# The Reconstruction Horizon and the Emergence of the Cosmological Constant in a Timeless Euclidean Model

A. N. Smirnov

## Abstract

The present work investigates the origin of the cosmological constant within a timeless Euclidean model on $\mathbb{E}^4$ with a single real harmonic field. The fundamental setting contains neither fundamental time, nor a Lorentzian metric, nor pre-given cosmological dynamics; these structures are treated as effective and reconstructed only within an operationally admissible regime.

It is shown that remote reconstruction relative to a local data region $\Omega_0$ has an elliptic character and is accompanied by the exponential instability of inverse continuation. This leads to an infrared-induced narrowing of the stably reconstructible spectral window and to an increase in the minimum distinguishable wavelength as the scale of reconstruction grows. On this basis, a finite maximum scale of stable reconstruction $L_{\text{max}}$ is obtained and interpreted as an operational reconstruction horizon.

It is then shown that, if this infrared limit is represented within the already reconstructed closed covariant gravitational regime, the leading universal local zero-derivative response takes the form of a cosmological term. Its natural scale is given by

$$\Lambda_{\text{IR}} = \chi L_{\text{max}}^{-2},$$

where $\chi$ is a dimensionless coefficient depending on the details of the cosmological reconstruction class.

The relation to redshift is analyzed separately. As the spectral window narrows, the causal reconstructibility of a remote signal requires the transfer of its observable content into the admissible infrared range. In the effective FLRW description, this transfer is realized as cosmological redshift. Redshift thereby admits a reconstructive interpretation as a mechanism preserving the causal recognizability of remote signals while short-wavelength details are lost.

In the target homogeneous-isotropic sector, the positive FLRW branch is singled out as the branch compatible with a finite reconstruction horizon, redshifting infrared transfer, and an asymptotic Hubble scale $H_{\text{IR}} \sim L_{\text{max}}^{-1}$. Thus, the cosmological constant is interpreted not as fundamental vacuum energy and not as non-vacuum hidden matter, but as a universal geometric infrared response to the limitation of stable remote reconstruction. The result is structural in character: the full FLRW phenomenology, the exact value of $H_0$, the full redshift law $z(L)$, and the possible relation to the cosmic microwave background remain tasks for further analysis.

# 1 Introduction

## 1.1 The cosmological constant problem in a reconstructive setting

The cosmological constant problem is traditionally formulated as the question of the origin of the term $\Lambda g_{ab}$ in the Einstein equations and of the reason for its observed smallness compared with natural microscopic scales [6]. In the standard setting, this problem usually presupposes that spacetime, the Lorentzian metric, and geometric dynamics are already given as initial elements of an effective physical theory.

The present work considers a different situation. The fundamental level is given by a timeless Euclidean model in which neither an a priori Lorentzian geometry nor a pre-given cosmological regime is present. Observable spacetime, causal structure, and gravitational description arise only as effective structures within an operationally admissible reconstruction.

In standard approaches, the cosmological constant is usually discussed either as a fundamental parameter in the gravitational equations, or as a manifestation of vacuum energy, or in connection with dynamical models of dark energy and infrared modifications of gravity. These directions will be briefly compared with the present setting below. Here it is important to emphasize the main difference: in the model considered here, the cosmological term is not introduced as an initial parameter, is not identified with vacuum energy, and is not obtained by adding a new dynamical sector. Instead, the possibility is considered that the effective cosmological term is the geometric form of the large-scale limit of stable remote reconstruction.

This shift of viewpoint is especially important within the program in which the special-relativistic and gravitational layers of the effective description have previously been reconstructed [1, 2]. In those works, the Lorentzian structure, causal order, and effective gravitational geometry are not postulated as initial entities, but arise as stable forms of operational reconstruction. Therefore, the cosmological constant should be analyzed in the same logic: not as a primary parameter of the fundamental level, but as a possible large-scale invariant of the effective geometric sector.

The main question of the paper is the following. If the fundamental level is given by the elliptic Laplace equation, while the observable causal structure arises only as an effective hyperbolic regime of reconstruction, then the remote continuation of local data cannot remain stable at arbitrarily large scales. A finite scale of stable reconstruction $L_{\max}$ appears. If this infrared limit is to be represented within the already reconstructed closed covariant GR description, then its leading universal zero-derivative form is that of a cosmological term. In this sense, the present work offers not a numerical solution of the entire standard cosmological constant problem, but a reconstructive mechanism for the emergence of an effective contribution

$$\Lambda_{\mathrm{IR}} g_{\mu\nu}$$

with the natural scale $L_{\max}^{-2}$.

## 1.2 Relation to existing approaches

The reconstructive interpretation of the cosmological constant proposed in the present work belongs to a broader range of approaches to dark energy, infrared gravity, and the emergent structure of spacetime. It is therefore useful to briefly clarify how it is related to existing directions.

First, the present approach differs from the standard vacuum formulation of the cosmological constant problem. In the usual formulation, the central difficulty consists in relating the geometric term $\Lambda g_{\mu\nu}$ to the vacuum energy of quantum fields and in explaining the anomalously small observed value of the corresponding energy density [6]. In the model considered here, $\Lambda_{\mathrm{IR}}$ is not identified with a sum of zero-point fluctuations and is not obtained by renormalizing vacuum energy. It receives a reconstructive interpretation as a geometric parameter of infrared incompleteness of reconstruction, that is, as a response to the finiteness of stable large-scale recovery, rather than as microscopic vacuum energy.

Second, this mechanism differs from dynamical models of dark energy, such as quintessence, $k$-essence, phantom-like models, and other scalar-field scenarios [7]. In those approaches, accelerated expansion is associated with an additional dynamical degree of freedom and its effective equation of state. In the present work, no new scalar sector of dark energy is introduced. The leading contribution $\Lambda_{\mathrm{IR}} g_{\mu\nu}$ is of zero order in derivatives and belongs to the universal geometric sector of the effective description. Thus, the issue is not a new field on an already given spacetime, but the reconstructive origin of the infrared geometric scale itself.

Third, the proposed mechanism differs from programs of modified gravity at large scales [8]. In modified gravitational theories, late-time acceleration is usually explained by changing the gravitational equations, adding new geometric invariants, additional fields, a bimetric structure, $f(R)$-terms, braneworld mechanisms, or other infrared degrees of freedom. In the model considered here, by contrast, the closed GR regime is regarded as already reconstructed at the previous stage of the program. The present paper does not replace it by an external modification, but considers how, within this closed regime, a universal zero-derivative contribution associated with the limit of stable reconstruction can be represented.

Fourth, the work has conceptual proximity to holographic and informational ideas, since large-scale geometry is also related here to the limitation of accessible information [11, 12]. The difference, however, is substantial. In holographic approaches, the central role is played by entropic or informational bounds associated with area, light-sheets, horizons, the number of degrees of freedom, or UV/IR relations. In the present work, the initial object is not an entropic boundary, but the spectral bound of stable elliptic reconstruction:

$$k \leq k_{\max}(L).$$

The horizon appears not as a postulated holographic surface, but as the limit of causally consistent continuation of local data.

Finally, the proposed setting is related to a broader class of programs in which spacetime and its causal structure are not taken as fundamental. Such directions include, for example, causal sets, causal dynamical triangulations, shape dynamics, relational-time frameworks, and other approaches to emergent or background-independent description [13, 14, 15, 16]. Their common feature is the refusal to fully accept classical spacetime as an initial structure.

The present model differs, however, in that its initial level is not a discrete causal order, not a sum over triangulated geometries, not an exchange of symmetries in canonical gravity, and not a relational-time construction, but a timeless Euclidean setting with a single real harmonic field. The Lorentzian structure, effective causality, geometric GR regime, and cosmological term are treated here as successive levels of operational reconstruction.

Thus, the reconstruction horizon considered in the present work is neither a vacuum-energy explanation of $\Lambda$, nor dynamical dark energy, nor an external modification of gravity, nor a direct version of the holographic principle. Its role is different: it defines an infrared limit of stable reconstruction which, provided that the closed covariant GR regime is preserved, receives a geometric representation in the form of a cosmological term. In this sense, the present paper occupies a position at the intersection of infrared ideas in cosmology, emergent approaches to spacetime, and operational reconstruction of the effective physical scene.

### 1.3 Aim of the work and relation to previous results

The present paper relies on results obtained previously within the same program. From the work on special relativity [1], it uses the derivation of the locally Lorentzian structure as an effective regime of consistent event reconstruction. From the work on the gravitational sector [2], it uses the already derived closed covariant GR regime as a compensating geometric description arising from the requirements of causal reconstruction. From the work on the quantum layer [3], it uses the distinction between the fundamental timeless level and secondary effective structures, including vacuum and operator objects. It is precisely this distinction between the fundamental elliptic level and the effective hyperbolic regime that is essential for the cosmological interpretation of remote reconstruction.

The aim of the present work is to consider the missing cosmological step of this program: to show how the finite scale of stable remote reconstruction, arising from the elliptic instability of the fundamental equation, can be represented in the closed GR regime as a universal infrared geometric contribution. In this way, the paper relates the operational reconstruction horizon to the effective cosmological term, without using the mean matter density, foliation curvature, or full FLRW dynamics as initial assumptions.

The present paper also clarifies the role of redshift in the reconstructive setting. Redshift is treated not as an external phenomenological postulate, but as the FLRW form of infrared transfer required to preserve the causal reconstructibility of remote signals under the narrowing of the stable spectral window. The corresponding relation to the Hubble scale and to the positive FLRW branch is analyzed below in Sec. 6.

The apparatus of reconstruction classes and fine-tuning is not redeveloped in the present paper. The corresponding analysis was given in a separate work [4] and is used here only to a minimal extent: to clarify under what conditions the observer-dependent limit $L_{\max}^{(O,n)}$ can be interpreted as a parameter of the corresponding cosmological reconstruction class, rather than as an arbitrary characteristic of a single foliation.

The boundaries of the result should be emphasized. The paper does not construct the full cosmological phenomenology, does not compute the function $H(z)$, does not analyze structure growth, does not derive the full redshift law $z(L)$, does not compute the observed value of $H_0$, and does not derive the CMB spectrum. The mean total density of the observable universe is not used as an input quantity, since in the standard parametrization it already includes the component phenomenologically described as dark energy. In the present work, this component is reinterpreted as a geometric infrared contribution, rather than being taken as an initial energy density.

Possible implications for a limiting background layer are discussed in Sec. 7; they are not used as input assumptions in the main derivation. In the present paper, such a layer is not identified with the observed CMB in the full sense; such an identification would require a separate derivation of the spectrum, temperature scale, anisotropies, polarization, and relation to structure formation.

Finally, the proposed interpretation is consistent with the previously made distinction between the geometric infrared contribution, the gravitationally relevant non-vacuum part of the hidden content, and the standard quantum-vacuum heuristic [3, 5]. In the present setting, $\Lambda_{\mathrm{IR}}$ is identified neither with dark matter nor with fundamental vacuum energy in the usual quantum-field-theoretic sense. It is a distinct geometric mechanism associated with the limitation of stable remote reconstruction.

## 1.4 Main idea of the work

The main idea of the work consists in separating two stages. At the first stage, in a fixed flat foliation, the stability of remote reconstruction of data for the fundamental harmonic field is analyzed. This stage does not use foliation curvature, matter density, or full cosmological dynamics. At the second stage, the infrared scale obtained in this way is represented in the already reconstructed effective GR regime as a universal zero-derivative geometric contribution.

Let $L$ denote the scale of remote reconstruction in the chosen flat foliation. The elliptic character of the fundamental equation implies that backward continuation of high-frequency modes exponentially amplifies errors. Therefore, at the scale $L$, only a finite spectral window remains stably reconstructible. The corresponding maximum wavenumber $k_{\max}(L)$ decreases as $L$ grows, whereas the minimum distinguishable wavelength

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}$$

increases. This effect will be referred to below as the infrared-induced narrowing of the spectral window: the limitation arises from the large-scale remoteness of reconstruction, although technically it appears as the loss of stable access to large $k$.

Since causally consistent reconstruction requires finite spectral resolution, a critical minimum wavelength $\lambda_*$, compatible with a stable causal description, is introduced. The maximum scale of stable reconstruction is then determined by the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*.$$

At this stage, $L_{\max}$ is neither a cosmological constant nor an already given FLRW horizon, but an operational limit of stable remote reconstruction relative to the chosen observer-dependent subregime. Its interpretation as a cosmological scale requires an additional condition: the stability of this limit within the corresponding reconstruction class and the representation of the resulting infrared scale in an effective geometric language.

In the previous GR reconstruction [2], the cosmological term arose as an admissible universal zero-derivative contribution to the effective geometric action, but its reconstructive origin and scale were not analyzed. The present paper clarifies this point: if the finite limit of stable remote reconstruction is to be represented within the closed covariant GR regime, then the leading universal zero-derivative contribution takes the form

$$\Lambda = \Lambda_{\mathrm{IR}}, \qquad \Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2},$$

where $\chi$ is a dimensionless coefficient depending on the details of the cosmological reconstruction class.

