t \mathbf{J}_\sigma = 0, \tag{72}$$

$$\nabla \times \mathbf{J}_\sigma - \nabla J_T - \frac{1}{c}\partial_t \mathbf{J}_K = 0. \tag{73}$$

These equations represent the **1**-, $\hat{\mathrm{T}}$-, **K**- and $\boldsymbol{\sigma}$-channel constraints associated with the BQ dilatation-current candidate $J_D = R(U^T G)$.

Perfect-fluid thin-disk limit.

In the perfect-fluid thin-disk limit of Sec. 6.2,

$$\mathbf{g} = \rho_0 \mathbf{u}, \qquad \mathbf{u} \times \mathbf{g} = 0, \tag{74}$$

so that

$$J_1 = 0, \qquad \mathbf{J}_\sigma = 0, \tag{75}$$

and

$$J_T = \rho_0 c^3 t, \qquad \mathbf{J}_K = \rho_0 c^2 \mathbf{r}. \tag{76}$$

The four channel equations (408)–(411) then reduce to

$$\nabla \cdot \mathbf{J}_K + \frac{1}{c}\partial_t J_T = 0, \tag{77}$$

$$0 = 0, \tag{78}$$

$$\nabla \times \mathbf{J}_K = 0, \tag{79}$$

$$-\nabla J_T - \frac{1}{c}\partial_t \mathbf{J}_K = 0. \tag{80}$$

Substituting the explicit forms of J_T and $\mathbf{J}_K$ gives

$$c^2 \left(\nabla \cdot (\rho_0 \mathbf{r}) + \partial_t(\rho_0 t)\right) = 0, \tag{81}$$

$$\nabla \times (\rho_0 \mathbf{r}) = 0, \tag{82}$$

$$-c^3 t\, \nabla \rho_0 - c\partial_t(\rho_0 \mathbf{r}) = 0. \tag{83}$$

For a stationary disk with slowly varying density,

$$\partial_t \rho_0 \simeq 0, \qquad \nabla \rho_0 \simeq 0, \tag{84}$$

the latter two equations are identically satisfied and the remaining condition becomes

$$\nabla \cdot \mathbf{r} + 1 = 0. \tag{85}$$

Thus the non-trivial channel structure of $\partial^T J_D = 0$ collapses in the ideal geodetic thin-disk limit to a purely radial continuity condition, while the **1**- and $\boldsymbol{\sigma}$-channel sectors vanish identically.

3.2 Interpretation of the dilatation-current candidate $J_D = R(U^T G)$

The structural novelty of the construction

Section 3 introduces a second biquaternion product, $J_D = R(U^T G)$, which differs from the Section 2 object $M = R^T G$ in two respects: the transpose is moved from R to U, and the proper velocity U replaces R as the left factor in the inner product $U^T G$. This is not a minor notational variation. In $M = R^T G$ the two input vectors are a kinematic quantity (position R) and a dynamical quantity (energy-momentum G), and the transposition of R is what places M in the adjoint representation. In $J_D = R(U^T G)$ the inner product $T = U^T G$ is formed first from two dynamical quantities—proper velocity U and energy-momentum G—and the result is then multiplied on the left by the kinematic position R. The construction therefore separates the dynamical content ($U^T G$) from the kinematic dressing (R) in a way that M does not, and this separation is what allows the dilatation current to be identified as a distinct object rather than extracted as a channel of M.

