

# Empirical Relations Between Masses and Mixing Angles in the Standard Model

Belkacem Lichani

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## Abstract

We report the discovery of a systematic empirical relation connecting the six mixing angles of the Standard Model to particle masses:  $\sin \theta = \sqrt{m_{\text{light}}/m_{\text{heavy}}}$ . The relation holds with a mean deviation of 1.49% across all six angles. A Monte Carlo simulation with 100,000 trials yields  $p < 10^{-5}$ , excluding a random origin. Four cross-consistency checks on quark masses are verified to better than 1.5%. We also report satellite relations among boson masses:  $m_t/m_W = 15/7$ ,  $m_t \times m_W = (5/3)m_Z^2$ , and  $m_H/m_Z + \alpha = m_t/m_H$ . A falsifiable prediction is given:  $m_u = 2.23 \pm 0.10$  MeV, testable by future lattice QCD measurements. The modular symmetry framework  $\Gamma(5)$  is discussed as a possible theoretical foundation.

## 1 Introduction

The Standard Model contains 22 free parameters in the flavor sector. No fundamental theory explains the hierarchy of fermion masses or the values of the mixing angles. A few empirical relations have been noted in the literature: Koide (1982) discovered  $Q(e, \mu, \tau) = 2/3$ , holding to 0.0009%. Cabibbo (1968) found  $\sin \theta_C = \sqrt{m_d/m_s}$ , accurate to 0.6%. Both remain unexplained and have motivated decades of theoretical work.

We extend these relations to all six mixing angles of the Standard Model and uncover additional connections among the electroweak boson masses.

## 2 The Main Relation

The six mixing angles are given by:

$$\boxed{\sin \theta = \sqrt{\frac{m_{\text{light}}}{m_{\text{heavy}}}}} \quad (1)$$

Input data (PDG 2024, pole masses in MeV):  $m_u = 2.16$ ,  $m_d = 4.67$ ,  $m_s = 93.4$ ,  $m_c = 1270$ ,  $m_b = 4180$ ,  $m_t = 172500$ ,  $m_Z = 91187.6$ .

The pair assignments are not arbitrary. Among the 15 possible quark-quark pairs, 5 of the 6 pairs used are the unique best choices for their respective angles. The probability of randomly selecting this optimal quintuplet is  $1/759375$  (0.00013%). The sixth angle ( $\theta_{23}^{\text{PMNS}}$ ) requires the  $Z$  boson because no quark pair can reach  $\sin \theta > 0.68$ .

Table 1: The six relations  $\sin \theta = \sqrt{m/m}$ . Mean deviation: 1.49%.

Angle	Formula	$\sqrt{m_1/m_2}$	$\sin \theta$ (PDG)	Deviation
$\theta_{12}$ CKM (Cabibbo)	$\sqrt{d/s}$	0.22361	0.22529	0.75%
$\theta_{23}$ CKM	$\sqrt{u/c}$	0.04124	0.04118	0.15%
$\theta_{13}$ CKM	$\sqrt{u/t}$	0.00354	0.00367	3.45%
$\theta_{12}$ PMNS (solar)	$\sqrt{c/b}$	0.55121	0.55063	0.11%
$\theta_{13}$ PMNS (reactor)	$\sqrt{s/b}$	0.14948	0.14850	0.66%
$\theta_{23}$ PMNS (atmosph.)	$\sqrt{Z/t}$	0.72707	0.75585	3.81%

### 3 Statistical Validation

#### 3.1 Monte Carlo Simulation

We performed a global Monte Carlo test with 100,000 trials. For each trial, we randomly drew 5 pairs from the 15 quark-quark combinations and 1 pair from the 36 quark-boson combinations. We computed the mean relative deviation from the observed mixing angles.

**Result:** Zero trials achieved a mean deviation  $\leq 1.49\%$ .  $p < 10^{-5}$ .

The optimal possible deviation using only quark pairs is 2.52%. The observed value of 1.49% is mathematically impossible to reproduce by random selection within the quark sector alone.

#### 3.2 Bootstrap Stability Analysis

We performed 50,000 bootstrap simulations per angle, varying all masses and angles within their PDG uncertainties (Gaussian sampling).

Table 2: Bootstrap results (50,000 simulations per angle).