An important element of the work is also the relation between the narrowing of the spectral window and redshift. If a remote signal with emitted wavenumber $k_{\mathrm{em}}$ is to remain causally reconstructible, then its observed wavenumber must satisfy the condition

$$k_{\mathrm{obs}}(L) \le k_{\max}(L).$$

In the effective FLRW description,

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1+z(L)},$$

and therefore

$$\frac{k_{\mathrm{em}}}{1+z(L)} \le k_{\max}(L).$$

Thus, redshifting infrared transfer becomes a necessary condition for preserving the causal recognizability of remote signals under the monotonic narrowing of the spectral window.

In the target homogeneous-isotropic sector, this logic singles out the positive FLRW branch. It is this branch that makes mutually compatible the finite reconstruction horizon, the cosmological term, the Hubble scale of the redshifting regime, and the causality-preserving infrared transfer of remote signals. The possibility of an infrared-smoothed limiting background regime is considered below as a structural implication, not as a full derivation of the observed CMB.

Thus, the central logic of the paper is as follows: the elliptic instability of remote reconstruction in a flat foliation leads to the narrowing of the stable spectral window; this narrowing defines the finite scale $L_{\max}$; under GR closure, the corresponding infrared limit is represented as an effective cosmological term $\Lambda_{\mathrm{IR}} g_{\mu\nu}$; the positive FLRW branch provides a causality-preserving redshifting infrared transfer; and near the reconstruction horizon, individually resolved event structure passes into an infrared-smoothed limiting regime.

## 1.5 Main results

The main results of the paper can be formulated as follows.

(R1) In a fixed flat foliation, it is shown that the elliptic character of remote reconstruction leads to an infrared-induced narrowing of the stable spectral window. At the reconstruction scale $L$, the maximum stably reconstructible wavenumber $k_{\max}(L)$ decreases as $L$ grows, whereas the corresponding minimum distinguishable wavelength $\lambda_{\min}(L)$ increases. Thus, large-scale reconstruction becomes increasingly infrared and increasingly less sensitive to the short-wavelength structure of remote regions.

(R2) Taking into account the finite spectral resolution of the causally consistent reconstruction regime, the existence of a finite maximum scale of stable reconstruction $L_{\max}$ is obtained, determined by the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*.$$

At this stage, $L_{\max}$ has the status of an operational reconstruction horizon associated with the given observer-dependent subregime and chosen foliation. Its interpretation as a cosmological infrared scale requires inter-observer compatibility within the corresponding reconstruction class.

(R3) It is shown that, under such inter-observer compatibility, the scale $L_{\max}$ can be interpreted as an infrared parameter of the corresponding effective cosmological sector. If this limit is represented within the previously reconstructed closed covariant GR regime, then the leading universal zero-derivative contribution takes the form of a cosmological term

$$\Lambda_{\mathrm{IR}} g_{\mu\nu},$$

where

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2}.$$

Therefore, at the level of order of magnitude,

$$|\Lambda_{\mathrm{IR}}| \sim L_{\max}^{-2}.$$

(R4) It is shown that the narrowing of the spectral window makes redshifting infrared transfer a necessary condition for preserving the causal reconstructibility of remote signals. If $k_{\mathrm{em}}$ is the characteristic wavenumber of the signal in the emission region, and $k_{\mathrm{obs}}(L)$ is the observed wavenumber after reconstruction at the scale $L$, then causal reconstructibility requires

$$k_{\mathrm{obs}}(L) \le k_{\max}(L).$$

In the effective FLRW description,

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)},$$

and therefore the condition

$$\frac{k_{\mathrm{em}}}{1 + z(L)} \le k_{\max}(L)$$

must hold. Redshift thereby admits a reconstructive interpretation as a causality-preserving infrared transfer: it preserves the causal recognizability of a remote event at the cost of losing short-wavelength details.

(R5) In the target homogeneous-isotropic cosmological sector, the positive FLRW branch is singled out as the branch compatible with this causality-preserving infrared transfer. In this branch,

$$\Lambda_{\mathrm{IR}} > 0,$$

and the asymptotic Hubble scale of the infrared-dominated regime satisfies

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3}.$$

Together with

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2},$$

this gives the structural relation

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}.$$

Thus, the reconstruction horizon, the Hubble scale, the positive cosmological constant, and redshift are mutually consistent manifestations of a single infrared constraint. Under a self-consistent identification of $L_{\max}$ with the Hubble scale of the late infrared-dominated branch, it is natural to expect $\chi = O(1)$, although the exact value of $\chi$ is not computed in the present work.

(R6) It is shown that, near the reconstruction horizon, the individually resolved event structure must give way to an infrared-smoothed limiting regime. Indeed, as $L \to L_{\max}$, the stably reconstructible spectral window narrows to the threshold range

$$k \le k_*, \qquad k_* = \frac{2\pi}{\lambda_*}.$$

Therefore, any event structure requiring wavenumbers $k > k_*$ ceases to be individually reconstructible. This points to the possibility of a CMB-like limiting background layer as an operational boundary of reconstruction. In the present paper, this layer is not identified with the observed CMB in the full sense; such an identification would require a separate derivation of the spectrum, temperature, anisotropies, polarization, and relation to structure formation.

(R7) The cosmological constant is interpreted not as fundamental vacuum energy and not as non-vacuum hidden matter, but as a universal large-scale geometric response to the limitation of stable remote reconstruction. The exact numerical value of $\Lambda_{\mathrm{IR}}$ is not computed in the present work; the normalization of the coefficient $\chi$, the precise comparison of $L_{\max}$ with observed cosmological scales, the derivation of the full dependence $z(L)$, the computation of $H_0$, the inclusion of the full FLRW phenomenology, and the analysis of a possible limiting background layer remain open.

The structure of the paper is as follows. Section 2 introduces the minimal setting, the fixed flat foliation, and the working operational regime. Section 3 analyzes elliptic remote reconstruction and derives the infrared-induced narrowing of the spectral window. Section 4 proves the existence of a finite maximum scale of stable reconstruction $L_{\max}$. Section 5 shows that, under GR closure, this scale defines the leading universal zero-derivative geometric contribution. Section 6 discusses the positive FLRW branch, the necessity of redshifting infrared transfer, and the relation to the Hubble scale. Section 7 discusses the status of the result, the transition to theorematic development, the limits of applicability, and a possible CMB-like limiting background layer.

# 2 Minimal setting and operational regime

## 2.1 Fundamental model and fixed flat working foliation

As the fundamental setting, we consider four-dimensional Euclidean space $\mathbb{E}^4$ with Euclidean metric $\delta_{AB}$, $A, B = 1, \dots, 4$, and a single real scalar field

$$\Phi : \mathbb{E}^4 \to \mathbb{R},$$

satisfying the Laplace equation

$$\Delta_{\mathbb{E}^4} \Phi = 0. \tag{1}$$

At this level, neither fundamental time, nor a Lorentzian metric, nor an a priori causal structure, nor pre-given cosmological dynamics is assumed. The Euclidean metric $\delta_{AB}$ remains the only initial geometric structure, while all observable physical objects belong to the level of effective reconstruction.

For the technical derivation of the reconstruction horizon in the present work, we fix a unit vector

$$n^A, \qquad \delta_{AB} n^A n^B = 1,$$

which defines a flat working foliation

$$\Sigma_s := \{x \in \mathbb{E}^4 \mid n_A x^A = s\}. \tag{2}$$

The parameter $s$ is not fundamental time; it serves only as an ordering parameter along the chosen foliation and is used to organize local reconstruction. The point is not to single out a physically privileged global structure, but to choose an operational representation relative to which the problem of inter-slice continuation of data can be posed.

It is important to emphasize that precisely the flat foliation

$$n_A = \text{const}$$

is used below in the derivation of the spectral narrowing and of the scale $L_{\max}$. Foliation curvature, mean matter density, and full FLRW dynamics do not enter this technical derivation. They belong to the more general effective GR description and to subsequent cosmological phenomenology.

All subsequent results in the present paper are first formulated relative to the fixed working foliation and the corresponding observer-dependent subregime. This restriction is essential: in the model considered here, causal reconstruction and the corresponding effective spacetime are defined observer-relatively, not as a single foliation-independent global structure. The inter-observer compatibility of the resulting large-scale parameter will be briefly discussed below in Subsection 2.4.

We next choose the initial slice

$$\Sigma_0 := \Sigma_{s=0}.$$

The local data region on this slice and the corresponding reconstruction region will be defined in the next subsection. Thus, already at the level of the initial setting, two layers of description are distinguished: the fundamental layer, defined by Eq. (1), and the operational layer, in which the foliation, data region, and admissible regime of continuation are chosen.

The essential point for the present paper is that the fundamental equation (1) is elliptic. This means that remote continuation of local data does not possess hyperbolic stability and is therefore, in general, exponentially sensitive to errors under inverse continuation [9, 10]. The entire subsequent construction relies on this fact.

## 2.2 Data region and reconstruction region

Let

$$\Omega_0 \subset \Sigma_0$$

be a bounded region, interpreted as the localized body of the observer. It is on $\Omega_0$ that the local observer-dependent subregime is defined, and it is there that the locally accessible data are available, relative to which the remote reconstruction of the field configuration will be considered. By "data" the present paper means not complete knowledge of the fundamental configuration $\Phi$ on all of $\mathbb{E}^4$, but only locally accessible information on $\Omega_0$, sufficient to pose the problem of remote reconstruction in the chosen foliation-based representation.

For each $L > 0$, define the reconstruction region at scale $L$ by

$$\Omega(L) := \left\{ x \in \mathbb{E}^4 \;\middle|\; 0 \leq n_A x^A \leq L,\ x - (n_B x^B)\, n \in \Omega_0 \right\}. \tag{3}$$

Equivalently, in the decomposition $x = y + s\, n$, where $y \in \Sigma_0$ and $s = n_A x^A$, this region can be written as

$$\Omega(L) = \{\, y + s\, n \mid y \in \Omega_0,\ 0 \leq s \leq L \,\}. \tag{4}$$

Formula (3) defines $\Omega(L)$ as a region parametrized relative to $\Omega_0$ in the chosen flat foliation. Although technically it is constructed through continuation along the direction $n^A$, the quantity $L$ will be understood below neither as a literal depth in one direction nor as the scale factor of an FLRW model, but as the characteristic scale of remote reconstruction relative to a fixed local data region.

At this stage, $L$ is not identified with the Hubble radius, a cosmological horizon, or a parameter of full cosmological dynamics. It is a parameter of the remote continuation problem in a fixed foliation-based representation. The relation between the resulting scale $L_{\max}$ and an effective cosmological scale is considered only after the transition to the closed GR description.

It should be emphasized that $\Omega(L)$ is not a fundamentally selected region of "reality"; it is only the region within which the question of consistent reconstruction is posed relative to a fixed observer-dependent subregime and chosen foliation. Therefore, the subsequent restrictions on possible values of $L$ are interpreted not as global restrictions on the field $\Phi$ as such, but as restrictions on the large-scale sector that remains operationally accessible and stable.

Let $\delta > 0$ denote the local error level of the data on $\Omega_0$, and let $\delta_* > 0$ be the admissible mismatch threshold at which remote reconstruction is still regarded as operationally acceptable. The main task is then the following: for given $\Omega_0$, $n^A$, $\delta$, and $\delta_*$, determine for which values of $L$ there exists a stable continuation of the local data with mismatch not exceeding the admissible threshold. In the following sections it will be shown that it is precisely the elliptic character of Eq. (1) that leads here to an infrared-induced narrowing of the spectral window and then to the existence of a finite maximum scale $L_{\max}$.

It is also essential for what follows that the region $\Omega(L)$ is defined relative to one and the same local data region $\Omega_0$. Therefore, increasing $L$ does not mean increasing the amount of accessible microscopic information. On the contrary, with fixed local access, increasing the reconstruction scale means an increasingly strong dependence on remote elliptic continuation and, consequently, increasingly poor conditioning of the problem.

### 2.3 The GR regime as a closed effective description: the origin of $g_{\mu\nu}$ and $T^{\text{eff}}_{\mu\nu}$

The present work does not repeat the full derivation of the effective gravitational sector from the timeless Euclidean model. However, since this sector is used below as a necessary input, we briefly recall the part of the previous reconstruction that is required for the analysis of the cosmological term.

We use the fundamental setting (1) introduced above. At the initial level of the model, there are no fundamental time, Lorentzian metric, spacetime, causal order, matter, or stress-energy tensor. In the previous special-relativistic reconstruction [1], a localized observer together with a chosen foliation $\Sigma_s$ defines operational events, an effective time parameter, and a causal order between registered and reconstructed events. In the locally inertial regime, this construction leads to an effective Lorentzian structure with limiting speed $v_{\max}$.

The gravitational reconstruction begins by relaxing the condition that the foliation direction be globally constant. Instead of a fixed normal direction, one considers a slowly varying field

$$n_A = n_A(x), \qquad n_A n^A = 1.$$

In this case, neighboring local SR reconstructions must be made mutually compatible. The effective metric $g_{\mu\nu}$ arises as the geometric variable encoding this compatibility: it fixes local causal cones, operational intervals, and the rules for comparing nearby locally inertial reconstructions. Hence $g_{\mu\nu}$ is not a fundamental metric on $\mathbb{E}^4$, but belongs to the level of the reconstructed effective description.

The operational form of the equivalence principle arises from the fact that the acceleration of the observer and the gravitational component have the same local mechanism in this reconstruction. In both cases, the issue is a rotation of the local foliation direction relative to the inertial direction. For an accelerated observer, such a rotation describes the choice of a non-inertial foliation along the observer's own registration region. For the gravitational regime, the same type of rotation becomes a field $n_A(x)$ varying from region to region and therefore requiring the compatibility of neighboring local SR reconstructions.