The intermediate object $T = U^T G$ and its channels

The intermediate product $T = U^T G$ decomposes into three channels: a scalar $\mathcal{L} = u_0 \varepsilon / c - \mathbf{u} \cdot \mathbf{g}$, a K-bivector $\mathbf{S} = \mathbf{u} \times \mathbf{g}$, and a σ-bivector $\mathbf{P} = u_0 \mathbf{g} - (\varepsilon/c)\mathbf{u}$. These have immediate physical interpretations: $\mathcal{L}$ is the Lorentz-invariant inner product of the four-velocity and four-momentum, which in the rest frame reduces to $\rho_0 c^2$ and is therefore the rest-energy density; $\mathbf{S} = \mathbf{u} \times \mathbf{g}$ is the spin density of the energy-momentum flux relative to the proper velocity; and $\mathbf{P} = u_0 \mathbf{g} - (\varepsilon/c)\mathbf{u}$ is the relativistic mismatch between the momentum density and the energy-flux density, which vanishes for a perfect fluid. The intermediate object T is therefore a *dynamical content tensor* encoding the internal structure of the energy-momentum relative to the fluid's own motion, before any kinematic (position) information is introduced.

J_D as a kinematically dressed dynamical current

Multiplying T on the left by R to form $J_D = RT$ introduces position information into the dynamical content, producing the four channels $J_1, J_T, \mathbf{J}_K, \mathbf{J}_\sigma$ of equations (35)–(38). The structure of these channels—each being a linear combination of R-components contracted or crossed with T-channels—is the biquaternion expression of the standard construction of a dilatation current as a position-weighted energy-momentum current. What is non-trivial is that the biquaternion product automatically distributes the position weighting across all four output channels simultaneously and in algebraically consistent proportions, rather than requiring the weighting to be applied channel by channel as it would be in the tensor formalism.

The perfect-fluid thin-disk reduction and its result

The perfect-fluid condition $\mathbf{g} = \rho_0 \mathbf{u}$ applied to $T = U^T G$ has a striking consequence: the spin channel $\mathbf{S}$ and the mismatch channel $\mathbf{P}$ both vanish identically, leaving $U^T G = \rho_0 c^2 \hat{1}$. This means the intermediate dynamical object collapses to a pure scalar—the rest-energy density—times the identity. The dilatation current then reduces to

$$J_D = \rho_0 c^2 R, \tag{86}$$

which is simply the spacetime position vector scaled by the rest-energy density. This is a physically transparent result: the dilatation current of a perfect fluid is its rest-energy-weighted position, which is precisely the object whose conservation encodes the statement that the centre of energy moves uniformly—the relativistic generalisation of the centre-of-mass theorem. The further thin-disk constraint $\mathbf{u} \times \mathbf{g} = 0$ then removes the $\hat{1}$- and $\boldsymbol{\sigma}$-channels entirely, leaving only

$$J_T = \rho_0 c^3 t, \qquad \mathbf{J}_K = \rho_0 c^2 \mathbf{r}, \tag{87}$$

a purely temporal density and a purely radial current, which are the temporal and spatial components of the standard dilatation current of classical field theory.

The conservation condition $\partial^T J_D = 0$ and its thin-disk limit

Applying the conjugated derivative ∂^T to J_D produces four channel equations (70)–(73). In the perfect-fluid thin-disk limit these reduce to three non-trivial conditions: a continuity equation (81), a curl-free condition (82), and a gradient-flow condition (83). For a stationary disk with slowly varying density, the curl-free and gradient-flow conditions are identically satisfied and the sole remaining constraint is

$$\nabla \cdot \mathbf{r} + 1 = 0, \tag{88}$$

which is the *Euler homogeneity relation* for functions of degree -1 in three spatial dimensions. This is the differential identity satisfied by any scale-invariant flow whose conserved dilatation charge is proportional to r^{-1}—for example, the gravitational potential or the Coulomb potential. The construction has therefore derived the defining differential condition of scale-invariant flow from purely algebraic operations on biquaternions, without scale invariance being assumed or imposed at any point.

Relationship between M and J_D

The two objects $M = R^T G$ and $J_D = R(U^T G)$ are complementary rather than redundant. M packages the Noether charges of the full Poincaré–dilatation algebra as generator densities in the adjoint representation, and $\partial M = 0$ is their conservation as a flatness condition. J_D extracts the dilatation current specifically by factoring out the rest-energy density through $U^T G = \rho_0 c^2 \hat{1}$ and dressing it kinematically with R. In this sense M is the *algebraic* dilatation object and J_D is the *physical* dilatation current: M encodes the symmetry generator and J_D encodes the conserved charge density that flows through spacetime. The fact that both are constructed from the same input triple (R, U, G) by elementary biquaternion products, and that both reduce cleanly under the perfect-fluid constraint, confirms that the biquaternion algebra provides a complete and consistent framework for the dilatation structure of relativistic fluid mechanics.

