Empirical Relations Between Quark Mass Ratios and CKM Mixing Angles

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Abstract

We document a numerical regularity in PDG data: the three mixing angles of the CKM matrix are approximately given by $\sin\theta_{ij} = \sqrt{m_i/m_j}$, extending the historical Gatto-Sartori-Tonin (1968) relation for the Cabibbo angle to all three angles. Evaluated in the space of possible combinatorial configurations ($C(15,3) = 455$ triplets of pairs), this configuration stands out with a statistical significance of $p = 0.0022$. Internal consistency is verified at 0.90σ by cross-extraction of the up quark mass. This relation is compatible with the Froggatt-Nielsen (1979) mechanism – it corresponds to an ordinary U(1) charge assignment among other valid configurations, without exceptional character. The real positive texture cannot generate a complex phase; the relation $|\sin\delta_{CP}(\mathrm{CKM})| = m_W/m_Z$ ($\delta_{CP} \approx 61.82°$) is a distinct conjecture, testable by future LHCb and Belle II measurements. The θ_{23} anomaly (use of 1-2 masses for 2-3 mixing) remains documented. This work does not constitute a complete dynamical theory.

1 Introduction

The flavor problem – the origin of fermion masses and their mixings – remains one of the deepest mysteries of the Standard Model. The 19 free parameters of the flavor sector (masses, mixing angles, CP phases) are measured experimentally but not derived from any theoretical principle.

Historically, a simple empirical relation was proposed by Gatto, Sartori and Tonin (1968) [1] and developed by Weinberg (1977) [2]: the Cabibbo angle is related to the down quark mass ratio by $\sin\theta_C \approx \sqrt{m_d/m_s}$. This relation emerges from a texture zero in the mass matrix – a structure where the diagonal (1,1) element vanishes.

The Froggatt-Nielsen (1979) mechanism [3] provides a theoretical framework for such relations: a U(1) flavor symmetry, broken by a scalar field (flavon), generates mass hierarchies and mixing angles through powers of a small parameter $\varepsilon = \langle\phi\rangle/\Lambda$.

In this work, we show that the Gatto-Sartori-Tonin relation extends to all three CKM matrix angles, and we evaluate its statistical significance and theoretical context.

2 The Formula

The proposed relation reads:

$$\boxed{\sin\theta_{ij} = \sqrt{\frac{m_i}{m_j}} \qquad (m_i < m_j)} \tag{1}$$

Applied to the three CKM angles with PDG 2024 masses [4] (MS-bar at 2 GeV for light quarks, pole mass for the top):

Angle	Formula	Predicted	Measured (UTfit 2023) [5]	Tension
$\sin\theta_{12}$	$\sqrt{m_d/m_s}$	0.2242 ± 0.0011	0.2251 ± 0.0008	0.65σ
$\sin\theta_{23}$	$\sqrt{m_u/m_c}$	0.0412 ± 0.0004	0.04193 ± 0.00041	1.31σ
$\sin\theta_{13}$	$\sqrt{m_u/m_t}$	0.00354 ± 0.00003	0.00370 ± 0.00008	1.85σ

Masses used: $m_u = 2.16 \pm 0.04$ MeV, $m_d = 4.70 \pm 0.04$ MeV, $m_s = 93.5 \pm 0.5$ MeV, $m_c = 1.2730 \pm 0.0028$ GeV, $m_b = 4.18 \pm 0.03$ GeV, $m_t = 172.4 \pm 0.7$ GeV.

The formula has no continuous adjustable parameters. Quark masses are experimental inputs, and predictions for the three angles follow directly. The most notable tension concerns θ_{13} (1.85σ), which should be monitored as precision measurements improve (LHCb upgrade, Belle II).

3 Statistical Test

To evaluate whether this coincidence could be due to chance, we performed an exhaustive permutation test.

From the 6 quark masses, one forms $C(6,2) = 15$ distinct pairs. For each pair, $\sqrt{m_{\min}/m_{\max}}$ is computed, yielding 15 numerical values. The number of ways to choose 3 distinct pairs and assign them to the 3 CKM angles, respecting the observed hierarchy $\theta_{12} > \theta_{23} > \theta_{13}$, is $C(15,3) = 455$.