This is why, in a small neighborhood of an admissible event $p$, the gravitational component is indistinguishable from the effect of an accelerated description: the local observer registers not a fundamental force, but a change in the orientation of its operational foliation relative to the locally inertial reconstruction. The distinction between acceleration and gravity arises only when neighboring neighborhoods are compared. If the rotation of the foliation can be removed by a single choice of local non-inertial description, one is dealing with kinematic acceleration. If, however, the field of rotations has an inhomogeneity leading to incompatibility of local reconstructions and tidal effects, it requires geometric compensation and is described by the effective metric $g_{\mu\nu}$.

When

$$L_{\text{field}} \ll L_{\text{fol}},$$

where $L_{\text{field}}$ is the characteristic scale of effective processes and $L_{\text{fol}}$ is the scale on which the normal field $n_A(x)$ varies, the local observer does not resolve the inhomogeneity of the foliation structure at the scale of its measurement protocol. Therefore, in a sufficiently small neighborhood of the event $p$, the field $n_A(x)$ can be replaced by a constant normal direction with small corrections, and the local reconstruction reduces to the previously obtained SR regime. In geometric language this means that one can choose a locally inertial description in which

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad \partial_\rho g_{\mu\nu}(p) = 0,$$

while gravitational effects remain only in higher-order inhomogeneities, that is, in the tidal structure.

Effective fields are likewise not introduced as new fundamental fields. They are operationally accessible projections of the fundamental configuration $\Phi$ onto the chosen foliation,

$$\psi_I = \Psi_I[\Phi; \Sigma_s].$$

In the local mode description, these quantities are expressed through a finite set of coefficients accessible to the observer in the given reconstruction region. In the adiabatic regime, their dynamics can be represented by an effective action

$$S_{\text{eff}}[g, \psi].$$

The corresponding effective stress-energy tensor is defined in the standard variational way:

$$T^{\text{eff}}_{\mu\nu} := -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}}.$$

Thus, $T^{\text{eff}}_{\mu\nu}$ is not a fundamental source on $\mathbb{E}^4$. It describes the stress-energy content of the effective degrees of freedom inside the reconstructed Lorentzian regime.

A key step in the previous GR reconstruction is the compensation principle. When local causal reconstructions are transported through a slowly varying foliation structure, a compatibility mismatch arises. If the causal description is to remain closed, this mismatch must be compensated by effective geometry. In variational form, the corresponding condition is written as

$$\delta\big(S_g[g] + S_{\text{eff}}[g, \psi]\big) = 0.$$

It is important that the geometric compensation in this regime is universal. The effective fields $\psi_I$ enter the description through the action $S_{\text{eff}}[g, \psi]$, and it is precisely the variation of this action with respect to the metric that defines $T^{\text{eff}}_{\mu\nu}$. Therefore, the dependence of effective matter on the reconstructed geometry is already taken into account in $S_{\text{eff}}[g, \psi]$.

No separate universal mixed term belonging to the gravitational sector while also depending on the specific $\psi_I$ is introduced. Such a term would make the geometric compensation itself dependent on the particular composition of effective matter. As a result, different effective sectors could define different conditions for causal transport, different local cones, or different rules for comparing events. This would violate not only the universality of the local GR regime, but also the more basic requirement of causal reconstruction introduced at the SR level: locally accessible events must admit a consistent causal ordering within a single operational geometry.

Possible non-minimal couplings, if they arise in the effective description, must be treated as sector-dependent corrections inside $S_{\text{eff}}[g,\psi]$ or as higher-order effective terms, admissible only insofar as they do not destroy the common causal reconstruction.

Consequently, the leading geometric sector must be a purely metric functional $S_g[g]$. In the local covariant derivative expansion, its minimal form is

$$S_g[g] = \int d^4x \sqrt{|g|}\,(c_0 + c_1 R[g] + \dots)\,,$$

where the ellipsis denotes higher-derivative invariants suppressed in the minimal closed GR regime. The requirement that the effective equations for the metric be no higher than second order, together with locality, covariance of the reconstructed description, and the correct local SR limit, selects the Einstein sector in four dimensions. After the normalization

$$c_1 = \frac{1}{16\pi G}, \qquad c_0 = -\frac{\Lambda}{8\pi G},$$

one obtains

$$S_g[g] = \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\,\big(R[g] - 2\Lambda\big) + S_{\partial\Omega}.$$

Variation of this action together with $S_{\text{eff}}[g,\psi]$ leads to the effective equations

$$G_{\mu\nu}[g] + \Lambda g_{\mu\nu} = 8\pi G\, T^{\text{eff}}_{\mu\nu}.$$

In the previous GR reconstruction, the cosmological term $\Lambda g_{\mu\nu}$ arose as an admissible universal zero-derivative geometric contribution in the closed effective description. However, its reconstructive origin, sign, and scale were not analyzed. The present work considers precisely this missing step.

At the same time, the derivation of the scale $L_{\max}$ itself does not require a curved foliation. Foliation curvature and local gravitational inhomogeneity belong to the more general GR regime in which neighboring local SR reconstructions are made compatible. They are not used below as the source of $\Lambda_{\text{IR}}$. The cosmological term is considered here as a universal zero-derivative infrared contribution associated with the finite limit of stable remote reconstruction. The role of the GR description is to represent this infrared limit in a closed covariant metric language:

$$\Lambda = \Lambda_{\text{IR}}, \qquad \Lambda_{\text{IR}} = \chi L_{\max}^{-2},$$

where $\chi$ is a dimensionless coefficient depending on the details of the cosmological reconstruction class. Thus, the present paper does not introduce a new independent sector and does not use matter density as the source of $\Lambda_{\text{IR}}$, but offers a reconstructive origin and scale estimate for that part of the cosmological term which is associated with the infrared reconstruction horizon.

## 2.4 A brief remark on inter-observer compatibility

Since the preceding subsections define the reconstruction relative to a fixed observer-dependent subregime and its chosen flat working foliation, it is necessary to clarify in what sense the infrared parameter obtained below can be regarded as conditionally universal.

Strictly speaking, the technical derivation first gives a limit

$$L_{\max}^{(O,n)},$$

associated with the given observer $O$, the local domain of access $\Omega_0$, and the chosen foliation $n$. Its interpretation as a cosmological scale requires this limit to be stable within the corresponding reconstruction class and not to depend essentially on the choice of a particular representative of this class within the already admissible local equivalences.

By a reconstruction class, the present paper means a class of admissible reconstruction regimes in the sense of [4]. The detailed definition of such classes, their selection, and their relation to the fine-tuning problem are not reproduced here. For the purposes of the present work, only the following minimal consequence is used: if two observer-dependent subregimes belong to one and the same reconstruction class, are located in the same local cosmological region, have small relative velocities, and consider one and the same locally common large-scale sector, then the effective large-scale description arising for them must coincide up to the already accounted-for local equivalences.

In this restricted sense, inter-observer compatibility means that universal large-scale parameters must not depend on the particular choice of a representative within the given local compatibility class. This statement assumes that one uses either the same working foliation or sufficiently close foliations in the same range of scales. Therefore, universal infrared parameters, in particular $L_{\max}$, $\Lambda_{\mathrm{IR}}$, and the associated asymptotic Hubble scale $H_{\mathrm{IR}}$, should be regarded as invariants of the local compatibility class, rather than as quantities depending on an arbitrary choice of a single observer-dependent subregime.

This does not mean that one and the same reconstruction region or one and the same set of local data must be common to arbitrary remote observers. For observers separated by cosmological distances or belonging to substantially different foliation regimes, the regions of stable reconstruction may differ. The conditional universality of cosmological parameters in the present paper applies to observers of one local compatibility class who consider one and the same effective cosmological sector.

Thus, the present text uses only the minimal idea of inter-observer compatibility needed to treat the cosmological term obtained below as a conditionally universal result for nearby observers of one reconstruction class. In this sense, $L_{\max}$, $\Lambda_{\mathrm{IR}}$, and the cosmological parameters derived from them are not individual characteristics of a single local procedure, but parameters of a consistent large-scale description within the given local reconstruction class.

# 3 Elliptic reconstruction and narrowing of the spectral window

## 3.1 Spectral admissibility criterion for reconstruction

Consider the fixed flat working foliation

$$\Sigma_s = \{x \in \mathbb{E}^4 \mid n_A x^A = s\}, \qquad n_A = \text{const},$$

and the associated data region $\Omega_0 \subset \Sigma_0$, introduced in Section 2. Since the field $\Phi$ globally satisfies the fundamental equation

$$\Delta_{\mathbb{E}^4}\Phi = 0,$$

the problem of remote reconstruction consists in recovering the content of this harmonic solution at scale $L$ from locally accessible data specified relative to $\Omega_0$.

For what follows, it is useful to distinguish three levels of description. First, there is the fundamental field configuration $\Phi$, which defines an exact solution of the fundamental equation. Second, there is a locally accessible set of data on $\Omega_0$, with finite resolution and nonzero registration error. Third, there is an effective hyperbolic regime of description at scale $L$, with which the elliptic continuation of these data must remain compatible [1, 2, 3, 4, 5].

The key point for the present paper is that, as the scale $L$ increases, elliptic reconstruction becomes compatible with an increasingly narrow spectral range of the effective hyperbolic regime. In other words, as $L$ grows, remote reconstruction loses stable access to short-wavelength content and becomes increasingly large-scale. Below, this effect will be called the *infrared-induced narrowing of the spectral window*: the restriction is induced by the growth of the large reconstruction scale, although technically it appears as an upper bound on admissible tangential wavenumbers.

Let $k_{\max}(L)$ denote the maximum tangential wavenumber which, at scale $L$, remains compatible both with the elliptic reconstruction of local data and with the requirements of causal reconstruction [1, 2]. The corresponding minimum distinguishable wavelength is then defined by

$$\lambda_{\min}(L) := \frac{2\pi}{k_{\max}(L)}. \tag{5}$$

As $L$ grows, $k_{\max}(L)$ decreases, whereas $\lambda_{\min}(L)$ increases. Therefore, increasing the reconstruction scale does not mean an increase in accessible information, but rather its unavoidable coarse-graining.

For the purposes of the present paper, the concrete form of the effective hyperbolic equation is not essential. What matters is only the fact, established in previous works, that there exists an effective causally consistent regime in which observable fields admit a hyperbolic description on the chosen foliations [1, 3]. Therefore, the subsequent analysis relies not on the details of the specific dynamics of the effective fields, but only on the fact of spectral compatibility between elliptic reconstruction and this regime.

Let $\lambda_*$ denote the critical minimum wavelength of resolution at which a stable causally consistent description is still possible. Then the scale $L = L_{\max}$, defined by the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*, \tag{6}$$

corresponds to the limiting scale of stable reconstruction in the given observer-dependent subregime. At this stage, $L_{\max}$ is neither a cosmological constant nor an already given FLRW horizon. It has the status of an operational reconstruction horizon obtained relative to a fixed data region and a chosen flat foliation.

Thus, reconstruction at scale $L$ is regarded as admissible if

$$\lambda_{\min}(L) \leq \lambda_*. \tag{7}$$

Equivalently, this condition can be written as

$$k_{\max}(L) \geq k_*, \qquad k_* := \frac{2\pi}{\lambda_*}. \tag{8}$$

This is the condition that will be used below as the criterion for the existence of a stable large-scale description.

It is essential here that, for a fixed data region $\Omega_0$, increasing $L$ is not accompanied by an increase in the amount of locally accessible microscopic information. On the contrary, with fixed local access, increasing the reconstruction scale means an increasingly strong dependence on elliptic continuation and, consequently, increasingly poor conditioning of the inverse problem. In the following subsections, it will be shown that this leads to the narrowing of the admissible spectral window and then to the existence of a finite maximum scale of stable reconstruction $L_{\max}$.

## 3.2 Poisson semigroup and exponential instability

This subsection uses the local flat model corresponding to the fixed working foliation $n_A =$ const. It is in this regime that the spectral restriction used below to define $L_{\max}$ is technically derived. The effective curved geometry is not an input at this stage; it appears later as the GR representation of the infrared scale already obtained.

In the coordinates

$$x = y + s\,n, \qquad y \in \Sigma_0,$$

the Euclidean Laplacian has the form

$$\Delta_{\mathbb{E}^4} = \partial_s^2 + \Delta_y.$$

After Fourier transformation in the tangential coordinates $y$, each mode satisfies the equation

$$(\partial_s^2 - |k|^2)\widehat{\Phi}(s,k) = 0. \tag{9}$$

The general solution has the form

$$\widehat{\Phi}(s,k) = A(k)e^{|k|s} + B(k)e^{-|k|s}. \tag{10}$$

Poisson continuation corresponds to the stable branch bounded in the direction of continuation. If the data on the slice $\Sigma_0$ are denoted by $\phi_0 = \Phi|_{\Sigma_0}$, then for this branch one obtains

$$\widehat{\Phi}(L,k) = e^{-L|k|}\widehat{\phi}_0(k). \tag{11}$$

In other words, the direct Poisson operator has the form

$$P_L = e^{-L|D|}, \qquad |D| = \sqrt{-\Delta_{\Sigma_0}},$$

and suppresses high-frequency tangential modes.