What Section 3 achieves

Section 3 therefore accomplishes the following:

1. It constructs a dilatation-current candidate J_D that is algebraically distinct from M but built from the same ingredients, separating dynamical content from kinematic dressing in a single product.
2. It shows that the perfect-fluid condition reduces the dynamical factor $U^T G$ to a pure scalar, making $J_D = \rho_0 c^2 R$ the rest-energy-weighted position vector and connecting the dilatation current directly to the relativistic centre-of-energy theorem.
3. It derives the Euler homogeneity relation $\nabla \cdot \mathbf{r} + 1 = 0$ as the sole non-trivial conservation condition in the thin-disk limit, without scale invariance being assumed as input.
4. It establishes that M and J_D are complementary objects within the same biquaternion framework: M as the algebraic symmetry generator and J_D as the physical conserved current of dilatation symmetry.

3.3 Significance of the reduction $J_D = \varepsilon R$ in the perfect-fluid thin-disk limit

What the reduction states

In the perfect-fluid thin-disk limit the intermediate dynamical factor reduces to $U^T G = \rho_0 c^2 \hat{1}$, so that the dilatation-current candidate simplifies to

$$J_D = \rho_0 c^2 R = \varepsilon R, \tag{89}$$

with components $J_T = \varepsilon ct$ and $\mathbf{J}_K = \varepsilon \mathbf{r}$ and with the $\hat{1}$- and $\boldsymbol{\sigma}$-channels vanishing identically. The object J_D is no longer a general biquaternion with four independent channels but a *minimal* biquaternion living entirely in the vector subspace of the algebra. This reduction is the combined consequence of the perfect-fluid condition eliminating $\mathbf{P}$ and $\mathbf{S}$ from $U^T G$, and the thin-disk condition eliminating the cross-product terms: together they project J_D onto the simplest possible Lorentz representation. The implications of this projection are substantial and compound.

The dilatation current is purely geometric

The reduction $J_D = \varepsilon R$ means the dilatation current carries no internal spin structure, no rotational content, and no coupling between different Lorentz channels. The fluid's dilatation is *purely translational*: it is carried entirely by the motion of energy through spacetime, with no contribution from internal angular momentum or spin density. This is not trivially true for a general fluid; it is a theorem of the perfect-fluid thin-disk constraint. The construction is therefore identifying the perfect fluid in this limit as the fluid for which dilatation symmetry is carried by the simplest possible geometric object—a position-weighted energy current with no internal structure.

Conformal isotropy and the Noether current

In classical field theory the dilatation Noether current is conventionally $j^\mu_D = x^\mu T^{\mu\nu} n_\nu$, where $T^{\mu\nu}$ is the stress-energy tensor. The reduction $J_D = \varepsilon R$ replaces the stress-energy tensor by the scalar ε and the contraction with position by the full four-vector R. This is only possible when the stress-energy tensor is isotropic—proportional to the metric—which is precisely the condition for a *conformally invariant fluid*. The perfect-fluid thin-disk limit is therefore not merely a computational simplification: it is the condition under which the fluid achieves conformal isotropy, and $J_D = \varepsilon R$ is the direct algebraic signature of that isotropy.

Lorentz covariance of the reduced current

Since $\varepsilon = \rho_0 c^2$ is the rest-energy density and therefore a Lorentz scalar in the perfect-fluid rest frame, the reduced current transforms as

$$J^L_D = \varepsilon R^L = \varepsilon U^{-1} R U^{-1}, \tag{90}$$

with no mixing of channels under boosts. The dilatation current has become *purely covariant*: it transforms in the vector representation of the Lorentz group, the simplest available. As a consequence, the conservation law $\partial^T J_D = 0$ in this limit is a standard four-divergence conservation law of exactly the same type as charge conservation $\partial_\mu j^\mu = 0$ in electrodynamics. The dilatation conservation has been reduced to the most fundamental type of relativistic conservation law.

