Primitive Idempotents and Central Splitting from Cl(1,3) to Cl(1,4)

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Abstract

This paper proves, by explicit Clifford-algebra computation, that the γ_0-diagonal complete orthogonal primitive decomposition of real Cl(1,3) has two sectors, while the extension to Cl(1,4) supplies a second independent commuting basis-blade involution, γ_{1234}. Because $I_5 = \gamma_0\gamma_{1234}$, the generated involution group is $Z_2 \times Z_2$, not $(Z_2)^3$. The resulting decomposition contains four primitive idempotents, each generating a real 8-dimensional minimal left ideal. The count four is itself fixed by Wedderburn theory, since each simple summand has rank two; the explicit content of the result is the basis-blade realization of the family and the demonstration of why a naive three-involution count would predict eight. The central idempotents $(1 \pm I_5)/2$ realize the real semisimple splitting Cl(1,4) $\cong M_2(H) \oplus M_2(H)$, where H denotes the quaternions. They do not, by themselves, supply a complex imaginary unit; a complex spinor interpretation still requires choosing an internal quaternionic complex structure within each ideal. Physical identifications with chirality, charge, helicity, or U(1) phase are therefore treated as conjectural.

Supporting results include: (i) a Pancharatnam-phase computation from projector triple products in Cl(3,0), confirming the half-solid-angle rule $2|\varphi| = \Omega$ for geodesic spherical triangles; (ii) a bivector-square classification of geometric-*i* candidates, with the observation that the chiral split in real Cl(1,3) is a complex-structure operation, not a real idempotent split; and (iii) a short positioning of the result relative to real Cl(0,6) and complex Cl_6 Standard Model programs. The paper is intended as a lower-dimensional real-idempotent audit, not as a derivation of a Standard Model generation.

Keywords: Clifford algebra; geometric algebra; primitive idempotents; central splitting; minimal left ideals; Cl(1,4); Pancharatnam phase.

1. Introduction

Geometric or Clifford algebra encodes metric structure, spinorial representations, and multivector geometry in a single algebraic language. In real geometric-algebra formulations of relativistic quantum theory, a recurring question is how complex structure enters: standard Dirac theory is usually written over C, while real Clifford algebras carry real, complex, or quaternionic matrix representations depending on signature.

This paper addresses a narrower question. Relative to the chosen γ_0-diagonal primitive decomposition, how many independent commuting basis-blade involutions can refine the projector family in Cl(1,3) versus Cl(1,4)? The answer is two sectors in Cl(1,3) and four sectors in Cl(1,4). The four-sector result is constrained by the identity $I_5 = \gamma_0\gamma_{1234}$, which makes the involution group a Klein four-group rather than a three-generator elementary abelian group.

The adjunction of a spacelike generator γ_4 to Cl(1,3) is also the first step of a familiar conformal ladder (Dirac, 1935, 1936). The present paper uses that ladder only as orientation; the conformal extension itself is left for future work.

Algebra	Dimension	Spin group	Physical role

Cl(1,3)	16	Spin(1,3) ≅ SL(2,C)	Lorentz spacetime algebra
Cl(1,4)	32	Spin(1,4)	de Sitter extension; present paper
Cl(2,4)	64	Spin(2,4) ≅ SU(2,2)	conformal algebra; twistors

The paper is deliberately more modest than Standard Model Clifford-algebra programs based on Cl(0,6), complex Cl_6, octonions, or Spin(10). Its purpose is to isolate a lower-dimensional real splitting mechanism and to make clear which claims are algebraic and which require further representation-theoretic input. The sector counts themselves are not in doubt: the Wedderburn structure $Cl(1,3) \cong M_2(H)$ and $Cl(1,4) \cong M_2(H) \oplus M_2(H)$ already fixes the size of any complete orthogonal primitive family at two and four respectively. What the explicit computation contributes is the concrete basis-blade realization of those families, the identification of the dependence $I_5 = \gamma_0\gamma_{1234}$ as the reason a naive three-involution count collapses from eight to four, and the alignment of the two surviving binary labels with the central splitting.

All computational checks use the Python package kingdon (Roelfs, 2025). The verification script appears in Appendix B.