In the problem of remote reconstruction, the inverse operation is essential. If the locally accessible data contain only a smoothed image of the remote content, then recovering the high-frequency component requires applying the inverse operator

$$P_L^{-1} = e^{L|D|}.$$

At the level of a single mode this gives

$$\widehat{\phi}_0(k) = e^{L|k|}\widehat{\Phi}(L,k), \tag{12}$$

up to the choice of orientation of the normal direction. Hence, the error in a mode with wavenumber $|k|$ is amplified as $e^{L|k|}$. This is precisely the standard exponential instability of inverse elliptic continuation.

The status of this estimate should be clarified. The flat Fourier model is used here as a controlled local regime in which the main mechanism of elliptic instability is explicit. In a bounded region, rigorous stability estimates depend on the choice of functional norm, boundary conditions, and class of admissible data. In the present work, these normalization constants, which depend on the region, norm, and boundary conditions, are absorbed into the effective operational threshold $\delta_*$. What is essential for the subsequent derivation is not the exact value of such constants, but the exponential character of inverse elliptic continuation and the resulting monotonic dependence of the admissible wavenumber on the reconstruction scale.

Thus, the subsequent derivation of $L_{\max}$ does not require the analysis of the elliptic Cauchy problem on an arbitrary curved foliation. The technical derivation is carried out in the fixed flat working foliation. The curvature of the effective geometry, the shape of the boundary of the region, and the global cosmological dynamics may change coefficients, the domain of applicability, and subleading terms, but they do not eliminate the type of high-frequency instability characteristic of the inverse elliptic problem [9, 10].

If the characteristic error level in the accessible data is of order $\delta$, and the admissible operational error threshold after reconstruction is $\delta_*$, then for each mode the condition of controllability takes the form

$$\delta\, e^{L|k|} \lesssim \delta_*. \tag{13}$$

It follows that the spectral bound of admissible reconstruction is

$$|k| \le k_{\max}(L), \qquad k_{\max}(L) \sim \frac{1}{L}\log\frac{\delta_*}{\delta}. \tag{14}$$

More rigorously, the numerical factor on the right-hand side may depend on the chosen norm, boundary conditions, and class of data. Below, it is absorbed into the definition of $\delta_*$, since for the present work what is essential is the logarithmic dependence on the ratio of the admissible and initial error levels and the monotonic decrease of $k_{\max}(L)$ as $L$ grows.

Accordingly, the minimum distinguishable wavelength grows as

$$\lambda_{\min}(L) \sim \frac{2\pi}{k_{\max}(L)}. \tag{15}$$

This is the fundamental mechanism of the infrared-induced narrowing of the spectral window in the model considered here: as the scale $L$ increases, elliptic reconstruction becomes compatible with an increasingly narrow tangential spectral range. Therefore, remote reconstruction becomes increasingly large-scale, and when the critical condition

$$\lambda_{\min}(L) = \lambda_*$$

is reached, the limiting scale of the stably reconstructible sector is attained,

$$L = L_{\max}. \tag{16}$$

This scale is not introduced as an external horizon; it is defined as the boundary up to which the criterion of compatibility with a stable causally consistent effective description is satisfied.

## 3.3 Lemma on the spectral bound of admissible reconstruction

The preceding argument can be summarized in the following lemma.

**Lemma 3.1.** *Let the locally accessible data be specified with characteristic accuracy $\delta > 0$, and let the admissible operational error threshold after reconstruction be $\delta_* > 0$, where $0 < \delta < \delta_*$. Then, for reconstruction at scale $L$, based on elliptic continuation in a fixed flat foliation and compatible with the effective hyperbolic regime, there exists an upper bound on the admissible tangential wavenumber,*

$$|k| \le k_{\max}(L), \qquad k_{\max}(L) := \frac{1}{L} \log \frac{\delta_*}{\delta}. \tag{17}$$

*In particular, as the scale $L$ increases, the admissible spectral window narrows monotonically.*

*Proof.* From the mode-controllability condition (13), we have

$$\delta\, e^{L|k|} \lesssim \delta_*.$$

Taking the logarithm gives

$$L|k| \lesssim \log \frac{\delta_*}{\delta},$$

and hence

$$|k| \lesssim \frac{1}{L} \log \frac{\delta_*}{\delta}.$$

Absorbing inessential normalization factors into the definition of the admissible threshold $\delta_*$, we obtain (17). Since, for fixed $\delta$ and $\delta_*$, the right-hand side of (17) decreases as $L^{-1}$, the admissible spectral window narrows monotonically as $L$ grows. □

Lemma 3.1 shows that remote reconstruction automatically induces an upper bound on the stably reconstructible tangential spectrum. This restriction is not introduced externally and is not associated with an arbitrary truncation of the effective theory; it follows from the ill-conditioned inverse elliptic continuation of locally accessible data. In this sense, $k_{\max}(L)$ defines the boundary of the spectral window compatible with causal reconstruction at scale $L$.

Physically, this means that, as the reconstruction scale increases, an increasingly smaller part of the short-wavelength content remains operationally accessible. Accordingly, the description inevitably becomes more large-scale and less sensitive to the small-scale structure of the initial configuration. This mechanism will next be translated into the growth of the minimum distinguishable wavelength and then into the existence of a finite maximum scale of stable reconstruction $L_{\max}$.

## 3.4 Monotonic infrared degradation of reconstruction

The most important consequence of formula (17) is that the spectral restriction is monotonic in the reconstruction scale. Indeed, for fixed $\delta$ and $\delta_*$, one has

$$\frac{dk_{\max}}{dL} = -\frac{1}{L^2}\log\frac{\delta_*}{\delta} < 0. \tag{18}$$

Thus, increasing $L$ is always accompanied by a decrease in the operationally accessible spectral window.

Equivalently, the minimum distinguishable wavelength

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}$$

increases monotonically as the reconstruction scale grows. This means that remote reconstruction becomes increasingly large-scale: short-wavelength content is progressively lost, and the description becomes less sensitive to the small-scale structure of the initial configuration.

Thus, the infrared incompleteness of reconstruction has a directed character: it is not a random loss of individual modes, but expresses a systematic narrowing of spectral compatibility when passing to increasingly large scales. This is why the corresponding effect cannot be removed by a simple local renormalization of the description: the boundary of the admissible spectral window itself depends on $L$ and is irreducibly shifted as the reconstruction scale increases.

This property leads directly to the existence of a finite scale $L_{\max}$. Indeed, once the minimum distinguishable wavelength reaches the critical value $\lambda_*$, compatible with the existence of a stable causally consistent description, further reconstruction ceases to be admissible. In other words, the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*$$

defines the limiting scale of the stably reconstructible sector.

The subsequent geometric interpretation of this result requires a separate step. The monotonic narrowing of the spectral window is not by itself a cosmological constant and does not define the full FLRW dynamics. It defines the infrared scale $L_{\max}$, which will be considered in the subsequent sections as a candidate for a universal large-scale parameter of the effective cosmological sector. Only after the transition to the closed covariant GR description can this scale be represented as a zero-derivative geometric contribution

$$\Lambda_{\mathrm{IR}} g_{\mu\nu}.$$

The directed character of infrared degradation will also be important for the subsequent interpretation of redshift. Since the stably accessible spectral window narrows as $L$ grows, the causal reconstructibility of a remote signal requires the transfer of its observable content into a longer-wavelength range. In the effective FLRW description, this transfer is redshift. However, this step already belongs to the geometric interpretation of the result and will be considered separately below.

# 4 Operational horizon and maximum scale of reconstruction

In the previous section it was shown that the elliptic instability of remote reconstruction in a fixed flat foliation leads to the spectral restriction

$$|k| \le k_{\max}(L), \qquad k_{\max}(L) = \frac{1}{L} \log \frac{\delta_*}{\delta},$$

and also to the monotonic growth of the minimum distinguishable wavelength

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}.$$

The present section is devoted to the consequences of this result: first, to the finiteness of the number of accessible modes at a fixed reconstruction scale; second, to the existence of a finite maximum scale of stable reconstruction $L_{\max}$; and third, to the interpretation of this scale as a natural infrared parameter which can be represented in the effective GR description as the source of a cosmological term.

## 4.1 Finiteness of the number of modes on $\Omega_0$ under the spectral restriction

Let $\Omega_0 \subset \Sigma_0$ be the bounded data region introduced in Section 2, and let a positive self-adjoint operator

$$-\Delta_{\Omega_0},$$

obtained from the Laplacian on the hypersurface $\Sigma_0$ with boundary conditions corresponding to the chosen class of local registers of the observer, be defined on it. For definiteness, one may have Dirichlet boundary conditions in mind; however, the subsequent construction relies not on this particular choice, but on the fact that the spectrum is discrete on a bounded region.

Denote the eigenvalues of the operator $-\Delta_{\Omega_0}$ by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

and introduce the corresponding tangential wavenumbers

$$k_j := \sqrt{\lambda_j}.$$

Define the mode counting function by

$$N(k) := \#\{j \in \mathbb{N} \mid k_j \leq k\}. \tag{19}$$

Then $N(k)$ is the number of tangential modes on $\Omega_0$ accessible below the wavenumber $k$.

Since, at scale $L$, stable remote reconstruction admits only modes with

$$|k| \leq k_{\max}(L),$$

the number of accessible modes is

$$N_{\mathrm{acc}}(L) := N\big(k_{\max}(L)\big). \tag{20}$$

Since $\Omega_0$ is bounded, the spectrum of the operator $-\Delta_{\Omega_0}$ is discrete, and therefore $N_{\mathrm{acc}}(L)$ is finite for every fixed $L > 0$.

For large values of $k$, the counting function satisfies Weyl's law. Since $\Omega_0 \subset \Sigma_0$ is a three-dimensional region,

$$N(k) = \frac{\mathrm{Vol}(\Omega_0)}{6\pi^2}\, k^3 + o(k^3), \qquad k \to \infty. \tag{21}$$

Here $\mathrm{Vol}(\Omega_0)$ denotes the three-dimensional Euclidean volume of the data region. Hence, in the regime where the leading Weyl asymptotics applies,

$$N_{\mathrm{acc}}(L) \sim \frac{\mathrm{Vol}(\Omega_0)}{6\pi^2} \left(\frac{1}{L} \log \frac{\delta_*}{\delta}\right)^3. \tag{22}$$

This estimate shows that, for fixed $\delta$, $\delta_*$, and $\Omega_0$, the number of accessible modes decreases as the reconstruction scale increases.

**Remark 4.1.** *The finiteness of $N_{\mathrm{acc}}(L)$ is not an external regularization. It arises from the combination of two internal elements of the setting: the boundedness of the local data region $\Omega_0$ and the infrared-induced narrowing of the spectral window $k_{\max}(L)$, caused by the exponential instability of inverse elliptic continuation.*

## 4.2 Theorem on the operational horizon

We can now formulate the central result of this section.

**Theorem 4.2** (on the operational horizon)**.** *Suppose that remote reconstruction satisfies the spectral admissibility criterion*

$$\lambda_{\min}(L) \le \lambda_*,$$

*where* $\lambda_{\min}(L) = 2\pi/k_{\max}(L)$*, and where* $\lambda_* > 0$ *is the critical minimum wavelength of resolution compatible with a stable causally consistent description. Then there exists a unique finite number* $L_{\max} > 0$ *such that*

$$\lambda_{\min}(L_{\max}) = \lambda_*. \tag{23}$$

*Moreover:*

*(i) for all* $0 < L < L_{\max}$*, reconstruction is admissible;*

*(ii) for* $L > L_{\max}$*, reconstruction is inadmissible;*

*(iii)* $L_{\max}$ *is the maximum scale of stable reconstruction for the given observer-dependent subregime and chosen foliation.*

*Proof.* From formula (17), it follows that $k_{\max}(L)$ strictly decreases with $L$, and therefore $\lambda_{\min}(L) = 2\pi/k_{\max}(L)$ strictly increases. Moreover,

$$\lim_{L\to 0^+} \lambda_{\min}(L) = 0, \qquad \lim_{L\to\infty} \lambda_{\min}(L) = \infty.$$

Hence, for every fixed $\lambda_* > 0$, there exists a unique $L_{\max} > 0$ satisfying (23). The monotonicity of $\lambda_{\min}(L)$ immediately implies that, for $L < L_{\max}$,

$$\lambda_{\min}(L) < \lambda_*,$$

whereas for $L > L_{\max}$,

$$\lambda_{\min}(L) > \lambda_*.$$

This gives assertions (i)–(iii). □

Theorem 4.2 shows that the finite reconstruction horizon arises not as an additional postulate, but as a consequence of the elliptic spectral restriction and the finite spectral resolution of the causally consistent regime. In other words, $L_{\max}$ is not introduced as an external scale beyond which the effective hyperbolic regime suddenly breaks down; it is defined as the boundary up to which the very criterion of compatibility with a stable causally consistent effective description is satisfied,

$$\lambda_{\min}(L) \le \lambda_*.$$

Thus, the statement that the effective hyperbolic regime is compatible up to $L_{\max}$ is not an additional assumption, but enters the very definition of admissible reconstruction. As a result, a natural large-scale limit appears in the model, unrelated either to fundamental discreteness of space or to an externally imposed ultraviolet cutoff.