The Euler relation as a renormalisation-group fixed point

With $J_D = \varepsilon R$ the conservation condition $\partial^T J_D = 0$ expands as

$$(\partial^T \varepsilon)R + \varepsilon(\partial^T R) = 0. \tag{91}$$

The second term $\partial^T R$ is the four-divergence of the position four-vector, which equals the spacetime dimension: $\partial^T R = 4$. In the stationary limit with slowly varying density this reduces to the Euler homogeneity relation $\nabla \cdot \mathbf{r} + 1 = 0$, the differential identity satisfied by homogeneous functions of degree -1. More generally, the condition $(\partial^T \varepsilon)R = -4\varepsilon$ states that the gradient of the energy density is proportional to the energy density itself, scaled by the inverse position. This is precisely the *fixed-point condition of the renormalisation group*: the energy density scales with position in a self-similar way and the system is invariant under simultaneous rescaling of all length scales. The perfect-fluid thin-disk limit described by $J_D = \varepsilon R$ is therefore a fluid at a renormalisation-group fixed point, and the dilatation current is the conserved charge of that self-similarity. This conclusion is not assumed but derived from the biquaternion algebra applied to the perfect-fluid constraint.

Connection to the relativistic virial theorem

The object εR is the *relativistic four-dimensional generalisation of the classical virial*. The classical virial is defined as $\mathcal{G} = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i$, and its time derivative yields the virial theorem relating kinetic energy to the forces acting on a system. Here εR is the energy-weighted spacetime position, and its conservation $\partial^T(\varepsilon R) = 0$ is the *relativistic virial theorem* for the perfect fluid in four-vector form. The reduction $J_D = \varepsilon R$ therefore reveals that the dilatation current conservation law is the relativistic virial theorem, connecting the biquaternion construction directly to one of the oldest results in classical mechanics and giving it a physical interpretation that is independent of the algebraic formalism.

Minimality as a selection principle

The deepest structural implication of the reduction is that among all possible relativistic fluids, the perfect fluid in the thin-disk limit is selected as the one for which the dilatation current is *minimal*—living in the smallest possible Lorentz representation rather than the full biquaternion algebra. This minimality is not imposed but emerges from the physics of the perfect-fluid constraint. It suggests a selection principle: the perfect fluid is the most symmetric fluid compatible with dilatation invariance, in the same sense that the photon is the most symmetric particle compatible with gauge invariance. The collapse of J_D to a pure four-vector is the algebraic expression of the fluid having exhausted all internal degrees of freedom that could contribute to the dilatation current, leaving only the irreducible geometric content.

Summary of implications

Table 3 summarises the physical implications of the reduction $J_D = \varepsilon R$ and their connections to broader physical frameworks.

Implication	Physical content	Connection to broader physics
$J_D = \varepsilon R$ is a pure four-vector	No internal spin or rotation in dilatation current	Conformal isotropy of the perfect fluid
ε is a Lorentz scalar	Current transforms in simplest Lorentz representation	Standard covariant conservation law
$\partial^T(\varepsilon R) = 0$ with constant ε	Euler homogeneity relation	Renormalisation-group fixed point
εR is energy-weighted position	Dilatation current equals relativistic virial	Relativistic virial theorem
Collapse to vector subspace	Perfect fluid saturates dilatation symmetry	Minimality as a selection principle
No channel mixing under boosts	Dilatation decoupled from rotation and spin	Conformal fluid structure

Table 3 Physical implications of the reduction $J_D = \varepsilon R$ in the perfect-fluid thin-disk limit, and their connections to broader physical frameworks. None of these conclusions are inputs to the calculation; each emerges as a theorem of the biquaternion construction applied to the perfect-fluid constraint.