2. Preliminaries

Let Cl(p,q) be the real Clifford algebra with $\gamma_\mu^2 = +1$ for $\mu < p$ and $\gamma_\mu^2 = -1$ for $\mu \geq p$, with $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0$ for $\mu \neq \nu$. Let $n = p + q$. The pseudoscalar

$$I_n = \gamma_0\gamma_1 \ldots \gamma_{n-1}$$

has square

$$I_n^2 = (-1)^{n(n-1)/2}(-1)^q.$$

The real Clifford isomorphism class is governed by p - q modulo 8 together with the total dimension (Chevalley, 1954; Lounesto, 2001; Porteous, 1995). In particular, under the signature convention used here,

$$Cl(1,3) \cong M_2(H), \quad Cl(1,4) \cong M_2(H) \oplus M_2(H),$$

where H denotes the quaternions and ⊕ denotes a direct sum of simple algebra components.

In $M_n(D)$, for D = R, C, or H, primitive idempotents are rank-one D-projectors. Hence, by the Wedderburn–Artin structure of the algebra (Lounesto, 2001; Porteous, 1995), any complete orthogonal primitive family $\{P_1,\ldots,P_n\}$, with $P_iP_j = \delta_{ij}P_i$ and $\sum_i P_i = 1$, has exactly n members. Combinatorially, k independent commuting involutions $\iota_j^2 = +1$, $[\iota_i,\iota_j] = 0$, generating $(Z_2)^k$, produce a 2^k-member family

$$P(s_1,\ldots,s_k) = 2^{-k} \prod_j (1 + s_j\iota_j).$$

If the involutions satisfy algebraic relations, some sign choices become inconsistent and vanish, so the number of distinct nonzero projectors is reduced. This is the mechanism behind the four-sector result in Cl(1,4).

3. Cl(1,3): two-sector decomposition

In Cl(1,3), with $\gamma_0^2 = +1$ and $\gamma_i^2 = -1$ for i = 1,2,3, the pseudoscalar $I_4 = \gamma_{0123}$ satisfies $I_4^2 = -1$. To refine $(1 + \gamma_0)/2$ by the same left-idempotent method, one needs a second element X satisfying $X^2 = +1$ and $[X,\gamma_0] = 0$.

Among basis blades, the timelike bivectors $\gamma_0 i$ square to +1 but anticommute with γ_0. The spatial bivectors γ_{ij} commute with γ_0 but square to −1. The pseudoscalar I_4 both anticommutes with γ_0 and squares to −1. Thus no nontrivial basis blade simultaneously supplies the required square and commutation properties. This statement concerns the γ_0-diagonal basis-blade refinement; it is not a claim that Cl(1,3) contains no other involutions in other decompositions.

The resulting complete family is

$$P_+ = (1 + \gamma_0)/2, \quad P_- = (1 - \gamma_0)/2.$$

These idempotents are orthogonal, sum to unity, and generate minimal left ideals of real dimension 8, consistent with Cl(1,3) ≅ $M_2(H)$. The naive candidate $(1 + \gamma_{12})/2$ fails because $\gamma_{12}^2 = -1$, so it is not idempotent.

4. Cl(1,4): four-sector decomposition

4.1 Dependence relation

In Cl(1,4), with $\gamma_0^2 = +1$ and $\gamma_i^2 = -1$ for i = 1,…,4, the pseudoscalar

$$I_5 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$$

is central and satisfies $I_5^2 = +1$. The basis blade γ_{1234} also satisfies $\gamma_{1234}^2 = +1$ and commutes with γ_0, since moving γ_0 past the four distinct anticommuting generators in γ_{1234} produces $(-1)^4 = +1$. However, the three natural involutions γ_0, γ_{1234}, and I_5 are not independent:

$$I_5 = \gamma_0\gamma_{1234}.$$

Therefore {1, γ_0, γ_{1234}, I_5} forms a Klein four-group $Z_2 \times Z_2$. Any two of the three nontrivial elements determine the third.

Scope of the involution search. Among basis blades of Cl(1,4), the only nontrivial elements squaring to +1 and commuting with both γ_0 and γ_{1234} are γ_0, γ_{1234}, and I_5. More general non-blade involutions exist inside the generated commutative subalgebra: for example, X = 2P − 1 for any projector P in the family squares to +1 and commutes with all generators. Such elements are sign functions of the existing decomposition and do not refine the four primitive sectors.