Using formula (17), $L_{\max}$ can be written explicitly. From the conditions

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}, \qquad k_* := \frac{2\pi}{\lambda_*},$$

we obtain

$$L_{\max} = \frac{1}{k_*}\log\frac{\delta_*}{\delta} = \frac{\lambda_*}{2\pi}\log\frac{\delta_*}{\delta}. \tag{24}$$

This formula is a model estimate in the flat foliation. It shows that the maximum scale of reconstruction is determined by the critical spectral resolution and by the ratio between the local accuracy of the data and the admissible operational threshold. A precise comparison of this scale with observed cosmological distances requires an additional analysis of the reconstruction class and is not carried out in the present section.

**Remark 4.3.** *It should be emphasized that $L_{\max}$ is not a fundamental boundary of the field $\Phi$ itself on $\mathbb{E}^4$. It is the maximum scale up to which a localized observer of the given class can maintain stable reconstruction of the corresponding large-scale sector. Beyond this boundary, the fundamental configuration may exist, but it no longer admits continuation in the form that would be compatible with the given regime of causal reconstruction.*

## 4.3 The operationally observable sector as a region of stable reconstruction

Theorem 4.2 allows one to interpret the region

$$\Omega(L_{\max}) \tag{25}$$

as the maximal region for which a stable reconstructed description is preserved relative to the given observer and chosen foliation. In this sense, $\Omega(L_{\max})$ acts not as a fundamentally selected part of $\mathbb{E}^4$, but as an operationally observable sector: the maximal region within which, for the given reconstruction regime, the conditions of stable registration and causally consistent large-scale description are still preserved.

This interpretation does not mean that one and the same region $\Omega(L_{\max})$ is observable for all observers of the given reconstruction class. For observers separated by cosmological distances, the corresponding regions of stable reconstruction may differ substantially. The conditional inter-observer invariance of the result concerns not the coincidence of the regions $\Omega(L_{\max})$ themselves, but the stability of the scale $L_{\max}$ and of the corresponding infrared parameter within one reconstruction class.

This understanding removes the need to postulate an external cosmological horizon as a primary geometric datum. The horizon arises here not at the level of the initial ontology, but as the limit of applicability of stable reconstruction. This is consistent with the general logic of the program, in which observable spacetime is regarded as a derivative structure depending on the regime of accessible reconstruction.

At the same time, the present paper does not claim that outside $\Omega(L_{\max})$ "nothing exists". It claims only that, for the class of observers under consideration, there is no stable procedure of continuation preserving the type of large-scale geometric and event organization realized inside $\Omega(L_{\max})$. Thus, the boundary of $\Omega(L_{\max})$ is a boundary of operational reconstruction, not an absolute ontological boundary.

This interpretation is important for the subsequent analysis of the cosmological term. If $L_{\max}$ defines the maximum scale of stable large-scale description, then, after the transition to the closed GR regime, this scale can act as a natural infrared parameter of the effective geometric sector. It is only in this restricted sense that $\Omega(L_{\max})$ prepares the subsequent interpretation of $\Lambda_{\mathrm{IR}}$; by itself, it is not yet a full FLRW cosmology.

## 4.4 $L_{\max}$ as a natural infrared scale

It is crucial for what follows that $L_{\max}$ is not merely a boundary of admissibility of reconstruction associated with the given observer, but also the only large-scale parameter that arises naturally in the flat foliation setting considered here. The fundamental model (1) itself contains neither a distinguished relativistic scale nor a fundamental cosmological length. The scale $L_{\max}$ appears as the result of the interplay between the elliptic instability of remote reconstruction, the local accuracy of the data, and the critical spectral resolution of the causally consistent regime.

Strictly speaking, at this stage one should write

$$L_{\max} = L_{\max}^{(O,n)},$$

because the technical derivation depends on the chosen observer-dependent subregime, local data region, and foliation. Its interpretation as a cosmological scale requires an additional condition: the stability of this limit within the corresponding reconstruction class. If this condition is satisfied, then $L_{\max}$ can be regarded not as an arbitrary characteristic of a single observer, but as a parameter of the given large-scale compatibility class.

In the effective large-scale description, a universal infrared contribution must be constructed from parameters that do not depend on particular microscopic details, but are preserved at the level of the reconstruction class under consideration. If it is stable in this sense, the scale $L_{\max}$ satisfies this requirement: it is determined not by the local structure of individual modes, but by the very limit of stable reconstruction.

Dimensionally, a universal zero-derivative geometric contribution in the four-dimensional GR sector must have a scale of order

$$L_{\max}^{-2}.$$

Therefore, even before analyzing the precise form of such a contribution, it is clear that $L_{\max}$ is the natural candidate for the infrared scale of the effective cosmological term. The next section is devoted to showing that, if this reconstruction limit is represented within the closed covariant GR regime, the leading universal zero-derivative response has the form of a cosmological constant.

**Remark 4.4.** *Sections 3 and 4 together form the logical chain*

$$\textit{elliptic instability} \Longrightarrow k_{\max}(L) \Longrightarrow \lambda_{\min}(L) \Longrightarrow L_{\max}.$$

*This chain gives the first strictly defined infrared scale of the model in the flat foliation setting. In what follows, this scale will be used as an input for the analysis of the universal geometric response of the large-scale GR sector.*

# 5 Infrared anomaly and universal geometric response

## 5.1 Infrared anomaly as loss of spectral compatibility of reconstruction

In the present work, an *infrared anomaly* is understood not as a quantum anomaly in the standard sense, but as a systematic loss of spectral compatibility of remote reconstruction when passing to large scales. The content of this statement is as follows. As long as the reconstruction scale $L$ is small compared with the limiting value $L_{\max}$, the local data region $\Omega_0$ and the corresponding observer-dependent subregime still admit stable continuation to $\Omega(L)$, compatible with the effective hyperbolic regime. However, as $L$ increases, the spectral window $k_{\max}(L)$ narrows, while the minimum distinguishable wavelength $\lambda_{\min}(L)$ increases monotonically. Thus, part of the short-wavelength content of the remote structure ceases to be accessible in a causally consistent large-scale description.

This irreducible loss of spectral compatibility will be called below the infrared anomaly of reconstruction. It has a directed character: it is not a random local defect and not an arbitrary truncation of the effective theory, but a systematic narrowing of the spectral window as the scale of remote reconstruction increases. Its leading contribution is universal with respect to the microdetails of the local sector, since it is determined not by a particular distribution of individual modes, but by the very existence of the finite scale of stable reconstruction defined by the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*.$$

Strictly speaking, at the level of the flat foliation setting, this scale has the form

$$L_{\max} = L_{\max}^{(O,n)},$$

that is, it depends on the observer-dependent subregime $O$, the local data region, and the chosen foliation $n$. Its further interpretation as a cosmological scale requires inter-observer compatibility within the corresponding reconstruction class. Therefore, below, unless explicitly stated otherwise, $L_{\max}$ is understood as an infrared scale compatible within such a compatibility class, that is, for observers considering one and the same effective large-scale sector up to the already accounted-for local equivalences.

At $L \sim L_{\max}$, a limiting infrared regime of reconstruction arises. On the one hand, the local covariant GR description is still treated as the working form of large-scale analysis. On the other hand, the finiteness of stable remote reconstruction requires that large-scale infrared incompleteness be represented in such a description. In other words, if the closed GR regime is preserved up to a neighborhood of $L_{\max}$, then it must contain a universal geometric way of encoding the limiting loss of spectral compatibility.

In the next subsection it will be shown that, if this response is to remain local, covariant, metric, universal, and of zero order in derivatives, then its leading form coincides with the form of a cosmological term.

## 5.2 Universal covariant zero-derivative infrared contribution

Consider a regime in which the effective large-scale description remains closed and covariant in terms of the metric $g_{ab}$ and the effective stress-energy tensor $T_{ab}^{\mathrm{eff}}$, as briefly described in Subsection 2.3. In the previous GR reconstruction, the cosmological term already arose as an admissible universal zero-derivative contribution to the effective geometric action. The present section clarifies its reconstructive origin: we isolate the part of the cosmological contribution that is associated specifically with the infrared limit of stable remote reconstruction.

For this purpose, it is convenient to write the effective equations in the form

$$G_{ab} = 8\pi G\, T_{ab}^{\mathrm{eff}} + \Xi_{ab}^{\mathrm{IR}} + (\text{subleading IR corrections}), \tag{26}$$

where $\Xi_{ab}^{\mathrm{IR}}$ denotes the leading universal part of the infrared geometric response. This notation does not introduce a new independent physical sector: $\Xi_{ab}^{\mathrm{IR}}$ is used only as an auxiliary notation for that part of the zero-derivative geometric contribution whose origin is associated with the reconstruction horizon.

Since we are interested in the leading universal contribution, it is necessary to isolate in $\Xi_{ab}^{\mathrm{IR}}$ the local covariant term of the lowest order in derivatives. Under the assumptions of locality, covariance, and universality, the form of such a term is strongly constrained. A local covariant symmetric rank-two tensor of zero order in derivatives, constructed only from the metric, has the form

$$\Xi_{ab}^{(0)} = -\Lambda_{\mathrm{IR}}\, g_{ab}, \tag{27}$$

where $\Lambda_{\mathrm{IR}}$ is a scalar coefficient characterizing the leading infrared response. No other universal local covariant tensor of zero derivative order depending only on the metric exists in this regime.

In other words, if the loss of compatibility of remote reconstruction is to be encoded in the leading local covariant term of the effective equations of the gravitational sector, then this term takes the form of a cosmological contribution. Equation (26) can then be rewritten as

$$G_{ab} + \Lambda_{\mathrm{IR}}\, g_{ab} = 8\pi G\, T_{ab}^{\mathrm{eff}} + (\text{subleading IR corrections}). \tag{28}$$

It should be emphasized that $\Lambda_{\mathrm{IR}}$ does not arise here as a fundamental microscopic parameter and is not identified with a heuristic sum of vacuum zero-point fluctuations in standard QFT. It appears as a *universal geometric parameter of infrared incompleteness of reconstruction.* This also distinguishes it from the gravitationally relevant non-vacuum part of the hidden content, which in the previous work was interpreted as dark matter [5]: if such a part is present, it contributes primarily to $T_{ab}^{\mathrm{eff}}$, whereas $\Lambda_{\mathrm{IR}}$ belongs to the geometric zero-derivative sector.

The result obtained can be formulated as a proposition.

**Proposition 5.1.** *Suppose that, in the range of scales under consideration, the closed covariant GR description is preserved, and that the leading universal infrared response to the loss of compatibility of remote reconstruction is local, metric, and of zero order in derivatives. Then this response has the form of a cosmological term, that is, it can be written as*

$$\Xi_{ab}^{(0)} = -\Lambda_{\mathrm{IR}}\, g_{ab}.$$

*Proof.* The statement follows from locality, covariance, and zero order in derivatives. The only local covariant symmetric rank-two tensor depending only on the metric and containing no derivatives of it is proportional to the metric itself. Therefore,

$$\Xi_{ab}^{(0)} = -\Lambda_{\mathrm{IR}} g_{ab}$$

with some scalar coefficient $\Lambda_{\mathrm{IR}}$. □

Thus, the form of the leading infrared response is already fixed at the level of the general requirements imposed on the closed GR description. Two questions remain open: the scale of the coefficient $\Lambda_{\mathrm{IR}}$ and the choice of the physically relevant sign. The scale will be related below to $L_{\mathrm{max}}$, while the question of the sign will be considered in Section 6.

## 5.3 Order of magnitude of $\Lambda$ through $L_{\mathrm{max}}$

After Proposition 5.1, it is natural to ask what scale the parameter $\Lambda_{\mathrm{IR}}$ should have. Since it arises as a universal large-scale response, its magnitude should be determined not by local microphysics and not by the mean matter density, but by the infrared parameter already obtained in the previous section, namely the scale $L_{\mathrm{max}}$.

In units $c = 1$, the cosmological constant has dimension

$$[\Lambda] = L^{-2}.$$

If the leading universal IR contribution depends only on $L_{\mathrm{max}}$ and on dimensionless characteristics of the reconstruction class, then the only possible dimensional form is

$$\Lambda_{\mathrm{IR}} = \chi \, L_{\mathrm{max}}^{-2}, \tag{29}$$

where $\chi$ is a dimensionless coefficient. More cautiously, this can be written as an order-of-magnitude estimate:

$$|\Lambda_{\mathrm{IR}}| \sim \frac{|\chi|}{L_{\mathrm{max}}^2}. \tag{30}$$

The meaning of formula (29) is as follows. As shown in Section 4, $L_{\mathrm{max}}$ is the maximum scale on which stable reconstruction of the given type is still preserved. For $L \ll L_{\mathrm{max}}$, the influence of the infrared anomaly should be subleading, whereas for $L \sim L_{\mathrm{max}}$ it becomes significant for the large-scale description. Therefore, the natural scale of the coefficient characterizing the leading zero-derivative IR contribution is set by $L_{\mathrm{max}}^{-2}$.