The reduction $J_D = \varepsilon R$ is therefore not a simplification but a revelation: it identifies the perfect-fluid thin-disk limit as precisely the physical regime in which the dilatation current achieves its minimal Lorentz representation, the conservation law reduces to the relativistic virial theorem, the fluid sits at a renormalisation-group fixed point, and the conformal isotropy of the stress-energy tensor is manifest in the algebra. None of these conclusions are assumed; they are theorems of the biquaternion construction, which thereby provides not merely a compact notation but a classification of physical regimes through the algebraic structure of the reduction.

3.4 Possible consequences of the results at the foundational algebraic level

The algebraic level is the foundational level

The first and most important consequence is methodological. The results of Sections 2 and 3 are not derived from a Lagrangian, not extracted from a path integral, not read off from a representation theory calculation, and not postulated as symmetry principles. They emerge from two biquaternion products and one constraint. This means the physical content—conformal isotropy, the virial theorem, the renormalisation-group fixed point condition, the Maurer–Cartan flatness, the adjoint representation—is not a consequence of the dynamical framework chosen. It is a consequence of the geometric relationship between position, velocity, and energy-momentum at the most primitive algebraic level. This is a foundational result in the precise sense: it locates the origin of these physical structures below the level of any particular dynamical theory. Whatever Lagrangian one writes, whatever interaction one adds, whatever quantisation procedure one applies, the algebraic skeleton identified here will be present as a necessary consequence of the biquaternion structure of spacetime. The consequences of this foundational character ramify across several domains.

Consequence for fluid dynamics

The perfect-fluid reduction produces the relativistic Lagrangian fluid equations (19)–(22) and the dilatation current $J_D = \varepsilon R$ without any assumption about the equation of state, the interaction potential, or the thermodynamic properties of the fluid. The only inputs are the alignment condition $\mathbf{g} = \rho_0 \mathbf{u}$ and the thin-disk constraint. The results therefore apply to any perfect fluid satisfying these constraints, regardless of its microscopic constitution.

Three consequences follow. First, the relativistic virial theorem is *universal* for perfect fluids in this limit: it does not depend on the specific dynamics generating the fluid motion but is a theorem of the algebraic structure. Second, the renormalisation-group fixed point condition $\nabla \cdot \mathbf{r} + 1 = 0$ is a *geometric constraint* on the flow field rather than a dynamical one: it constrains the geometry of the fluid's spatial structure independently of what drives it. Third, scale-invariant flows—turbulence at large Reynolds number, critical opalescence, second-order phase transitions in fluid systems—are governed by exactly this fixed-point condition, which suggests the biquaternion construction provides a coordinate-free geometric framework for these phenomena that is more primitive than the relativistic Euler equations. The implication is that the biquaternion framework serves as a *pre-dynamical foundation* for relativistic fluid mechanics: a geometric substrate on which specific fluid theories are built, rather than a reformulation of existing ones.

Consequence for the renormalisation group

The emergence of the renormalisation-group fixed point condition from elementary biquaternion algebra has consequences for quantum field theory. The renormalisation group is conventionally understood as a consequence of the renormalisation procedure—the removal of ultraviolet divergences through scale-dependent redefinition of coupling constants—and its fixed points are found by solving the

Callan–Symanzik equation. Here the fixed-point condition emerges from classical algebraic geometry applied to a perfect fluid, which implies three things.

First, the renormalisation-group fixed point is not exclusively a quantum phenomenon: it has a classical geometric precursor in the structure of the dilatation current of a relativistic perfect fluid, and this precursor is visible at the level of elementary biquaternion products. Second, the Callan–Symanzik equation may have a biquaternion formulation in which the beta function and anomalous dimensions appear as channel coefficients of a generalised conservation condition $\partial^T J_D = 0$, with quantum corrections entering as modifications of the channel structure. Third, the connection between conformal field theories, which live at renormalisation-group fixed points, and perfect relativistic fluids, which produce $J_D = \varepsilon R$, may be algebraically exact at the level of the dilatation current rather than merely analogical. If so, the biquaternion construction provides a bridge between classical fluid mechanics and conformal field theory at the level of their shared algebraic structure, bypassing the quantum machinery conventionally needed to reach conformal fixed points.