The proposed configuration (d/s for θ_{12}, u/c for θ_{23}, u/t for θ_{13}) ranks **first** among the 455, with a total relative error of 6.14%. The second-best configuration has an error of 24.8%, approximately 4 times larger.

$$\boxed{p = \frac{1}{455} \approx 0.0022} \tag{2}$$

This result was verified with two different metrics (relative deviations and χ^2 weighted by uncertainties). A Monte Carlo test (10,000 samples) varying quark masses within their PDG error bars confirms that the proposed configuration remains the best in all cases.

Limitation: this test is post-hoc (the data were used both to discover and to test the formula). Pre-registration of the methodology would strengthen its evidential value.

4 Internal Consistency

4.1 Cross-extraction of m_u

The formula allows extraction of the up quark mass in two independent ways:

$$m_u = m_c \times \sin^2\theta_{23} = 2.22 \pm 0.07 \text{ MeV} \tag{3}$$

$$m_u = m_t \times \sin^2\theta_{13} = 2.35 \pm 0.13 \text{ MeV} \tag{4}$$

These two estimates are consistent with each other at 0.90σ, and with the PDG value (2.16 ± 0.49 MeV).

Limitation: this test is partially circular – the formulas were chosen to reproduce the angles in forward fit; backward agreement is not an independent validation.

4.2 Scale Invariance

Mass ratios of quarks with the same electric charge (m_u/m_c, m_d/m_s) are quasi-invariant under QCD renormalization group evolution at leading order. The formula is therefore stable under scale evolution.

4.3 Jarlskog and Unitarity Triangle

With $\delta_{CP} = 61.82^\circ$ (Section 6), the predicted Jarlskog invariant is $J \approx 2.8 \times 10^{-5}$, consistent with the PDG measurement ($J \approx 3.0 \times 10^{-5}$). The unitarity triangle angles are $\alpha \approx 96.4^\circ$, $\beta \approx 21.8^\circ$ (0.5σ), $\gamma \approx 61.8^\circ$ (1.2σ).

5 Origin and Theoretical Context

5.1 Mass Textures and Texture Zeros

The $\sqrt{m_i/m_j}$ form is the signature of a mass matrix with texture zeros (Weinberg form). For a 2×2 matrix:

$$M = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \tag{5}$$

the eigenvalues are $\lambda_1 \approx A^2/B$, $\lambda_2 \approx B$, and the mixing angle $\sin\theta \approx A/B = \sqrt{\lambda_1/\lambda_2}$.

The matrix product $V_{\text{CKM}} = V_{\text{up}}^T \times V_{\text{down}}$ with Fritzsch-type M_{up} and Weinberg-type M_{down} (third generation decoupled) gives:

$$\theta_{12}(\text{CKM}) \approx \theta_{12}^{\text{down}} = \sqrt{m_d/m_s} \tag{6}$$

$$\theta_{23}(\text{CKM}) \approx \theta_{12}^{\text{up}} = \sqrt{m_u/m_c} \tag{7}$$

$$\theta_{13}(\text{CKM}) \approx \theta_{13}^{\text{up}} = \sqrt{m_u/m_t} \tag{8}$$

This mechanism is compatible with SO(10), but does not require it. Any theory with texture zeros produces this result.

5.2 Froggatt-Nielsen

The relation fits within the Froggatt-Nielsen mechanism [3]. The integer U(1) charge assignment:

$$q_u = 0,\ q_d = 0,\ q_s = 2,\ q_b = 3,\ q_c = 4,\ q_t = 10 \tag{9}$$

with $\varepsilon \approx 0.225$ simultaneously reproduces the mass hierarchies and mixing angles. The required $\mathcal{O}(1)$ coefficients (1.31, 1.42, 0.73) are all natural in the FN sense (within the interval [1/3, 3]).

This configuration is not unique. It is an ordinary point among other valid FN configurations. The fact that the coefficient is close to 1 is unremarkable in this framework – it is a banal point in the space of natural configurations.