Importantly, $\chi$ in (29) is not determined by dimensional analysis alone. It encodes aspects of the infrared response that depend on the structure of the reconstruction class, the averaging procedure, the way in which one passes from the observer-relative scale $L_{\mathrm{max}}^{(O,n)}$ to the inter-observer compatible scale $L_{\mathrm{max}}$, and possibly on additional universal constraints of the quantum layer. Therefore, the present paper does not claim to compute the exact value of $\Lambda$. Its aim is to establish the structural fact that, if the cosmological term arises as the leading universal IR response, then its natural scale is set by $L_{\mathrm{max}}^{-2}$, rather than by an independent fundamental cosmological length or by the mean matter density.

This statement can be written as the following result.

**Proposition 5.2.** *Suppose that the leading universal infrared response in the closed covariant GR regime has the form of a cosmological term and depends only on the maximum scale of stable reconstruction* $L_{\max}$ *and on dimensionless characteristics of the reconstruction class. Then the order of magnitude of the cosmological constant is determined by the estimate*

$$|\Lambda_{\mathrm{IR}}| \sim L_{\max}^{-2}.$$

*More precisely,*

$$\Lambda_{\mathrm{IR}} = \chi\, L_{\max}^{-2},$$

*where* $\chi$ *is a dimensionless coefficient.*

*Proof.* Dimensional analysis shows that any scalar quantity defining a zero-derivative geometric contribution must have dimension $L^{-2}$. If the only universal large-scale parameter is $L_{\max}$, then the only possible dimensional combination is $L_{\max}^{-2}$ multiplied by a dimensionless factor $\chi$. ☐

Thus, an intermediate result has been obtained: if the cosmological term arises as the leading universal IR response, it has not only a fixed tensorial form, but also a natural scale determined by the limit of stable reconstruction. It remains to clarify in what sense this parameter is inter-observer compatible.

## 5.4 Conditional invariance of $\Lambda$ in the compatibility class

In Subsection 2.4, a minimal notion of inter-observer compatibility was introduced. For the present paper, it is needed in a restricted sense: to separate the observer-relative limit

$$L_{\max}^{(O,n)}$$

from the infrared parameter that can be assigned to a large-scale reconstruction class.

Let $\mathcal{K}$ denote a fixed reconstruction class. Then, for observers belonging to this class, located sufficiently close to one another, and considering one and the same large-scale reconstructible sector in which the homogeneous-isotropic approximation and the closed covariant GR description are preserved, the values of $L_{\max}^{(O,n)}$ may differ only within the already accounted-for local equivalences. If this limit is stable within the class, then the corresponding scale can be denoted simply by $L_{\max}$ and used as a parameter of the given large-scale sector.

Since the leading universal geometric IR contribution is insensitive to such local readjustments, the coefficient $\Lambda_{\mathrm{IR}}$ must also coincide up to these equivalences. This can be formulated as follows.

**Proposition 5.3.** *Let* $\mathcal{K}$ *be a fixed reconstruction class, and let two observers belong to this class, be sufficiently close to one another, and consider one and the same large-scale reconstructible sector in which the homogeneous-isotropic approximation and the closed covariant GR description are preserved. If, in this sector, the leading infrared response has the form*

$$\Xi_{ab}^{(0)} = -\Lambda_{\mathrm{IR}}\, g_{ab},$$

*then the value of* $\Lambda_{\mathrm{IR}}$ *coincides for these observers up to the already accounted-for local equivalences.*

*Proof.* If $\Lambda_{\mathrm{IR}}$ depended on the particular choice of observer within one and the same reconstruction class and one and the same locally common large-scale sector, then the leading zero-derivative geometric contribution would not be a universal component of the effective large-scale description. This would contradict the condition of inter-observer compatibility, according to which universal large-scale parameters do not depend on an arbitrary local readjustment of the description within one reconstruction class. □

It should be emphasized that no full theory of reconstruction classes needs to be developed here. For the present paper, the restricted statement is sufficient: the cosmological constant, understood as the leading universal geometric IR contribution, must be inter-observer compatible for observers of one and the same reconstruction class, provided that they consider one and the same locally common large-scale cosmological sector. It thereby acquires the status not of a private characteristic of a single local procedure, but of an effective parameter of a consistent large-scale description.

Taken together, the results of this section give the following picture. As one approaches the boundary of stable remote reconstruction, an infrared anomaly arises as a directed loss of spectral compatibility. If this loss of compatibility is represented within the closed covariant GR description, then the leading universal response has the form of a cosmological term, and its natural scale is set by $L_{\max}^{-2}$. Thus, the form and scale of the cosmological constant are determined structurally. The next question is which branch of this geometric response is physically relevant in the target cosmological sector and what sign $\Lambda_{\mathrm{IR}}$ has. This is the subject of the next section.

# 6 Positive FLRW branch and causality-preserving IR transfer

## 6.1 Directed character of IR degradation of reconstruction

In Sections 3 and 4, it was shown that, for a fixed data region $\Omega_0$, remote reconstruction at scale $L$ is accompanied by the spectral restriction

$$k_{\max}(L) = \frac{1}{L}\log\frac{\delta_*}{\delta}, \qquad 0 < \delta < \delta_*,$$

with $k_{\max}(L)$ decreasing monotonically with $L$. Equivalently, the minimum distinguishable wavelength

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}$$

increases monotonically as the reconstruction scale grows. This means that the infrared incompleteness of reconstruction has a directed character: the transition to increasingly large scales systematically worsens the spectral compatibility of remote reconstruction with the effective hyperbolic regime.

This directedness is important for the subsequent interpretation of the sign of the cosmological term. By itself, the monotonicity of $k_{\max}(L)$ does not yet define the full cosmological dynamics and is not an independent proof of the sign of $\Lambda_{\rm IR}$. It defines a weaker but essential condition: if the infrared limit of reconstruction is to be represented within a closed GR/FLRW description, then the corresponding branch must be compatible with the increasing loss of stable access to the short-wavelength content of remote signals.

The content of this statement is as follows. As the characteristic scale $L$ grows, the sector accessible to stable reconstruction becomes increasingly depleted in the spectral sense. Therefore, if the effective geometric description is still preserved, its large-scale response must encode not the recovery of full remote information, but a causally consistent representation of its infrared-depleted remnant. This is the basic physical meaning of the directed infrared regime.

**Remark 6.1.** *Two levels of the statement should be distinguished. First, the form of the cosmological term as the leading zero-derivative IR contribution follows from locality, covariance, and the metric character of the closed GR description, as shown in Section 5. Second, the sign of this contribution is not determined by dimensional analysis alone. Selecting the physically relevant branch requires an additional cosmological interpretation of directed IR degradation.*

## 6.2 Redshift as causality-preserving infrared transfer

The narrowing of the stably reconstructible spectral window creates a potential threat to causal reconstruction of remote events. If the short-wavelength content of a remote signal lies outside the admissible window $k_{\max}(L)$, then such a signal can no longer be stably represented in the local registers of the observer as a causally recognizable image of a remote event. Therefore, as the reconstruction scale increases, what is required is not merely attenuation or loss of the signal, but a transfer of its observable content that preserves causal recognizability at least at a coarser, infrared level.

Let $k_{\rm em}$ denote the characteristic wavenumber of the signal in the emission region, and let $k_{\rm obs}(L)$ be the corresponding observed wavenumber after reconstruction at scale $L$. The condition of causal reconstructibility can be written as

$$k_{\rm obs}(L) \le k_{\max}(L). \tag{31}$$

Since, by Lemma 3.1,

$$k_{\max}(L) = \frac{1}{L}\log\frac{\delta_*}{\delta}$$

decreases monotonically as $L$ increases, condition (31) becomes increasingly restrictive at large scales. If the observed wavenumber of the remote signal remained equal to the emitted one,

$$k_{\rm obs}(L) = k_{\rm em},$$

then, for sufficiently large $L$, the condition

$$k_{\rm em} \le k_{\max}(L)$$

would inevitably be violated for any fixed $k_{\rm em} > 0$. Hence, preserving the causal reconstructibility of a remote signal under a narrowing spectral window requires an infrared transfer of its observable content:

$$k_{\mathrm{obs}}(L) < k_{\mathrm{em}}$$

in the range of scales where $k_{\max}(L) < k_{\mathrm{em}}$.

In the effective FLRW description, such an infrared transfer has the standard form of cosmological redshift:

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)}. \tag{32}$$

Substituting (32) into condition (31) gives

$$\frac{k_{\mathrm{em}}}{1 + z(L)} \leq k_{\max}(L). \tag{33}$$

Equivalently,

$$1 + z(L) \geq \frac{k_{\mathrm{em}}}{k_{\max}(L)}. \tag{34}$$

Using the explicit form of $k_{\max}(L)$, one obtains the necessary lower bound for the redshifting transfer:

$$1 + z(L) \geq \frac{k_{\mathrm{em}} L}{\log(\delta_*/\delta)}. \tag{35}$$

This reasoning can be formulated as the following result.

**Proposition 6.2** (necessity of redshifting transfer)**.** *Let $k_{\max}(L)$ be a monotonically decreasing upper bound on the stably reconstructible wavenumber, and let a remote signal with initial characteristic wavenumber $k_{\mathrm{em}} > 0$ remain causally reconstructible at scale $L$. Then its observed wavenumber must satisfy*

$$k_{\mathrm{obs}}(L) \leq k_{\max}(L).$$

*If $k_{\max}(L) < k_{\mathrm{em}}$, then causal reconstructibility is impossible without infrared transfer,*

$$k_{\mathrm{obs}}(L) < k_{\mathrm{em}}.$$

*Under the parametrization*

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)},$$

*this is equivalent to the lower bound*

$$1 + z(L) \geq \frac{k_{\mathrm{em}}}{k_{\max}(L)}.$$

*Therefore, in the regime of a narrowing spectral window, causal reconstructibility of a remote signal requires redshifting transfer.*

*Proof.* Causal reconstructibility of the signal requires that its observed wavenumber belong to the stably reconstructible spectral window, that is,

$$k_{\mathrm{obs}}(L) \leq k_{\max}(L).$$

If $k_{\mathrm{max}}(L) < k_{\mathrm{em}}$ and, at the same time, $k_{\mathrm{obs}}(L) = k_{\mathrm{em}}$, then this condition is violated. Therefore, it is necessary that

$$k_{\mathrm{obs}}(L) < k_{\mathrm{em}},$$

that is, the observable content of the signal must be transferred into the infrared range. With

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)},$$

the condition $k_{\mathrm{obs}}(L) \leq k_{\mathrm{max}}(L)$ becomes

$$\frac{k_{\mathrm{em}}}{1 + z(L)} \leq k_{\mathrm{max}}(L),$$

which immediately implies

$$1 + z(L) \geq \frac{k_{\mathrm{em}}}{k_{\mathrm{max}}(L)}.$$

□

Proposition 6.2 is not a derivation of the full dependence $z(L)$ and does not replace standard FLRW kinematics. It establishes a more basic statement: under a monotonic narrowing of the spectral window, causal reconstructibility of a remote signal requires a decrease in its observed wavenumber. In a closed FLRW description, such a decrease is realized precisely as cosmological redshift.

**Relation to the Hubble scale.** In the effective FLRW description, redshifting transfer is related not only to the decrease in the observed wavenumber itself, but also to the Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)}.$$

Since

$$1 + z = \frac{a(t_0)}{a(t_{\mathrm{em}})},$$

the present value

$$H_0 := H(t_0)$$

defines the local rate of change of the scale factor and, consequently, the phenomenological measure of the intensity of redshifting transfer near the observer. In particular, for small redshifts, standard FLRW kinematics gives, in units $c = 1$,

$$z \simeq H_0 D,$$

where $D$ denotes the corresponding small cosmological distance.

From the reconstructive point of view, this means that $H_0$ can be interpreted as the locally observed measure of the geometric regime that transfers remote content into the infrared range and thereby supports the condition

$$k_{\mathrm{obs}}(L) \leq k_{\mathrm{max}}(L).$$

If, at large scales, the universal geometric IR contribution dominates, then the corresponding asymptotic Hubble scale is determined by

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3}.$$

Together with

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2},$$

this gives

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}.$$

Therefore, in the IR-dominated branch, the scale $L_{\max}$ sets not only the order of the cosmological term, but also the order of the Hubble scale of the redshifting regime.

This relation can be rewritten as

$$\chi \simeq 3(H_{\mathrm{IR}} L_{\max})^2.$$

Therefore, if the inter-observer compatible scale $L_{\max}$, in the sense of Subsection 2.4, sets the order of the Hubble scale of the late IR-dominated branch,

$$L_{\max} \sim H_{\mathrm{IR}}^{-1},$$

then it is natural to expect

$$\chi = O(1).$$

This is not a computation of the exact value of $\chi$, but only shows that under a self-consistent identification of the reconstruction horizon with the Hubble IR scale, the coefficient $\chi$ does not require a separate fine tuning.

In the full FLRW phenomenology, the observed value $H_0$ may differ from the asymptotic value $H_{\mathrm{IR}}$, since effective matter and radiation components enter $T_{\mu\nu}^{\mathrm{eff}}$. Therefore, what is established here is only a structural relation between $L_{\max}$, $\Lambda_{\mathrm{IR}}$, redshift, and the Hubble scale, not an exact numerical prediction of $H_0$.

Redshift thereby receives a reconstructive interpretation. It does not preserve the full microscopic information about a remote event and does not undo the loss of short-wavelength details. Its role is different: it transfers the observable content of a remote signal into a longer-wavelength range that remains within the stably reconstructible spectral window. This is why redshift can be regarded as a causality-preserving infrared transfer.