Angle	Nominal deviation	5 <sup>th</sup> percentile	Status
$\theta_{12}$ CKM	0.75%	0.36%	Stable
$\theta_{23}$ CKM	0.15%	0.70%	Unstable ( $m_u$ 23%)
$\theta_{13}$ CKM	3.45%	0.71%	Stable
$\theta_{12}$ PMNS	0.11%	0.06%	Stable
$\theta_{13}$ PMNS	0.66%	0.30%	Stable
$\theta_{23}$ PMNS	3.81%	0.94%	Unstable

Four of six angles are stable under experimental uncertainties. The instability of  $\theta_{23}^{\text{CKM}}$  is driven entirely by the poorly known  $m_u$  (23% uncertainty). The instability of  $\theta_{23}^{\text{PMNS}}$  reflects the larger intrinsic deviation (3.81%) combined with the  $\pm 0.013$  uncertainty on  $\sin \theta_{23}$ .

#### 3.3 Look-Elsewhere Correction

The relations were found among 14,391 functional forms tested by the SAVEUR-AI code. The effective number of independent tests is estimated at  $\sim 60$  (accounting for correlations among functional forms). The Bonferroni-corrected significance is  $p < 0.001$  ( $3.3\sigma$ ).

## 4 Cross-Consistency Checks

Each quark mass appearing in two different formulas can be computed independently:

Table 3: Cross-consistency predictions for quark masses (in MeV).

Mass	Path 1	Path 2	Deviation
$m_s$	$m_b \sin^2 \theta_{13}^{\text{PMNS}} = 92.2$	$m_d / \sin^2 \theta_C = 93.4$	1.3%
$m_c$	$m_b \sin^2 \theta_{12}^{\text{PMNS}} = 1267$	$m_u / \sin^2 \theta_{23}^{\text{CKM}} = 1270$	0.2%
$m_b$	$m_c / \sin^2 \theta_{12}^{\text{PMNS}} = 4184$	$m_s / \sin^2 \theta_{13}^{\text{PMNS}} = 4180$	0.1%
$m_u$	$m_c \sin^2 \theta_{23}^{\text{CKM}} = 2.15$	$m_t \sin^2 \theta_{13}^{\text{CKM}} = 2.32$	7.3%

The two independent determinations of  $m_u$  are consistent at  $1.2\sigma$ , yielding a combined prediction:

$$\boxed{m_u = 2.23 \pm 0.10 \text{ MeV}} \quad (2)$$

The predicted ratio  $m_u/m_d = 0.478$  compares favorably with the PDG value  $0.474 \pm 0.056$  (deviation 0.8%).

## 5 Bosonic Satellite Relations

Independent of the mixing angle relations, we find simple numerical relations among the electroweak boson masses:

Table 4: Relations among boson masses.

Relation	Prediction	Measurement	Status
$m_t/m_W = 15/7$	2.14286	$2.1459 \pm 0.0038$	Compatible ( $0.8\sigma$ )
$m_t m_W = \frac{5}{3} m_Z^2$	13859	$13865 \pm 20$	Compatible ( $0.3\sigma$ )
$m_H/m_Z + \alpha = m_t/m_H$	1.3794	1.3780	0.17%
$m_Z/m_W = 3/\sqrt{7}$	1.13389	$1.13452 \pm 0.00017$	Ruled out ( $3.7\sigma$ )

The first three relations are mutually consistent. The relation  $m_Z/m_W = 3/\sqrt{7}$  is excluded by current data and is included only for completeness.

## 6 Connection Between the Two Sectors

The bosonic relations imply  $m_Z/m_t = \sqrt{7}/5 = 0.52915$ . Taking the square root:

$$\sqrt{\frac{m_Z}{m_t}} = \sqrt{\frac{\sqrt{7}}{5}} = 0.72707 \quad (3)$$

This is precisely the value used for  $\sin \theta_{23}^{\text{PMNS}}$  in the main relation. The two sectors—mixing angles and boson masses—are connected through  $m_Z$  and  $m_t$  with a coherence of 0.05%.

From  $m_H/m_Z + \alpha = m_t/m_H$  and  $\sin^2 \theta_{23}^{\text{PMNS}} = m_Z/m_t$ , one can derive  $m_H \approx 125.0$  GeV, consistent with the measured value of 125.2 GeV (0.16% deviation). The three laws—angles, bosons, and the  $H/Z + \alpha$  relation—are not independent.

## 7 Underlying Arithmetic Structure

The mass ratios used in the relations are simple fractions whose prime factors belong to the set  $\{2, 3, 5, 7, 17\}$ :

$$\begin{aligned} m_d/m_s &= 1/20 = 1/(2^2 \times 5) \\ m_s/m_b &= 1/45 = 1/(3^2 \times 5) \\ m_c/m_b &= 3/10 = 3/(2 \times 5) \\ m_Z/m_t &= 9/17 = 3^2/17 \\ m_t/m_W &= 15/7 = (3 \times 5)/7 \\ m_H/m_W &= 14/9 = (2 \times 7)/3^2 \end{aligned}$$

The number 5 appears in most denominators, suggesting a connection with the golden ratio  $\phi = (1 + \sqrt{5})/2$  and the modular symmetry group  $\Gamma(5)$ .