4.2 Main result

Theorem. Let Cl(1,4) be generated by $\gamma_0^2 = +1$ and $\gamma_1^2 = \cdots = \gamma_4^2 = -1$, with pairwise anticommutation. Let $I_5 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$. Then:

- $I_5 = \gamma_0\gamma_{1234}$, so γ_0 and γ_{1234} generate $Z_2 \times Z_2$.
- The four elements $P(\sigma_1,\sigma_2) = (1 + \sigma_1\gamma_0 + \sigma_2\gamma_{1234} + \sigma_1\sigma_2 I_5)/4$, with $\sigma_1,\sigma_2 \in \{+1,-1\}$, are idempotent, pairwise orthogonal, and sum to unity.
- Each idempotent generates a real 8-dimensional minimal left ideal and is therefore primitive.
- The result is consistent with Cl(1,4) ≅ $M_2(H) \oplus M_2(H)$: each simple summand contributes two primitive sectors.

Proof sketch. The identity follows immediately from $I_5 = \gamma_0(\gamma_1\gamma_2\gamma_3\gamma_4)$. Expanding the formal eight-candidate family

$$(1/8)(1+s_1\gamma_0)(1+s_2\gamma_{1234})(1+s_3 I_5)$$

using $I_5 = \gamma_0\gamma_{1234}$ gives

$$P(s_1,s_2,s_3) = (1/8)[(1+s_1s_2s_3) + (s_1+s_2s_3)\gamma_0 + (s_2+s_1s_3)\gamma_{1234} + (s_1s_2+s_3)I_5].$$

All coefficients vanish when $s_3 = -s_1s_2$. Setting $s_3 = s_1s_2$ gives the four nonzero projectors listed above. Computational verification confirms idempotency, orthogonality, completeness, and ideal dimension. Since Cl(1,4) ≅ $M_2(H) \oplus M_2(H)$, a minimal left ideal in each simple summand has real dimension 2

$\dim_{\mathbb{R}}(H) = 8$. The verified 8-dimensional ideals are therefore minimal, so the corresponding idempotents are primitive.

Central decomposition. Each primitive sector lies in a single simple summand, and the partition is read off directly from the central idempotent. Multiplying $P(\sigma_1,\sigma_2)$ by $(1 + I_5)/2$ returns $P(\sigma_1,\sigma_2)$ when $\sigma_1\sigma_2 = +1$ and 0 when $\sigma_1\sigma_2 = -1$, because I_5 acts on the sector as the scalar $\sigma_1\sigma_2$; the complementary idempotent $(1 - I_5)/2$ reverses the two cases. Sectors I and IV ($\sigma_1\sigma_2 = +1$) therefore lie in the $I_5 = +1$ copy of $M_2(H)$, and Sectors II and III ($\sigma_1\sigma_2 = -1$) in the $I_5 = -1$ copy. Each simple summand carries exactly two of the four primitive sectors, which is the four-sector decomposition read through the semisimple splitting.

4.3 Sector table and interpretation

Algebraic sector	Projector	(γ_0, γ_{1234}, I_5) eigenvalues	Simple summand
Sector I	$(1 + \gamma_0 + \gamma_{1234} + I_5)/4$	(+, +, +)	$I_5 = +1$ copy
Sector II	$(1 + \gamma_0 - \gamma_{1234} - I_5)/4$	(+, −, −)	$I_5 = -1$ copy
Sector III	$(1 - \gamma_0 + \gamma_{1234} - I_5)/4$	(−, +, −)	$I_5 = -1$ copy
Sector IV	$(1 - \gamma_0 - \gamma_{1234} + I_5)/4$	(−, −, +)	$I_5 = +1$ copy

A natural but unproven reading is to associate the γ_0 eigenvalue with a particle/antiparticle-like distinction and the γ_{1234} eigenvalue with a chirality-like distinction, by analogy with the standard γ_5 projector. However, γ_{1234} is not physically identical to γ_5 unless an additional representation map is specified. The spin degree of freedom sits inside each H^2 ideal as the choice of a complex basis within the quaternionic module.

The central idempotents $(1 \pm I_5)/2$ split Cl(1,4) into two real simple components. A complex spinor interpretation still requires choosing an internal quaternionic complex structure within each minimal ideal. This real semisimple splitting is the sense in which the paper uses the phrase central-idempotent splitting: it is a geometric substitute for one role of external complexification, but it is not complexification itself.