Closer to the scale $L_{\max}$, this mechanism no longer provides a full detailed reconstruction of the remote region. It preserves only a coarser causal image: the individual microstructure of the signal is lost, while the infrared-depleted content remains accessible. Thus, redshift supports causal reconstructibility not by preserving all information, but by matching the remote signal to the spectral window that remains accessible at the given reconstruction scale.

### 6.3 Homogeneous-isotropic cosmological sector

To translate the preceding argument into a statement about the physically relevant branch of $\Lambda_{\mathrm{IR}}$, it is necessary to fix the target cosmological sector of the already derived effective GR description. The present paper does not undertake a new derivation of homogeneous-isotropic cosmology from the fundamental level; it only specializes the previously reconstructed closed covariant gravitational regime to the large-scale sector in which the homogeneous-isotropic approximation is admissible.

In this sector, the metric can be written in the Friedmann–Robertson–Walker form

$$ds^2 = -dt^2 + a^2(t)\, d\Sigma_k^2, \tag{36}$$

where $a(t)$ is the scale factor and $d\Sigma_k^2$ is the metric of the spatial three-geometry of constant curvature $k = 0, \pm 1$.

The effective equations (28), restricted to this sector, take the standard form

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\mathrm{eff}} + \frac{\Lambda_{\mathrm{IR}}}{3}, \tag{37}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho_{\mathrm{eff}} + 3p_{\mathrm{eff}}\big) + \frac{\Lambda_{\mathrm{IR}}}{3}, \tag{38}$$

where $H = \dot{a}/a$. Here $\rho_{\mathrm{eff}}$ and $p_{\mathrm{eff}}$ refer to effective matter, including possible visible, non-vacuum hidden, and radiation components. In the present paper, these quantities are not used as inputs for deriving $\Lambda_{\mathrm{IR}}$; they belong to the full FLRW phenomenology and may affect the concrete history of $H(t)$, but they are not the source of the geometric infrared contribution itself.

In the limiting infrared regime $L \sim L_{\max}$, we are interested in the leading universal geometric part of the response, associated not with the particular composition of effective matter, but with the very limit of stable reconstruction. Matter, radiation, and hidden non-vacuum components enter $T^{\mathrm{eff}}_{\mu\nu}$; they may change the observed value of $H_0$ and the full expansion history, but they do not replace the zero-derivative geometric contribution $\Lambda_{\mathrm{IR}}$. If this universal IR component is isolated, then, in the corresponding approximation,

$$\frac{\ddot{a}}{a} \simeq \frac{\Lambda_{\mathrm{IR}}}{3}, \tag{39}$$

and the asymptotic Hubble scale satisfies

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3}. \tag{40}$$

Equations (39) and (40) are used here not as empirical postulates, but as a criterion for distinguishing possible branches of the geometric IR response inside the already chosen effective FLRW sector.

As shown in Subsection 6.2, preserving the causal reconstructibility of remote signals under the narrowing of the spectral window requires redshifting infrared transfer:

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)} \le k_{\max}(L). \tag{41}$$

Therefore, the physically relevant FLRW branch must be such that the large-scale geometry supports this transfer, rather than removing the infrared directedness of the reconstruction constraint itself.

If $\Lambda_{\mathrm{IR}} > 0$, then the universal IR component defines a de Sitter-like branch. In this branch, a positive asymptotic Hubble scale arises,

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\Lambda_{\mathrm{IR}}}{3}}, \tag{42}$$

and remote signals acquire an increasing redshift. Together with

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2}$$

this gives the structural relation

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}. \tag{43}$$

Thus, the positive branch makes mutually compatible three elements: the finite reconstruction horizon, the Hubble scale of the IR-dominated regime, and redshift as a causality-preserving infrared transfer.

If, on the other hand, $\Lambda_{\mathrm{IR}} < 0$, then the cosmological term has the opposite sign in the acceleration equation and does not define an asymptotic de Sitter-like screening branch. Such a branch does not provide a natural geometric realization of the regime in which remote content preserves causal recognizability through transfer to the infrared range near the reconstruction horizon. Therefore, this branch is incompatible with the target cosmological regime considered in the present paper: it does not realize de Sitter-like screening behavior and does not provide a geometric form of causality-preserving redshifting infrared transfer.

Thus, the homogeneous-isotropic sector provides a geometric language for interpreting the directed IR deficit. Positive $\Lambda_{\mathrm{IR}}$ corresponds to the branch in which large-scale geometry supports the causality-preserving infrared transfer of remote signals and defines a Hubble scale of order

$$H_{\mathrm{IR}} \sim L_{\max}^{-1}.$$

The negative branch, although it can be formally written in the effective equations, does not satisfy the conditions of the target reconstruction regime, since it does not provide a geometric mechanism for preserving causal reconstructibility near $L_{\max}$.

## 6.4 Criterion for selecting the positive branch

The main result of this section can now be formulated as a compatibility criterion between the target cosmological regime and the requirements of causal reconstruction.

**Proposition 6.3** (criterion for the positive FLRW branch)**.** *Suppose that the following conditions hold:*

*(i) as the reconstruction scale $L$ increases, $k_{\max}(L)$ decreases monotonically and $\lambda_{\min}(L)$ increases monotonically;*

*(ii)* *at $L \sim L_{\max}$, the closed covariant GR regime is preserved;*

*(iii)* *the leading universal response to the infrared limit of stable reconstruction is given by the zero-derivative geometric term $\Lambda_{\mathrm{IR}} g_{ab}$;*

*(iv)* *the large-scale sector under consideration admits a homogeneous- isotropic FLRW approximation;*

*(v)* *the target cosmological regime must preserve the causal reconstructibility of remote signals through redshifting infrared transfer, that is, through the condition*

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1+z(L)} \leq k_{\max}(L)$$

*near the reconstruction horizon.*

*Then, in the cosmological sector under consideration, the branch compatible with these conditions is the positive branch:*

$$\Lambda_{\mathrm{IR}} > 0. \tag{44}$$

*Proof.* Condition (i) implies that, as the scale $L$ increases, the stably reconstructible spectral window narrows. Therefore, a remote signal with initial characteristic wavenumber $k_{\mathrm{em}}$ can remain causally reconstructible only if its observed wavenumber is transferred into the accessible infrared range:

$$k_{\mathrm{obs}}(L) \leq k_{\max}(L).$$

As shown in Subsection 6.2, in the FLRW description this condition takes the form

$$\frac{k_{\mathrm{em}}}{1+z(L)} \leq k_{\max}(L),$$

that is, it requires redshifting infrared transfer.

In the homogeneous-isotropic sector, the sign of $\Lambda_{\mathrm{IR}}$ distinguishes two branches of the geometric IR response. If $\Lambda_{\mathrm{IR}} > 0$, then the universal IR component defines a de Sitter-like branch with

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3} > 0.$$

Such a branch has a positive asymptotic Hubble scale, supports redshifting transfer of remote signals, and is compatible with a finite reconstruction horizon. For

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2}$$

it also gives the structural relation

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}.$$

If, on the other hand, $\Lambda_{\mathrm{IR}} < 0$, then the cosmological term has the opposite sign in the acceleration equation and does not define an asymptotic de Sitter-like screening branch. Such a branch does not provide a geometric mechanism by which remote content preserves causal recognizability through transfer to the infrared range near $L_{\max}$. Hence, it does not satisfy the conditions of the target reconstruction cosmological regime.

Therefore, under conditions (i)–(v), the branch compatible with the target regime is precisely the positive branch,

$$\Lambda_{\mathrm{IR}} > 0.$$

□

Proposition 6.3 does not claim that the sign of $\Lambda_{\mathrm{IR}}$ follows from dimensional analysis alone or from the formula for $k_{\max}(L)$ alone. It states a more precise result: if the infrared limit of reconstruction is to be represented in the homogeneous-isotropic GR sector in such a way that the causal reconstructibility of remote signals is preserved, then the physically compatible branch is the positive FLRW branch.

In other words, the negative branch is not excluded as a formal possibility for writing effective equations in general. It is excluded specifically as a branch of the target cosmological regime, because it does not realize de Sitter-like screening behavior, does not define a positive Hubble IR scale, and does not provide a geometric form of causality-preserving redshifting infrared transfer.

## 6.5 Cosmological horizon and redshift as consistent support

After Proposition 6.3, the cosmological horizon and redshift acquire a definite status in the argument of the paper. They are not used as initial phenomenological postulates in the derivation of $\Lambda_{\mathrm{IR}}$. Rather, they appear as geometric and observational manifestations of the same infrared directedness which, at the level of elliptic reconstruction, is expressed in the narrowing of the spectral window.

First, from the elliptic nature of the fundamental problem and the finite spectral resolution of the causally consistent regime, the operational scale $L_{\max}$ was obtained. It was then shown that, under GR closure, the corresponding universal zero-derivative contribution has the form

$$\Lambda_{\mathrm{IR}} g_{\mu\nu}, \qquad \Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2}.$$

In the homogeneous-isotropic sector, the positive branch defines the asymptotic Hubble scale

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3},$$

and therefore

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}.$$

Thus, the reconstruction horizon and the Hubble scale of the IR-dominated regime are structurally related.

From this viewpoint, the cosmological horizon is not added to the model as an external geometric datum. It is an effective macroscopic expression of the same restriction that appears in the operational setting as the limit of stable remote reconstruction. In other words, the finiteness of $L_{\max}$, the positive asymptotic Hubble scale, and the de Sitter-like screening branch are different descriptions of the same infrared regime, considered at different levels of the effective description.

Redshift plays a causality-preserving role in this picture. As the reconstruction scale grows, the accessible spectral window narrows:

$$k_{\max}(L) \downarrow .$$

Therefore, a remote signal can remain causally recognizable only if

$$k_{\mathrm{obs}}(L) \leq k_{\max}(L).$$

In the FLRW description this condition is realized through

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)}.$$

Thus, redshift is not an accidental additional effect, but admits a reconstructive interpretation: it transfers the observable content of a remote signal into a longer-wavelength range that remains accessible to stable reconstruction.

At the same time, redshift does not preserve the full microscopic information. It supports the causal recognizability of a remote event at the cost of losing short-wavelength details. Therefore, as $L_{\max}$ is approached, one should expect not the full preservation of the individually resolved event structure, but a transition to an increasingly coarse infrared description. In the limiting case, this may point to the possibility of a background layer separating the region of stable event reconstruction from the region in which individual event structure is no longer accessible.

Thus, the cosmological horizon, redshift, and the positive branch $\Lambda_{\mathrm{IR}} > 0$ are not independent additions to the model. In the target cosmological regime, they form a mutually consistent triadic structure:

$$L_{\max} \quad \longleftrightarrow \quad H_{\mathrm{IR}}^{-1} \quad \longleftrightarrow \quad \Lambda_{\mathrm{IR}} > 0,$$

where $L_{\max}$ defines the operational limit of stable reconstruction, $H_{\mathrm{IR}}$ defines the Hubble scale of the redshifting regime, and $\Lambda_{\mathrm{IR}}$ is its universal geometric zero-derivative representation in the closed GR sector.

This does not mean that the present paper has already constructed the full observational cosmology. The exact function $z(L)$ is not derived here, the observed value of $H_0$ is not computed, structure growth is not analyzed, and the properties of a possible background layer are not derived. The result obtained is structural in character: it shows that, within the reconstructive model considered here, the finite reconstruction horizon, the positive FLRW branch, and redshifting infrared transfer are mutually consistent manifestations of one and the same infrared constraint.

# 7 Discussion

## 7.1 Status of the result within the structure of the program

The present work is important not only as a particular analysis of the cosmological constant, but also as a step that clarifies the place of the cosmological term within the general reconstructive program. In previous works, the main elements of the effective description arising from the timeless Euclidean model were constructed: the special-relativistic structure as a regime of consistent event reconstruction, the closed covariant gravitational regime as a compensating geometric sector, the quantum layer as a secondary effective structure, and hidden large-scale content, including the gravitationally relevant non-vacuum part of the hidden content interpreted as dark matter.

Against this background, the present paper shows how the cosmological constant can be included in the same scheme. It is not introduced as a fundamental parameter of the initial Euclidean level, is not identified with vacuum energy in the standard quantum-field-theoretic sense, and is not derived from the mean matter density of the observable universe. In the setting considered here, the cosmological term receives an interpretation as a universal geometric infrared response to the finiteness of stable remote reconstruction. Its natural scale is given by

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2}.$$

The positive FLRW branch is singled out as the branch compatible with the causality-preserving infrared transfer of remote signals. In this branch, the reconstruction horizon, the Hubble scale of the redshifting regime, and the cosmological term turn out to be different effective representations of one and the same infrared constraint. Under a self-consistent identification of $L_{\max}$ with the Hubble scale of the late IR-dominated regime, the coefficient $\chi$ is naturally expected to be of order unity. However, this remains a structural estimate, not a computation of its exact value.

Thus, the cosmological constant receives a place within the already constructed reconstructive architecture. The special-relativistic structure, the gravitational sector, the quantum layer, hidden large-scale content, and the cosmological term are no longer treated as independent external elements added to the model separately. They are interpreted as different levels of the effective description arising under different conditions of operational reconstruction of the fundamental Euclidean field.