The logarithmic differences  $\Delta_{ij} = |\ln(m_i) - \ln(m_j)|$  for the six pairs are approximately integer multiples of a fundamental unit  $\delta \approx 0.32$ , which itself satisfies  $\delta \approx \frac{2}{3} \ln \phi$  (deviation 0.6%).

## 8 Falsifiable Predictions

1.  $m_u = 2.23 \pm 0.10$  MeV. Testable by future lattice QCD calculations. Current precision ( $\pm 0.49$  MeV) is insufficient; a factor 5 improvement is required.
2.  $\sin \theta_{23}^{\text{PMNS}} = 0.727$ . The combined T2K + NO $\nu$ A update expected in 2026–2027 will reduce the uncertainty to approximately  $\pm 0.008$ . If the central value remains at 0.756, our prediction is in tension at  $3.6\sigma$ . If it shifts toward 0.740, the law survives.
3. **Sterile neutrino angle**  $\sim 43^\circ$ . If a sterile neutrino exists, its mixing angle should be approximately  $\sqrt{m_u/m_d} \approx \sqrt{m_W/m_t} \approx 0.68$ .
4.  $\Sigma m_\nu = 0.0593$  eV (normal hierarchy), from the logarithmic relation of Lichani (Article 1), testable by CMB-S4 and DUNE.

## 9 Theoretical Discussion

The  $\sqrt{m_1/m_2}$  form suggests that masses are squares of more fundamental quantities. In the modular symmetry framework (Feruglio 2017, Criado-Feruglio 2018, Novichkov et al. 2019), Yukawa couplings are modular forms  $f(\tau)$ . Masses are proportional to  $|f(\tau)|^2$ , and mixing angles are given by ratios of modular forms:

$$\sin \theta_{ij} \propto \frac{|f_i(\tau)|}{|f_j(\tau)|} = \sqrt{\frac{|f_i|^2}{|f_j|^2}} = \sqrt{\frac{m_i}{m_j}} \quad (4)$$

The  $\sqrt{m_1/m_2}$  form emerges naturally if masses are squares of modular forms. The prime factor set  $\{2, 3, 5, 7, 17\}$  corresponds to dimensions of modular form spaces for congruence subgroups. The modular group  $\Gamma(5)$  is particularly promising: its cusp structure, the appearance of  $\phi = (1 + \sqrt{5})/2$  in the associated number field  $\mathbb{Q}(\zeta_5)$ , and the pervasive factor 5 in the mass ratios all point toward a  $\Gamma(5)$  modular flavor symmetry.

No existing modular flavor model predicts  $\sin \theta = \sqrt{m_1/m_2}$ . Our law provides a new, strong constraint on such models.

## 10 Limitations

We acknowledge the following limitations:

1. **No derived Lagrangian.** The relations are empirical; no variational principle is known.
2.  $\theta_{23}^{\text{PMNS}}$  **is unstable.** The 3.81% deviation and bootstrap failure make this the weakest link.
3.  $\theta_{23}^{\text{CKM}}$  **is unstable.** This is driven entirely by the 23% uncertainty on  $m_u$ .
4. **Neutrino masses are absent** from the PMNS formulas. The relations use charged lepton-sector quarks ( $c, b, s$ ) and the  $Z$  boson, not the neutrino masses themselves.
5.  $m_Z/m_W = 3/\sqrt{7}$  **is ruled out** at  $3.7\sigma$  by current data. The other bosonic relations survive.
6. **The look-elsewhere correction** depends on an estimate of the effective number of independent tests ( $\sim 60$ ). A fully rigorous determination would require simulating the entire SAVEUR-AI search pipeline.

## 11 Conclusion

We have identified a systematic empirical structure in the Standard Model flavor sector: all six mixing angles are given by  $\sin \theta = \sqrt{m_{\text{light}}/m_{\text{heavy}}}$  with a mean deviation of 1.49%. The statistical significance ( $p < 0.001$  after look-elsewhere correction) excludes a trivial numerical coincidence. The structure is internally consistent across four cross-checks and connects naturally to bosonic mass relations through  $m_Z$  and  $m_t$ . Falsifiable predictions are provided for  $m_u$  and  $\theta_{23}^{\text{PMNS}}$ . The modular symmetry framework offers a promising theoretical foundation, though the explicit model remains to be constructed.

## References

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