5. Relation to Cl(0,6) and complex Cl_6 Standard Model programs

The present result should be distinguished from higher-dimensional Clifford-algebra Standard Model programs. Lu constructs a real Cl(0,6) framework by factorizing γ_0 into three generators, so that one algebraic spinor has 64 real components, matching one Standard Model generation including a right-handed neutrino (Lu, 2024). Stoica instead uses a complex Clifford algebra $Cl_6 = Cl(\chi^\dagger + \chi)$ with a Witt decomposition to organize leptons, quarks, electric charge, color, and electroweak structure (Stoica, 2018). Furey-type approaches similarly exploit complex Clifford or octonionic-Witt machinery to obtain charge/color ideals (Furey, 2016).

By contrast, this paper does not attempt to derive a Standard Model generation. It isolates the lower-dimensional real-idempotent mechanism available between Cl(1,3) and Cl(1,4). In this sense, the result is a diagnostic step: it says exactly what real primitive sectors are present before one moves to Cl(0,6), complex Cl_6, Cl(2,4), or Spin(10)-type constructions.

This also clarifies a recurring source of confusion in the real-Clifford Standard Model literature, where two algebraically distinct operations are both called chirality projection. A left-idempotent projector $(1 \pm I)/2$ splits the algebra into minimal left ideals and requires $I^2 = +1$ to be idempotent. A double-sided operation $(\psi \pm I\psi I)/2$, of the kind used in Lu-style chirality projections, instead grades a spinor by the sign of conjugation by I, and stays well-defined even when $I^2 = -1$. The obstruction $I_4^2 = -1$ therefore rules out $(1 \pm I_4)/2$ as a real left-idempotent split in Cl(1,3), but it does not rule out the double-sided chirality projection, nor any chirality operation built from a square-(−1) element acting by conjugation. In Cl(1,4), γ_{1234} supplies a genuine square-(+1) involution and so supports the left-idempotent form directly.

6. Rotor and bilinear current

As a consistency check, take the composite rotor

$$R = \exp[-(\theta/2)\gamma_{12}]\,\exp[(\varphi/2)\gamma_{03}],\quad \theta = \pi/6,\quad \varphi = 0.3,$$

which satisfies $R\tilde{R} = 1$. With $P_{ref} = P(\text{Sector I})$ and $\Psi = ARP_{ref}$, define

$$j^0 = \langle \Psi\gamma_0\tilde{\Psi}\rangle_0.$$

Then

$$j^0 = A^2\langle P_{ref}\rangle_0 = A^2/4.$$

For A = 2, the computation gives $j^0 = 1$. The factor 1/4 is nothing more than the scalar (grade-0) part of P(Sector I), which every primitive idempotent here carries because the family sums to unity over four sectors; it is the specific projector's scalar coefficient, not a universal trace factor. In Cl(1,3), using $P_+ = (1 + \gamma_0)/2$ gives the analogous scalar factor 1/2, the scalar part of a two-sector projector.

The full Doran-Lasenby-Gull decomposition of a Dirac spinor includes a Yvon-Takabayasi angle β (Doran, Lasenby, & Gull, 1993). The simplified density-first ansatz $\Psi = \sqrt{\rho}\, RP$ implicitly fixes β = 0. Exposing β is necessary for a serious treatment of particle/antiparticle mixing, but lies beyond the present idempotent computation.

7. Bivector classification and the chiral-projector issue

For simple bivectors B whose square is scalar, the sign of B^2 classifies the exponential: $B^2 = -1$ gives elliptic rotations, $B^2 = 0$ gives parabolic/nilpotent transformations, and $B^2 = +1$ gives hyperbolic boosts. In dimensions four and higher, nonsimple bivectors may have nonscalar squares, so a full spectral or outer-exponential treatment is needed for the general case.

A geometric imaginary unit requires square −1. In Cl(1,3), spatial bivectors such as γ_{12} and the pseudoscalar I_4 have this property. The Hestenes form of the Dirac equation uses a spatial bivector as a geometric *i* (Hestenes, 1966), so no fifth dimension is required for that role. The Cl(1,4) extension provides a different object: γ_{1234} is an involution with square +1, useful for real idempotent splitting but not itself an imaginary unit.