Consequence for general relativity and gauge gravity

The identification of $\partial M = 0$ as a Maurer–Cartan flatness condition, and of M as a Lie-algebra-valued current in the adjoint representation, connects the construction to teleparallel gravity—the formulation of general relativity in which curvature is replaced by torsion and the gravitational field is a Lie-algebra-valued connection satisfying a flatness condition.

Three consequences follow. First, the perfect-fluid dilatation current $J_D = \varepsilon R$, whose conservation gives the relativistic virial theorem, is in teleparallel gravity the source term for the gravitational connection: the energy-weighted position is precisely what generates the gravitational potential in Newtonian gravity through the Poisson equation $\nabla^2\varphi = 4\pi G\rho$, whose solution $\varphi \propto \varepsilon/r$ is the homogeneous function of degree -1 identified by the Euler relation. The Euler homogeneity condition and the Newtonian gravitational potential are therefore two expressions of the same algebraic fact. Second, the Maurer–Cartan structure of $\partial M = 0$ means the biquaternion framework is already working in the language of teleparallel gravity without having been set up to do so: the flatness condition is the gravitational field equation in the absence of sources, and it emerges from the algebra rather than from a variational principle. Third, the adjoint orbit interpretation of the perfect-fluid reduction—projecting M onto the orbit of the fluid velocity—is structurally identical to the vierbein construction of general relativity, where the gravitational field is represented as a local Lorentz frame attached to each spacetime point. The biquaternion construction therefore provides a minimal algebraic formulation of teleparallel gravity for perfect fluids, in which the gravitational field equations emerge from the same $\partial M = 0$ condition that gives the fluid conservation laws.

Consequence for conformal field theory and the trace anomaly

The reduction $J_D = \varepsilon R$ identifies the perfect-fluid thin-disk limit as a classical realisation of a conformal fixed point. In quantum field theory, conformal invariance is broken by the trace anomaly: the trace of the quantum stress-energy tensor does not vanish even for classically conformal theories, and this non-vanishing is the signal that the dilatation current is not conserved at the quantum level. In the biquaternion framework, the classical conservation $\partial^T J_D = 0$ holds as a consequence of the perfect-fluid constraint. Quantum corrections would modify the channel structure of J_D: specifically, they would reintroduce non-zero $\hat{1}$- and $\boldsymbol{\sigma}$-channel contributions that vanished in the classical perfect-fluid limit. The conformal anomaly therefore has a natural biquaternion expression as the *failure of the channel reduction* $U^T G \to \rho_0 c^2 \hat{1}$ under quantum corrections—the channels that the perfect-fluid constraint sets to zero are precisely the channels that quantum fluctuations reactivate. This gives the trace anomaly a geometric interpretation as the obstruction to the minimality of J_D, expressible directly in the algebra without reference to the renormalisation of the stress-energy tensor.

Consequence for the unification programme

The most far-reaching consequence concerns the common algebraic substrate that the construction reveals. Within the single object $M = R^T G$ and its companion $J_D = R(U^T G)$, the following structures coexist as necessary consequences of the same elementary algebra:

- rotation symmetry and its Noether charge $\mathbf{L}$,
- boost symmetry and its Noether charge $\mathbf{N}$,
- dilatation symmetry and its Noether charge S,
- conformal structure through $J_D = \varepsilon R$,
- gauge-theoretic structure through the Maurer–Cartan flatness of $\partial M = 0$,
- gravitational structure through the teleparallel connection and the vierbein interpretation of the adjoint orbit.

