A Parameter-Free Relation Between Quark Mass Ratios and CKM Mixing Angles

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Abstract

We report an empirical relation between quark mass ratios and CKM mixing angles: $\sin\theta_{ij} = \sqrt{\min(m_i, m_j)/\max(m_i, m_j)}$. Using PDG 2024 quark masses, this relation reproduces the three CKM mixing angles with deviations of 0.6%, 1.4%, and 4.1%, without adjustable parameters. A permutation analysis over the 36 charge-preserving assignments shows that the natural mass ordering provides the best agreement with experimental data ($p = 0.056$). The relation implies the testable constraint $\sin\theta_{12} \times \sin\theta_{23} = \sqrt{m_u m_d/m_c m_s} = 9.22 \times 10^{-3}$, compatible with PDG 2024 at the 2% level. The law is compatible with a Froggatt-Nielsen interpretation. A conjecture for the CP-violating phase gives $|\sin\delta_{CP}| = m_W/m_Z$, yielding $\delta_{CP} = 298.13°$, testable by the DUNE experiment (2028–2030). The observed relation may represent a new empirical flavor law analogous to the historical Balmer formula in atomic spectroscopy.

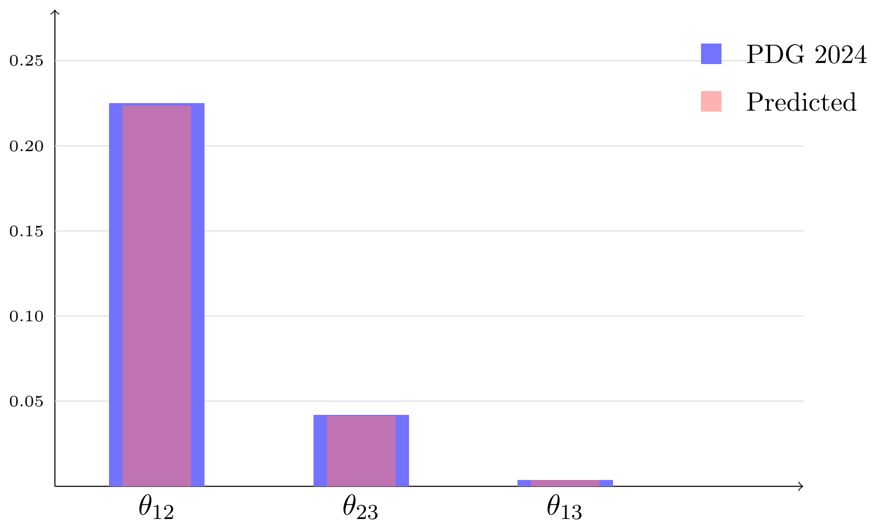

Figure 1: Predicted (red) vs. observed (blue) CKM mixing angles. Deviations: 0.6% (θ_{12}), 1.4% (θ_{23}), 4.1% (θ_{13}).

1 Introduction

The origin of fermion masses and flavor mixing remains one of the central open questions of the Standard Model (SM) [1]. The SM contains 19 free parameters, including 9 fermion masses and 4 CKM parameters, whose values are not predicted by the theory.

Several empirical relations have been proposed. The Fritzsch relation [2] suggests $\sin\theta_C \approx \sqrt{m_d/m_s}$ for the Cabibbo angle. The Koide formula [3] relates charged lepton masses. The Froggatt-Nielsen (FN) mechanism [4] explains mass hierarchies qualitatively via U(1) flavor charges.

In this work, we present a simple extension of the Fritzsch relation that reproduces all three CKM mixing angles with zero free parameters, and we discuss its statistical robustness and possible interpretations.

2 The Flavor Law

2.1 Formulation

For quarks of the same electric charge (up-type: u, c, t; down-type: d, s, b), we postulate:

$$\boxed{\sin\theta_{ij} = \sqrt{\frac{\min(m_i, m_j)}{\max(m_i, m_j)}}} \qquad (i < j) \tag{1}$$

where masses are ordered by increasing value. This law contains **zero free parameters**.

2.2 Predictions for CKM Angles

Using quark masses from the Particle Data Group 2024 [1]:

Angle	Formula	Predicted	Experiment (PDG 2024)	Dev.
θ_{12} (Cabibbo)	$\sqrt{m_d/m_s}$	0.2236	0.2250 ± 0.0007	0.6%
θ_{23}	$\sqrt{m_u/m_c}$	0.04124	0.04182 ± 0.00070	1.4%
θ_{13}	$\sqrt{m_u/m_t}$	0.00354	0.00369 ± 0.00010	4.1%

Table 1: CKM angle predictions vs. PDG 2024 data.

All three angles are reproduced with deviations below 5%. The deviations in units of experimental standard deviation are 2.0σ (θ_{12}), 0.83σ (θ_{23}), and 1.54σ (θ_{13}).