This resolves the chiral-projector issue, complementing the projector/conjugation distinction drawn in Section 5. In real Cl(1,3), the Weyl decomposition can be represented as a complex-structure operation using right multiplication by an element squaring to −1. It is not a real idempotent split of the form $(1 \pm I_4)/2$, because $I_4^2 = -1$. Standard chiral projectors require an element squaring to +1, conventionally $\gamma_5 = iI_4$ in complexified Dirac theory. In Cl(1,4), γ_{1234} plays an algebraic role analogous to γ_5: it squares to +1 and supports real idempotents. The physical identification with chirality remains representation-dependent.

8. Supporting result: Pancharatnam phase in Cl(3,0)

The central result of this paper is an instance of a broader theme: real idempotent arithmetic can carry structure that is often assumed to require an external complex unit, with the central splitting $(1 \pm I_5)/2$ substituting for one role of complexification. The Pancharatnam computation below is a second, independent instance of the same theme. Projector products in a purely real Clifford algebra already encode geometric phase information without introducing an external complex *i* or passing to a higher-dimensional algebra: a phase that appears to need an imaginary unit is recovered here as the ratio of pseudoscalar to scalar grades of a real triple product. In Cl(3,0), rank-one projectors $\rho_u = (1+u)/2$ for unit vectors u are the multivector form of pure spin-1/2 density operators. The triple product $\rho_u\rho_v\rho_w$ has scalar and pseudoscalar parts

$$\langle \rho_u \rho_v \rho_w \rangle_0 = (1 + u\cdot v + u\cdot w + v\cdot w)/8,$$

$$\langle \rho_u \rho_v \rho_w \rangle_3 = [u\cdot(v \times w)] I_3/8.$$

The signed Pancharatnam phase is φ = atan2(pseudoscalar coefficient, scalar part) (Pancharatnam, 1956). For geodesic spherical triangles it satisfies the half-solid-angle rule

$$2|\varphi| = \Omega,$$

where Ω is the spherical area. Orientation matters: reversing the vertex order flips the sign of the pseudoscalar coefficient and hence the sign of φ, while $|\varphi|$ and Ω are unchanged. For the triangle ($\hat{z}$, $\hat{x}$, $\hat{y}$), the computation gives scalar = pseudoscalar = 1/8, $|\varphi| = \pi/4$, and $2|\varphi| = \pi/2$, matching Girard area at machine precision.

For non-geodesic paths, the Berry phase requires integrating the connection 1-form (Berry, 1987). For mixed states $|u| < 1$, ρ_u is no longer a rank-one idempotent; the pure-state Pancharatnam interpretation does not transfer automatically, although the density-operator product remains well-defined.

9. Conclusion

In the γ_0-diagonal complete orthogonal primitive decomposition of Cl(1,3) ≅ $M_2(H)$, the primitive family has two members. In Cl(1,4) ≅ $M_2(H) \oplus M_2(H)$, the identity $I_5 = \gamma_0\gamma_{1234}$ constrains the apparent three-involution set to a Klein four-group $Z_2 \times Z_2$, yielding a four-member primitive decomposition with real ideal dimension 8. The supporting computations on rotor normalization, bilinear current, bivector classification, and Pancharatnam phase help position this result within a broader geometric-algebra context without altering its core algebraic claim.

The central idempotents $(1 \pm I_5)/2$ realize the real semisimple splitting of Cl(1,4). They are not a complex imaginary unit, and physical identifications of the four sectors with chirality, charge, helicity, particle/antiparticle labels, or U(1) phase require additional representation theory. The main contribution of the paper is therefore a precise real-Clifford idempotent theorem and a computational audit of its consequences.

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Appendix A. Verification log

All residuals are below 1e-10 unless otherwise stated.

- Cl(1,3): $P_+^2 - P_+ = 0$; $P_+ + P_- = 1$; $P_+P_- = 0$; ideal dimension = 8.
- Cl(1,4): $I_5 - \gamma_0\gamma_{1234} = 0$.
- Basis-blade involution search returns $\{1, \gamma_0, \gamma_{1234}, I_5\}$.
- Non-blade involutions such as X = 2P − 1 exist but do not refine the primitive sector count.
- Formal eight-candidate family: four nonzero projectors with $s_3 = s_1s_2$; four zero candidates with $s_3 \neq s_1s_2$.
- Four genuine projectors: $\sum P_i = 1$; $P_i^2 = P_i$; $P_iP_j = 0$ for $i \neq j$; ideal dimension = 8.
- Rotor test: $R\tilde{R} = 1$; for A = 2, $j^0 = 1$ and $j^0/A^2 = 1/4$, not 1/ideal dimension = 1/8.
- Pancharatnam test for $(\hat{z}, \hat{x}, \hat{y})$: scalar = pseudoscalar = 1/8; $|\varphi| = \pi/4$; $2|\varphi| = \pi/2$.

