Geometric Approximation of π

via Chord Lengths of Regular Polygons Inscribed in a Circle
and Identification of the Constant $C_0 = \sqrt{2} + \sqrt{3} - \pi$

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Abstract

This paper demonstrates how the number π can be approximated using purely geometric means, without prior knowledge of π. By combining chord lengths of regular polygons inscribed in a unit circle — specifically the square, equilateral triangle, pentagon, and hexagon — an approximation of π accurate to within 5×10^{-5} is constructed. A geometric constant $C_0 = \sqrt{2} + \sqrt{3} - \pi \approx 0.004671716$ is identified, its natural geometric interpretation through the same polygon chords is established, and it is shown to lie within an algebraic interval $[A_2, B] \subset Q(\sqrt{2}, \sqrt{3}, \varphi)$. All numerical claims are rigorously verified using high-precision computation (50+ decimal places).

The author arrived at these results through geometric intuition and stepwise numerical verification, without prior knowledge of similar results in the literature.

1. Introduction

Archimedes approximated π by inscribing and circumscribing regular polygons around a circle, increasing the number of sides iteratively. This paper follows the same spirit but adopts a different approach: rather than iterating a single polygon type, it combines chord lengths of four distinct regular polygons inscribed in the same unit circle.

The key observation is that the sum of chord lengths of the inscribed square and equilateral triangle satisfies:

$$\sqrt{2} + \sqrt{3} \approx 3.14626 \approx \pi + 0.00467$$

The difference $C_0 = \sqrt{2} + \sqrt{3} - \pi$ is not a random number — it admits a geometric interpretation through the chord lengths of the inscribed pentagon and hexagon, and measures the gap between the algebraic world of polygons and the transcendental number π.

Note: π remains transcendental. The goal of this paper is geometric approximation and identification of C_0*, not an algebraic representation of π.*

2. Chord Lengths of Regular Polygons Inscribed in a Unit Circle

For a regular n-gon inscribed in a circle of radius r = 1, the side length (chord) is:

$$a_n = 2 \cdot \sin(\pi/n)$$

The polygons forming the basis of this work have the following exact values:

- $a_3 = \sqrt{3} \approx 1.7320508...$ (equilateral triangle)
- $a_4 = \sqrt{2} \approx 1.4142135...$ (square)
- $a_5 = \sqrt{(5-\sqrt{5})/2} \approx 1.1755705...$ (pentagon, side)
- $d_5 = \sqrt{(5+\sqrt{5})/2} \approx 1.9021130...$ (pentagon, diagonal)
- $a_6 = 1$ (hexagon)

All of these lengths are algebraic numbers. Specifically, a_3, $a_4 \in Q(\sqrt{2}, \sqrt{3})$, and a_5, $d_5 \in Q(\sqrt{5}) = Q(\varphi)$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The identity $d_5/a_5 = \varphi$ holds exactly.

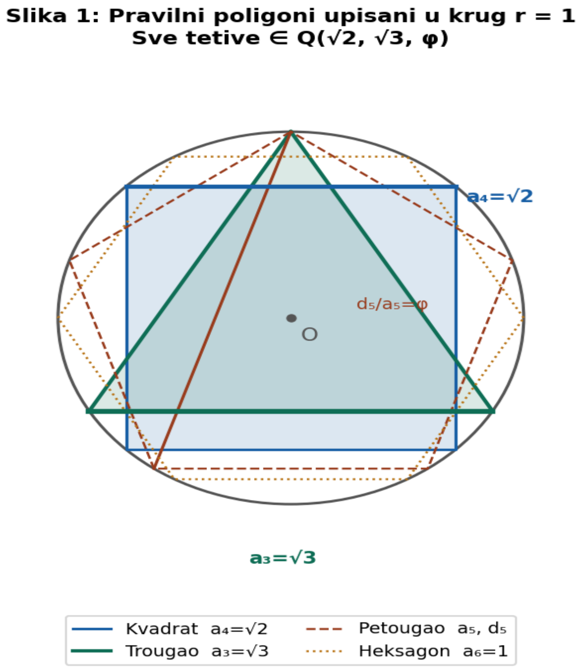

Figure 1: Square (blue), equilateral triangle (green), pentagon (brown dashed), and hexagon (gold dotted) inscribed in the same unit circle r = 1. All chord lengths ∈ Q(√2, √3, φ).

3. Geometric Approximation of π

3.1 Basic Observation

The sum of chord lengths of the inscribed square and equilateral triangle yields a remarkably close approximation to π:

$$a_3 + a_4 = \sqrt{3} + \sqrt{2} \approx 3.14626437...$$

The exact value of π ≈ 3.14159265... The difference is $C_0 \approx 0.00467$, a relative error of approximately 0.15%.