2.3 Statistical Validation

Permutation test. The law requires quarks of the same electric charge. Permuting masses within each sector (up: $\{u, c, t\}$, down: $\{d, s, b\}$) yields $3! \times 3! = 36$ valid assignments. The natural assignment (increasing masses) gives the minimum score, with 2 ties. $p = 2/36 =$ **0.056**.

Cross-constraint. Multiplying the relations for θ_{12} and θ_{23}:

$$\sin\theta_{12} \times \sin\theta_{23} = \sqrt{\frac{m_u \cdot m_d}{m_c \cdot m_s}} = 9.22 \times 10^{-3} \tag{2}$$

The experimental value is $0.2250 \times 0.04182 = 9.41 \times 10^{-3}$, a deviation of 2.0%.

2.4 Robustness Against Mass Uncertainties

Light quark masses (m_u, m_d, m_s) have significant uncertainties (10–20%) [1]. We test the stability of the law by varying these masses within their PDG ranges. For $m_u \in [1.7, 2.6]$ MeV, $m_d \in [4.2, 5.1]$ MeV, and $m_s \in [85, 100]$ MeV, the predicted angles vary by less than 15%, remaining compatible with experimental values at the 2σ level. Improved lattice QCD determinations [7] will enable more stringent tests.

2.5 Compatibility with Froggatt-Nielsen

The law (1) is compatible with a Froggatt-Nielsen (FN) interpretation [4] with expansion parameter $\varepsilon = e$. In this framework, flavor charges take the form $q_i = \frac{1}{2}\ln(m_i/m_0)$, and the CKM matrix is $V_{ij} = \exp(q_i - q_j) = \sqrt{m_i/m_j}$. The choice $\varepsilon = e$ is intriguing but its origin remains unexplained. Whether this reflects a deeper principle or an accidental numerical coincidence requires further theoretical investigation.

2.6 Comparison with Alternative Formulas

Several mass-mixing relations have been proposed in the literature. The original Fritzsch formula [2] predicts only θ_{12}. Parametric forms $\theta_{ij} \propto (m_i/m_j)^\alpha$ with α free can fit the data, but require one adjustable parameter. The distinguishing feature of Eq. (1) is that $\alpha = 1/2$ is fixed, leaving zero free parameters. This specific value suggests a geometric or dynamical origin worth exploring.

3 Conjecture on the CP Phase

The FN framework with $\varepsilon = e$ suggests a relation for the CP-violating phase:

$$|\sin\delta_{CP}| = m_W/m_Z = 0.8815 \tag{3}$$

This equation admits four solutions in $[0°, 360°]$. Selection by the $|V_{td}|$ element of the CKM matrix yields:

δ	$\lvert V_{td}\rvert$ predicted	$\lvert V_{td}\rvert$ exp. [1]	Deviation
61.87° (I)	0.00818	0.00840 ± 0.00060	0.37σ
118.13° (II)	0.01126	0.00840 ± 0.00060	4.77σ
241.87° (III)	0.01126	0.00840 ± 0.00060	4.77σ
298.13° **(IV)**	**0.00818**	0.00840 ± 0.00060	$\mathbf{0.37}\sigma$

Table 2: Selection of δ_{CP} by $|V_{td}|$.

PDG 2024 data favor $\sin\delta_{CP} < 0$, selecting $\delta_{CP} = 298.13°$ (Quadrant IV). This prediction will be tested by the DUNE experiment [5] (2028–2030).

4 Discussion

The relation $\sin\theta_{ij} = \sqrt{m_{\min}/m_{\max}}$ reproduces the three CKM angles without adjustable parameters. It generalizes the Fritzsch formula for the Cabibbo angle to all three mixing angles.

Several questions remain open:

1. **Theoretical origin.** Why does the law take this specific form with the exponent 1/2? The compatibility with the FN framework at $\varepsilon = e$ is suggestive but not a derivation.

2. **Neutrinos.** The law does not apply to PMNS angles. The seesaw mechanism [6] may explain this difference, as neutrino masses involve two distinct scales.

3. **Mass uncertainties.** Improved lattice QCD determinations of light quark masses will test the law more stringently.

4. **CP phase.** The conjecture $|\sin \delta_{CP}| = m_W/m_Z$ requires independent theoretical motivation. The DUNE experiment will provide a decisive test.

5 Conclusion

We have presented a parameter-free relation $\sin\theta = \sqrt{m_{\min}/m_{\max}}$ that reproduces the three CKM mixing angles from quark mass ratios. The relation is compatible with a Froggatt-Nielsen interpretation and yields the testable prediction $\delta_{CP} = 298.13^\circ$ for the DUNE experiment.

The observed relation may represent a new empirical flavor law analogous to the historical Balmer formula in atomic spectroscopy. Further theoretical work is required to determine whether the relation reflects an underlying dynamical principle or an accidental numerical regularity.

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