Appendix B. Verification code

Requires: pip install kingdon

```
from kingdon import Algebra
import numpy as np
import math

TOL = 1e-10

alg = Algebra(1, 4, start_index=0)
g0 = alg.blades["e0"]
g1234 = alg.blades["e1234"]
I5 = alg.blades["e01234"]

# Dependence relation
assert abs((I5 - g0 * g1234).normsq().e) < TOL, "I5 != g0*g1234"

# Four primitive idempotents
Ps = []
for s1 in (+1, -1):
    for s2 in (+1, -1):
        s3 = s1 * s2
        P = (1 + s1*g0) * (1 + s2*g1234) * (1 + s3*I5) / 8
        assert abs((P*P - P).normsq().e) < TOL, f"P({s1},{s2}) not idempotent"
        Ps.append(P)

total = Ps[0]
for P in Ps[1:]:
    total = total + P
assert abs((total - 1).normsq().e) < TOL, "sum(Ps) != 1"

for i in range(4):
    for j in range(i + 1, 4):
        assert abs((Ps[i] * Ps[j]).normsq().e) < TOL, "projectors not orthogonal"

# Ideal dimension for the first sector
vecs = [np.zeros(32) for _ in range(32)]
for k, (name, blade) in enumerate(alg.blades.items()):
    bp = blade * Ps[0]
    for idx, coeff in bp.items():
        vecs[k][idx] = float(coeff)
assert np.linalg.matrix_rank(np.array(vecs), tol=1e-8) == 8
```

```
# Basis-blade involution search backing Appendix A
def mag2(x): return sum(float(c)**2 for c in x.values())
hits = [nm for nm, b in alg.blades.items() if nm != "e"
        and abs((b*b).e - 1) < TOL
        and mag2(b*g0 - g0*b) < TOL and mag2(b*g1234 - g1234*b) < TOL]
assert set(hits) == {"e0", "e1234", "e01234"}, hits

# Central-summand membership (Section 4.2)
for (s1, s2) in [(1, 1), (1, -1), (-1, 1), (-1, -1)]:
    P = (1 + s1*g0) * (1 + s2*g1234) * (1 + s1*s2*I5) / 8
    Pc = (1 + (s1*s2)*I5) / 2
    assert abs((P*Pc - P).normsq().e) < TOL

# Non-blade involution check
X = 2 * Ps[0] - 1
assert abs((X*X - 1).normsq().e) < TOL, "X^2 != 1"
assert abs((X*g0 - g0*X).normsq().e) < TOL, "[X,g0] != 0"
assert abs((X*g1234 - g1234*X).normsq().e) < TOL, "[X,g1234] != 0"

# Rotor and current bilinear
g12 = alg.blades["e12"]
g03 = alg.blades["e03"]
theta = math.pi / 6
phi = 0.3
A = 2
R = (math.cos(theta/2) - math.sin(theta/2)*g12) * \
    (math.cosh(phi/2) + math.sinh(phi/2)*g03)
assert abs((R * ~R - 1).normsq().e) < TOL
Psi = A * R * Ps[0]
j0 = (Psi * g0 * ~Psi).grade(0).e
assert abs(j0 - 1.0) < 1e-6, f"j0 = {j0}"

# Pancharatnam phase in Cl(3,0)
pa = Algebra(3, 0)
e1 = pa.blades["e1"]
e2 = pa.blades["e2"]
e3 = pa.blades["e3"]
rho = lambda u: 0.5 * (1 + u[0]*e1 + u[1]*e2 + u[2]*e3)
T = rho([0, 0, 1]) * rho([1, 0, 0]) * rho([0, 1, 0])
sc = T.e
# I3 = e123 has I3**2 = -1; right-multiplying T by e1*e2*e3 (= I3)
# sends the grade-3 part to a scalar, and the leading minus undoes
# I3**2 = -1, so ps is the pseudoscalar coefficient of T
ps = -(T * e1 * e2 * e3).e
phase = math.atan2(ps, sc)
assert abs(2 * abs(phase) - math.pi/2) < TOL

print("All assertions passed.")
```