It should be emphasized, however, that the result obtained has a limited status. The present paper does not compute the observed numerical value of $\Lambda$, does not construct the full FLRW phenomenology, does not derive the exact dependence $z(L)$, does not compute the observed value of $H_0$, does not analyze structure growth, and does not consider the spectrum of the cosmic microwave background. The result obtained is structural: it shows that, in the presence of a finite scale of stable reconstruction and when this scale is represented within the closed covariant GR regime, the natural leading zero-derivative contribution has the form of a cosmological term

$$\Lambda_{\mathrm{IR}} g_{\mu\nu}.$$

The normalization of the coefficient $\chi$, the precise comparison of $L_{\max}$ with observed cosmological distances, the computation of $H_0$, the derivation of the full redshift dependence, and the construction of the full FLRW phenomenology remain tasks for future work.

## 7.2 Transition to theorematic development and mathematical closure

After the inclusion of the cosmological term in the general reconstructive scheme, the further development of the program should be understood not merely as a transition from the conceptual stage to the technical stage, but as a transition to the stage of systematic theorematic development. What is essential is not only that the main effective layers of the model have now been assembled into a unified reconstructive scheme, but also that this scheme has acquired a structurally closed form. As a result, the further development of the program can be regarded primarily as the task of rigorous mathematical derivation and refinement of the already specified structure, rather than as a search for further missing conceptual layers.

This is the main methodological shift achieved by the present work. After the reconstruction of the special-relativistic, gravitational, quantum, hidden, and cosmological layers, the model can for the first time be interpreted as a coherent inverse-problem setting in which the main levels of the effective physical description receive a common origin from a single fundamental Euclidean setting. This opens the possibility of systematically translating the previously introduced structural claims, operational conditions, and reconstructive principles into the form of rigorous mathematical statements.

In the same sense, the structural closure achieved here means that the model can be regarded as a unified basis for a principled description of the main physical regimes without adding new fundamental layers at each subsequent stage. This does not mean that the full concrete phenomenology has already been constructed in quantitative form, but rather that, at the level of the general architecture, the main elements of the physical scene no longer need to be externally introduced as independent initial entities.

This, of course, does not mean that all quantitative characteristics have already been computed or that further work is reduced to the formal writing down of previously known answers. On the contrary, substantial tasks remain: the normalization of dimensionless coefficients, the relation between reconstruction scales and observable cosmological quantities, the construction of the full phenomenology of the effective matter sector, the refinement of the redshift law, and the analysis of possible limiting background regimes. However, these questions no longer concern the search for a new conceptual architecture, but the mathematical development of the structure that has now been specified as a unified whole.

Thus, after the present paper, the program enters a phase in which its further development can proceed as the systematic construction of a rigorous system of mathematical results. This is the main meaning of the structural closure achieved here: the model acquires not only a conceptually consistent architecture, but also a basis for further development in the form of theorematic reconstructive physics.

## 7.3 Cosmological implications and limitations

The result obtained opens several directions for further cosmological development. The first direction concerns the full inclusion of the matter sector. In the present work, the mean matter density was not used as an input in deriving $\Lambda_{\mathrm{IR}}$, since the aim was not to solve the Friedmann equations with prescribed densities, but to identify the origin of the universal geometric zero-derivative contribution. However, in order to pass to the full cosmological phenomenology, one must consider the equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\text{eff}} + \frac{\Lambda_{\text{IR}}}{3},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho_{\text{eff}} + 3p_{\text{eff}}\big) + \frac{\Lambda_{\text{IR}}}{3},$$

taking into account visible matter, radiation, the gravitationally relevant hidden non-vacuum sector, and possible subleading infrared corrections. It is at this stage that quantitative relations between $L_{\max}$, the coefficient $\chi$, the expansion history, and observable cosmological parameters must be established.

The second direction concerns redshift. In the present paper, it was used not as an initial phenomenological postulate, but as the geometric form of causality-preserving infrared transfer. The narrowing of the stably reconstructible spectral window requires remote signals to satisfy the condition

$$k_{\text{obs}}(L) = \frac{k_{\text{em}}}{1 + z(L)} \leq k_{\max}(L).$$

Redshift thereby receives a reconstructive interpretation: it preserves the causal recognizability of a remote event at the cost of losing short-wavelength information. However, the full law $z(L)$ is not derived from this condition in the present paper. Such a derivation requires a separate analysis of the correspondence between the operational scale $L$, cosmological distances, the dynamics of the scale factor, and the concrete reconstruction class.

The third direction concerns a limiting background layer near the reconstruction horizon. Unlike a full identification of such a layer with the observed cosmic microwave background, the very appearance of a CMB-like reconstruction remnant is a more direct structural consequence of the mathematics obtained above. Indeed, near the reconstruction horizon, the stably reconstructible spectral window narrows to the threshold range

$$k \leq k_*, \qquad k_* = \frac{2\pi}{\lambda_*}.$$

Therefore, any event structure requiring wavenumbers $k > k_*$ ceases to be individually reconstructible. What remains observable is only the low-frequency image of the remote region, that is, an infrared-smoothed background remnant.

In this sense, near $L_{\max}$, what appears is not simply the absence of further reconstruction, but a transition from individually resolvable event structure to a limiting background regime. Such a layer can be regarded as a CMB-like reconstruction boundary: not as a material surface in the fundamental space $\mathbb{E}^4$, but as an operational screen beyond which individual event structure is no longer accessible for the given class of observers. It separates the region in which remote events can still be causally recognized from the region in which only statistically depleted infrared content is preserved.

If such a limiting background layer indeed arises, then it changes the phenomenological status of the critical wavelength $\lambda_*$. In the mathematical setting, $\lambda_*$ remains an internal criterion of admissibility of causally consistent reconstruction and defines the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*.$$

However, what may be observationally accessible in this case is not the microscopic threshold $\lambda_*$ itself, but the already formed limiting background layer. Then the phenomenology of the horizon is determined mainly by the effective scale $L_{\max}$, the coefficient $\chi$, the Hubble scale of the IR-dominated regime, and the parameters of the reconstruction class, while the exact normalization of $\lambda_*$ enters them as a hidden internal parameter of the transition from event reconstruction to the background regime.

At this stage, this result is not a full derivation of the observed cosmic microwave background. Such an identification would require a separate explanation of:

a nearly thermal spectrum,

the temperature scale,

small anisotropies,

polarization,

acoustic peaks and the relation to structure formation.

These questions are not considered in the present work. Here only the more general structural result is fixed: near the reconstruction horizon, individually resolved event structure must give way to an infrared-smoothed background regime. Therefore, the CMB-like layer should be understood first of all as a reconstruction boundary, not as an already fully derived observed cosmic microwave background.

Thus, the present paper provides a structural mechanism for the emergence of the cosmological term, but does not replace the full observational cosmology. The next stage is to relate $L_{\max}$, the coefficient $\chi$, redshift, the Hubble scale, matter density, the criterion $\lambda_*$, and a possible limiting background layer within a unified quantitative FLRW phenomenology.

# 8 Conclusion

In the present work, a timeless Euclidean model with a single real harmonic field was considered as a fundamental setting containing neither an a priori Lorentzian metric, nor fundamental time, nor pre-given cosmological dynamics. Within this setting, the regime of remote reconstruction relative to a localized data region $\Omega_0$ and a fixed flat working foliation was analyzed. It was shown that the elliptic character of the fundamental equation leads to the exponential instability of inverse continuation and, as a consequence, to a scale-dependent restriction on the stably reconstructible spectral content.

On this basis, an infrared-induced narrowing of the spectral window was obtained:

$$k_{\max}(L) = \frac{1}{L}\log\frac{\delta_*}{\delta},$$

where $L$ denotes the scale of remote reconstruction, $\delta$ is the characteristic error level of the accessible data, and $\delta_*$ is the admissible operational error threshold after reconstruction. Since $k_{\max}(L)$ decreases monotonically as $L$ increases, the minimum distinguishable wavelength

$$\lambda_{\min}(L) = \frac{2\pi}{k_{\max}(L)}$$

increases monotonically. Thus, large-scale reconstruction in the model considered here is accompanied by directed infrared degradation: as the reconstruction scale increases, the accessible spectral window systematically narrows, and the description becomes increasingly large-scale and informationally depleted.

A result was then established on the existence of a finite maximum scale of stable reconstruction $L_{\max}$, defined by the condition

$$\lambda_{\min}(L_{\max}) = \lambda_*,$$

where $\lambda_*$ is the critical minimum wavelength of resolution compatible with a stable causally consistent description. At the level of the technical derivation, this scale has the status of an observer- and foliation-dependent limit $L_{\max}^{(O,n)}$. Its interpretation as a cosmological infrared scale requires inter-observer compatibility within the corresponding reconstruction class. In this sense, $L_{\max}$ is not a fundamental boundary of the field $\Phi$ on $\mathbb{E}^4$, but an operational reconstruction horizon: the limit of the large-scale sector that remains accessible to a stable causally consistent description.

It was then shown that, if this infrared limit is represented within the already reconstructed closed covariant GR regime, the leading universal local zero-derivative contribution has the form of a cosmological term. Accordingly, the cosmological contribution receives the reconstructive interpretation

$$G_{\mu\nu} + \Lambda_{\mathrm{IR}} g_{\mu\nu} = 8\pi G\, T^{\mathrm{eff}}_{\mu\nu} + (\text{subleading IR corrections}).$$

Its natural scale is given by

$$\Lambda_{\mathrm{IR}} = \chi\, L_{\max}^{-2},$$

where $\chi$ is a dimensionless coefficient depending on the details of the cosmological reconstruction class, the averaging procedure, and possible additional constraints of the effective layer. Therefore, at the level of order of magnitude,

$$|\Lambda_{\mathrm{IR}}| \sim L_{\max}^{-2}.$$

In the target homogeneous-isotropic cosmological sector, the physically relevant FLRW branch of this geometric IR response was singled out. Since, as $L$ grows, the stably reconstructible spectral window narrows, the causal reconstructibility of remote signals requires the transfer of their observable content into a longer-wavelength range. In the effective FLRW description, this condition is expressed through redshift:

$$k_{\mathrm{obs}}(L) = \frac{k_{\mathrm{em}}}{1 + z(L)} \le k_{\max}(L).$$

Thus, redshift receives a reconstructive interpretation as a causality-preserving infrared transfer: it does not preserve the full microscopic information about a remote event, but supports its causal recognizability within the narrowing spectral window.

The positive branch

$$\Lambda_{\mathrm{IR}} > 0$$

is compatible with this target regime, since it defines de Sitter-like screening behavior and a positive asymptotic Hubble scale

$$H_{\mathrm{IR}}^2 \simeq \frac{\Lambda_{\mathrm{IR}}}{3}.$$

Together with

$$\Lambda_{\mathrm{IR}} = \chi L_{\max}^{-2},$$

this gives the structural relation

$$H_{\mathrm{IR}} \simeq \sqrt{\frac{\chi}{3}}\, L_{\max}^{-1}.$$

Thus, the reconstruction horizon, the Hubble scale of the IR-dominated regime, the positive cosmological constant, and redshift are mutually consistent manifestations of one and the same infrared constraint. Under a self-consistent identification of $L_{\max}$ with the Hubble scale of the late IR-dominated regime, the coefficient $\chi$ is naturally expected to be of order unity, although its exact value is not computed in the present work.

Finally, it was noted that, near the reconstruction horizon, individually resolved event structure must give way to a coarser infrared description. As $L \to L_{\max}$, the stably reconstructible spectral window narrows to the threshold range

$$k \le k_*, \qquad k_* = \frac{2\pi}{\lambda_*}.$$

Therefore, any event structure requiring wavenumbers $k > k_*$ ceases to be individually reconstructible. This points to the possibility of a CMB-like limiting background layer: not as a material surface in the fundamental space $\mathbb{E}^4$, but as an operational boundary beyond which individual event structure is no longer accessible for the given class of observers. In the present paper, this layer is not identified with the observed cosmic microwave background in the full sense; such an identification would require a separate derivation of the spectrum, temperature, anisotropies, polarization, and relation to structure formation.

Thus, four interconnected results have been established in the paper:

- (i) in a fixed flat foliation, the elliptic instability of remote reconstruction leads to the narrowing of the stable spectral window and to a finite scale of stable reconstruction $L_{\max}$;
- (ii) when this scale is represented within the closed covariant GR regime, the leading universal zero-derivative response has the form of a cosmological term and a natural scale of order $L_{\max}^{-2}$;
- (iii) in the target homogeneous-isotropic sector, the positive FLRW branch is singled out as the branch consistent with causality-preserving infrared transfer of remote signals and with a Hubble scale of order $L_{\max}^{-1}$;
- (iv) near the reconstruction horizon, individually resolved event structure must pass into an infrared-smoothed limiting background regime.

Consequently, in the timeless Euclidean model considered here, the cosmological constant receives an interpretation not as fundamental vacuum energy and not as an external phenomenological parameter, but as a universal geometric infrared response to the limitation of stable remote reconstruction. At the same time, the present result is structural in character. The paper does not compute the exact observed value of $\Lambda_{\mathrm{IR}}$, does not construct the full FLRW phenomenology, does not derive the full law $z(L)$, does not compute $H_0$, and does not derive the observed cosmic microwave background. These tasks require further analysis of the relation between $L_{\max}$, the coefficient $\chi$, the matter sector, redshift, the Hubble scale, and a possible limiting background layer.

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