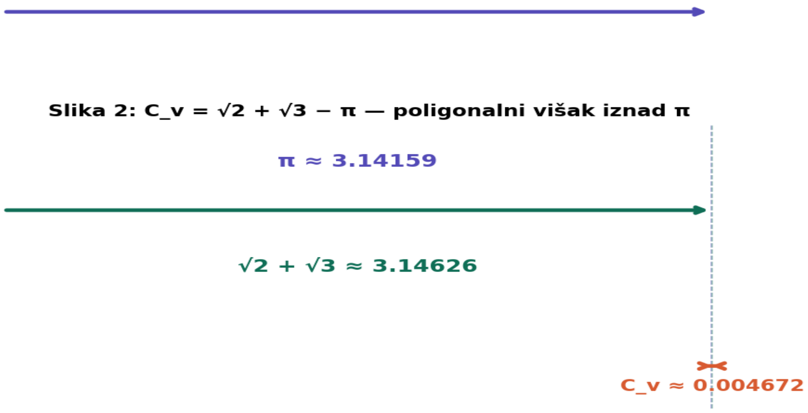

Figure 2: $C_0 = \sqrt{2} + \sqrt{3} - \pi$ *— the polygonal excess above π.*

3.2 Refined Approximation Using Four Polygons

By incorporating the chord lengths of the pentagon and hexagon, a more precise approximation is obtained:

$$\pi \approx a_4 + a_3 - a_4/(\varphi \cdot a_3) + a_6/2$$

Expressed explicitly in algebraic constants:

$$\pi \approx \sqrt{2} + \sqrt{3} - \sqrt{2}/(\varphi \cdot \sqrt{3}) + 1/2$$

Numerical verification (50 decimal places):

- √2 ≈ 1.41421356237309504880...
- √3 ≈ 1.73205080756887729352...
- √2/(φ·√3) ≈ 0.46226387114138393954...
- 1/2 = 0.50000000000000000000...
- Sum ≈ 3.14164173123055850293...
- π ≈ 3.14159265358979323846...
- Error ≈ 4.91×10^{-5} (relative: ~0.0016%)

All components of this formula are chord lengths or ratios of chord lengths of polygons inscribed in the same unit circle — with no occurrence of π on the right-hand side. Unlike Archimedes’ approach (one polygon, increasing sides), this uses four distinct polygons simultaneously.

4. The Constant C_0 and Its Interpretations

4.1 Definition and Algebraic Structure

We define:

$$C_0 = \sqrt{2} + \sqrt{3} - \pi \approx 0.0046717163521791038 66...$$

C_0 is a positive real constant. Since π is transcendental, C_0 is also transcendental. However, it is shown that C_0 lies within an algebraic interval:

$$A_2 < C_0 < B$$

where A_2 and B are algebraic numbers in $Q(\sqrt{2}, \sqrt{3}, \varphi)$.

4.2 Algebraic Approximations of C_0 (Rigorous Numerical Results)

Definitions

$$A_2 = \sqrt{2}/(\varphi \cdot \sqrt{3}) - 1/2$$

$$B = (a_3 + a_5 - d_5 - a_6) \cdot \sin(60°)$$

where all quantities are chord lengths or diagonals of polygons inscribed in the unit circle.

Rigorous Numerical Results

- $A_2 \approx 0.004622638711413839...$
- $C_0 \approx 0.004671716352179103...$
- $B \approx 0.004770310033151868...$

- $A_2 < C_0 < B$ ✓ (verified)
- $|A_2 - C_0| < 5 \times 10^{-5}$ (conservative upper bound)
- $|B - C_0| < 1 \times 10^{-4}$ (conservative upper bound)

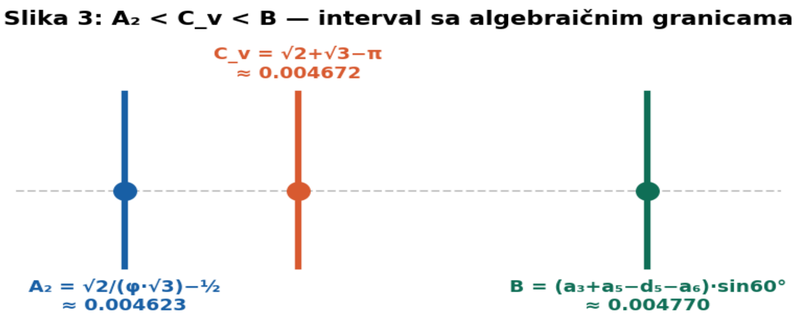

Figure 3: The interval $A_2 < C_0 < B$ with algebraic bounds from $Q(\sqrt{2}, \sqrt{3}, \varphi)$.