6 Conjecture for δ_{CP}

The real positive texture imposes $V_{\text{CKM}} \in \text{SO}(3)$ at leading order, hence $J_{CP} = 0$. The CP-violating phase cannot be generated by the main texture.

We propose a distinct conjecture:

$$\boxed{|\sin\delta_{CP}(\text{CKM})| = \frac{m_W}{m_Z} \implies \delta_{CP}(\text{CKM}) = 61.82^\circ} \tag{10}$$

This value is compatible with current measurements of the unitarity triangle angle γ ($\gamma = 65.5^\circ \pm 3.0^\circ$, PDG 2024). Future measurements by LHCb (upgrade) and Belle II, targeting degree-level precision on γ by 2030, will test this conjecture.

Important note: δ_{CP}(CKM) – the CP-violating phase in the quark sector – is distinct from δ_{CP}(PMNS) – the phase in the lepton sector measured by neutrino oscillations. The

DUNE experiment measures δ_{CP}(PMNS), not δ_{CP}(CKM). This conjecture concerns only the quark sector and is not testable by DUNE.

Limitation: the relation $\arcsin(m_W/m_Z) = 90° - \arccos(m_W/m_Z)$ is a trigonometric identity. The conjecture is not derived from a principle – it is a numerical observation whose value lies in its falsifiability.

7 The θ_{23} Anomaly

The angle θ_{23} physically corresponds to $|V_{cb}|$, mixing between the second and third generations. The "natural" formula would be $\sqrt{m_c/m_t}$ (up sector) or $\sqrt{m_s/m_b}$ (down sector). Our formula uses $\sqrt{m_u/m_c}$ – the 1-2 up masses to describe 2-3 mixing.

The standard Fritzsch construction (two full sectors + relative phase) gives a mathematical floor of 0.0636 for $\sin\theta_{23}$ – incompatible with the measurement (0.0419). The choice of $\sqrt{m_u/m_c}$ circumvents this exclusion by ignoring the down sector contribution.

Ten explanatory hypotheses have been tested without success. This anomaly remains documented as an open question.

8 Limitations

1. **Tension on θ_{13} (1.85σ)** – to be monitored with future precision measurements
2. **Post-hoc test** – the formula was discovered and tested on the same CKM data
3. **Dominant prediction uncertainties** – 3 to 15 times wider than measurements for θ_{12} and θ_{23}
4. **Non-universality** – the formula does not apply to the lepton sector (PMNS)
5. **Absence of fundamental mechanism** – the formula is an empirical regularity (Balmer, not Bohr)
6. **Underived δ_{CP} conjecture** – numerical observation, not a consequence of a principle

9 Invalidation Criteria

To guarantee the falsifiability of this work, we explicitly define the conditions under which it would be refuted:

- If the measurement of θ_{13} deviates from $\sqrt{m_u/m_t}$ beyond 3σ with future LHCb/Belle II data, the basic formula will be considered falsified.
- If δ_{CP}(CKM) stabilizes at a value significantly different from 61.82° (e.g., $70° \pm 2°$), the conjecture will be invalidated.
- If future precision measurements reduce the uncertainty on γ and exclude 61.82° beyond 3σ, this pillar falls.

10 Conclusion

We have documented a numerical regularity in PDG data: the three CKM mixing angles are approximately given by $\sin\theta_{ij} = \sqrt{m_i/m_j}$. This relation, extending the historical Gatto-Sartori-Tonin (1968) formula to all three angles, obtains $p = 0.0022$ in an exhaustive permutation test ($C(15, 3) = 455$ configurations).

This configuration fits within the Froggatt-Nielsen (1979) mechanism as an ordinary U(1) charge assignment – it is neither unique nor exceptional. The conjecture $|\sin\delta_{CP}(\text{CKM})| = m_W/m_Z$ ($\delta_{CP} \approx 61.82°$) is testable by LHCb and Belle II.

The θ_{23} anomaly (use of 1-2 masses for 2-3 mixing) remains unresolved after the failure of ten explanatory hypotheses.

This work does not constitute a dynamical theory. It documents an empirical regularity, compatible with the Froggatt-Nielsen framework, and defines explicit invalidation criteria.

References

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