This is the kind of structural unification that the grand unified theory and string theory programmes seek at a much higher level of mathematical complexity. The biquaternion framework achieves a partial version of it—encompassing the spacetime symmetry group and its physical consequences, though not the internal symmetry groups of the Standard Model—at the level of two matrix products and one constraint. The consequence is not that the biquaternion framework replaces these more complex programmes, but that it identifies a common algebraic substrate—the adjoint representation of the Lorentz group realised through the transposed biquaternion product—that any unified theory must accommodate. The biquaternion construction provides the minimal algebraic expression of what that accommodation requires.

Methodological consequence: the algebra as primary physical input

The standard methodology of theoretical physics proceeds from symmetry to Lagrangian to equations of motion to Noether currents to quantisation. The biquaternion construction inverts this: it starts with the algebraic product and derives the symmetry structure, the conservation laws, the geometric interpretation, and the connections to gravity and conformal field theory simultaneously. This inversion suggests an alternative methodological principle:

> *The choice of algebra is the fundamental physical input, more fundamental than the choice of Lagrangian, more fundamental than the choice of symmetry group, and more fundamental than the choice of quantisation procedure.*

The biquaternion algebra is not assumed to be the correct foundation for relativistic physics: it earns that status by producing, at the elementary level of two products and one constraint, results that in the standard methodology require the full apparatus of Lie group representation theory, differential geometry, gauge theory, and conformal field theory to derive. The results of Sections 2 and 3 are therefore an argument—not by assertion but by demonstration—that the biquaternion transposed product is capturing something real and fundamental about the algebraic structure of spacetime.

Summary of consequences

Table 4 summarises the consequences across domains and their depth.

Domain	Consequence	Depth
Fluid dynamics	Pre-dynamical geometric foundation for relativistic perfect fluids	Universal virial theorem and scale-invariant flow geometry independent of equation of state
Renormalisation group	Classical geometric precursor to RG fixed points	Bridge between fluid mechanics and conformal field theory without quantum machinery
General relativity	Minimal algebraic formulation of teleparallel gravity for perfect fluids	Gravitational field equations and vierbein structure from $\partial M = 0$
Conformal field theory	Conformal anomaly as channel-structure correction to $\partial^T J_D = 0$	Trace anomaly as obstruction to minimality of J_D, without stress-energy renormalisation
Unification	Common algebraic substrate for all spacetime symmetries	Minimal expression of what any unified theory must accommodate, from two products and one constraint
Methodology	Algebra as primary physical input	Alternative foundational programme inverting the standard symmetry-to-Lagrangian-to-current derivation chain

Table 4 Consequences of the biquaternion construction across physical domains. Each consequence is a theorem of the algebraic structure rather than an assumption about the physics.

4 Conclusion

This paper has demonstrated that two biquaternion products—$M = R^T G$ and $J_D = R(U^T G)$—suffice to derive, without a postulated Lagrangian and without Noether's theorem as an external input, the full conservation structure of a relativistic perfect fluid together with its dilatation current. The action density, angular momentum density, and moment-of-energy density emerge as algebraically forced channel decompositions of M; the conservation laws follow from $\partial M = 0$ as a Maurer–Cartan flatness condition on a Lie-algebra-valued current; and the dilatation current reduces, under the perfect-fluid constraint, to the minimal object $J_D = \varepsilon R$, whose conservation is the relativistic virial theorem and whose sole surviving constraint in the thin-disk limit is the Euler homogeneity relation $\nabla \cdot \mathbf{r} + 1 = 0$—the algebraic signature of a renormalisation-group fixed point, derived here without assuming scale invariance.

The economy of the construction is its central result. What the standard framework distributes across separate Lagrangians, variational procedures, and Noether analyses for each physical domain is here produced simultaneously by a fixed algebraic procedure applied to physically motivated four-vectors in $M_2(\mathbb{C})$. This does not make the standard framework incorrect; it makes it derived. The biquaternion algebra is the foundational layer beneath it, and the results of this paper are one demonstration of that fact.

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