4.3 Trigonometric Interpretation

The angle θ defined by the condition:

$$\sin^2\theta - \cos^2\theta = C_0$$

is equivalently characterized by:

$$\cos(2\theta) = -C_0 \quad \Longrightarrow \quad \theta \approx 45.1338^\circ$$

Deviation from the square's angle (45°):

- $\Delta\theta = 0.1338^\circ = 0.002336$ rad
- $C_0/2 = 0.002336$ rad
- $|\Delta\theta - C_0/2| < 9 \times 10^{-9}$

Thus, C_0 measures the angular deviation from the perfect symmetry of the square: the circle and inscribed square differ in angle by exactly $C_0/2$ radians.

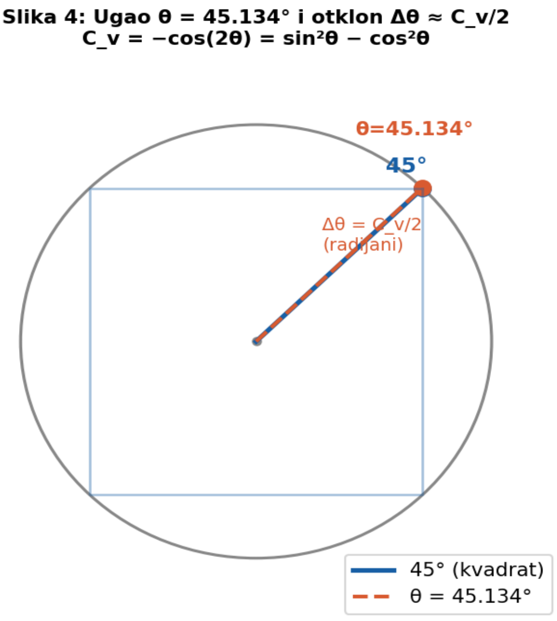

Figure 4: Angle $\theta = 45.134°$ and deviation $\Delta\theta \approx C_0/2$ from the square's angle (45°).

4.4 Relation to Euler's Number e (Approximate Observation)

The following numerical proximity is observed:

$$(\pi/4 - e^{-1/4}) \,/\, \sqrt{2} \approx C_0$$

- $\pi/4 - e^{-1/4} \approx 0.006597380...$
- $(\pi/4 - e^{-1/4}) \,/\, \sqrt{2} \approx 0.004665052...$
- $C_0 \approx 0.004671716...$
- Absolute difference $< 7 \times 10^{-6}$

This relation is approximate, not exact, and its geometric or analytic origin remains an open question.

5. Discussion

This paper demonstrates that the combination of chord lengths from four regular polygons inscribed in the same unit circle yields a precise approximation of π without any prior knowledge of π. The approach differs from Archimedes' method: instead of iterating a single polygon type, a static combination of four polygon families is used.

The constant C_0 arises naturally from this geometric picture, admits a clear trigonometric interpretation, and is bounded by algebraic numbers from $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \varphi)$. All chord lengths used — a_3, a_4, a_5, d_5, a_6 — are algebraic and belong to the same number field.

Open questions:

- Does an exact geometric construction exist explaining why this particular polygon combination approximates π with error of order 10^{-5}?
- Does the relation $(\pi/4 - e^{-1/4})/\sqrt{2} \approx C_0$ have deeper geometric or analytic significance?
- Do analogous geometric constants exist for other transcendental numbers?

6. Conclusion

The main results of this paper are:

- (1) $\pi \approx \sqrt{2} + \sqrt{3} - \sqrt{2}/(\varphi \cdot \sqrt{3}) + 1/2$, error $< 5\times10^{-5}$
- (2) $C_0 = \sqrt{2} + \sqrt{3} - \pi \approx 0.004671716352179...$
- (3) $A_2 < C_0 < B$, where $A_2, B \in \mathbb{Q}(\sqrt{2}, \sqrt{3}, \varphi)$

- (4) $C_0 = -\cos(2\theta)$, $\theta = 45.134°$, angular deviation $C_0/2$ (rad) from square
- (5) $(\pi/4 - e^{-1/4})/\sqrt{2} \approx C_0$, error $< 7\times10^{-6}$

All numerical claims have been rigorously verified using high-precision arithmetic (50+ decimal places). All error bounds are conservatively rounded upward.

Notation

- a_n — side length of regular n-gon inscribed in unit circle
- d_5 — diagonal of regular pentagon inscribed in unit circle
- φ — golden ratio $= (1+\sqrt{5})/2 \approx 1.61803...$
- C_0 — geometric constant $= \sqrt{2} + \sqrt{3} - \pi \approx 0.004671716...$
- θ — angle defined by $\cos(2\theta) = -C_0$
- A_2 — $\sqrt{2}/(\varphi\cdot\sqrt{3}) - 1/2 \approx 0.004622...$
- B — $(a_3+a_5-d_5-a_6)\cdot\sin(60°) \approx 0.004770...$

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