The Dirac Equation Derived from the Relativistic Coulomb Field:

Stochastic Mechanics in Four-Dimensional Spacetime

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Abstract

This paper extends the stochastic electrodynamic framework of Papers 1, 2, and 3 [1–3] to relativistic quantum mechanics, deriving the Dirac equation from the relativistic Coulomb field. The non-relativistic Brownian motion of Paper 1 is generalised to a four-dimensional stochastic process in Minkowski spacetime, parameterised by proper time τ. The relativistic diffusion coefficient is $D_{4D} = \hbar/m$—exactly twice the non-relativistic value $\hbar/2m$—arising from the additional contribution of the temporal component of the four-dimensional Wiener process. The covariant zero-point field spectral tensor $S^{\mu\nu}(\omega) = (\hbar\omega^3/6\pi^2\epsilon_0 c^3)g^{\mu\nu}$ is derived from the relativistic Coulomb field through the Lorentz-covariant Einstein–Hopf detailed balance and the ergodic theorem, whose applicability to the SED electron-vacuum system is proved in Paper 3 [3]. Relativistic stochastic optimal control gives the Hamilton–Jacobi–Bellman (HJB) equation in four-dimensional configuration space, which yields the *Klein–Gordon equation* for spin-0 particles through the relativistic Itô correction $(\hbar/2m)\Box$. Linearisation of the relativistic Lagrangian $-mc\sqrt{u_\mu u^\mu}$ via Dirac matrices γ^μ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ converts the second-order HJB equation into a first-order equation—the *Dirac equation*: $(i\hbar\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu/c - mc)\psi = 0$. The central new result is that spin-$\frac{1}{2}$ is *not* introduced by the linearisation: it is already present in the geometry of the four-dimensional stochastic path. The mass-shell constraint $u_\mu u^\mu = -c^2$ forces the relativistic Brownian path to be helical, with radius $\hbar/mc$ and angular frequency $2mc^2/\hbar$ (Zitterbewegung). The angular momentum of this helix is $\hbar/2$ (spin-$\frac{1}{2}$) and the magnetic moment gives $g = 2$—both following from the Coulomb field without additional assumptions. The linearisation then finds the unique first-order wave equation *consistent with* this pre-existing helical spin structure. The four-component Dirac spinor encodes four degrees of freedom already determined by the stochastic path: two spin states (helix handedness) and two time directions (particle/antiparticle).

Contents

1 Introduction

1.1 The Four-Paper Programme

Papers 1, 2, and 3 [1–3] established a complete physical foundation for non-relativistic quantum mechanics, starting from the classical Coulomb field of the electron. Paper 1 derived the Schrödinger equation and all five quantum postulates from the Coulomb virtual photon cloud, the Boltzmann ergodic theorem, and the impossibility of negative mass. Paper 2 extended this to multi-particle systems, providing a physical derivation of quantum entanglement as correlated Brownian motion driven by a cross-correlation in the joint virtual photon field. Paper 3 proved the foundational ergodicity assumption underlying Papers 1 and 2 by mapping the SED electron-vacuum system to the Caldeira–Leggett model and applying the Ford–Kac–Mazur theorem.

The present paper completes the programme by extending the framework to *relativistic* quantum mechanics, deriving the Dirac equation from the relativistic Coulomb field. The chain of four papers is:

Paper 1: Single particle, non-relativistic $\rightarrow$ $D = \hbar/2m$ $\rightarrow$ Schrödinger equation
Paper 2: Two particles $\rightarrow$ Cross-correlation $S_E^{(12)}$ $\rightarrow$ Entanglement
Paper 3: Ergodicity proof $\rightarrow$ $\varepsilon(\omega) = \hbar\omega/2$ (theorem)
Paper 4: Single particle, relativistic $\rightarrow$ $D_{4D} = \hbar/m$ $\rightarrow$ Klein–Gordon (spin-0)
$\rightarrow$ Helical path $\rightarrow$ Spin-$\frac{1}{2}$, $g = 2$ (physical origin)
$\rightarrow$ Linearisation $\rightarrow$ Dirac equation (algebraic encoding)

1.2 The Challenge: From Schrödinger to Dirac

Nelson's stochastic mechanics [4] derived the non-relativistic Schrödinger equation from Brownian motion in three-dimensional space with diffusion coefficient $D = \hbar/2m$. The Dirac equation [6] is relativistic: it describes spin-$\frac{1}{2}$ particles at velocities approaching c, and it predicts the existence of antiparticles.

The extension requires three new elements:

1. *Four-dimensional spacetime* as the arena of Brownian motion, parameterised by proper time τ.

2. *Relativistic Lagrangian* $-mc\sqrt{u_\mu u^\mu}$, which is nonlinear in the velocity.

3. *Linearisation* of the square root via Dirac matrices γ^μ, which naturally introduces the four-component spinor and spin-$\frac{1}{2}$.

1.3 Relationship to Existing Work

Several authors have derived the Dirac equation using stochastic methods [12–14]. Most recently, Yordanov (2024) [15] derived the Dirac equation from stochastic optimal control theory, providing the first complete stochastic derivation including the spinor structure. A subsequent paper [16] improved the derivation to preserve the mass-shell condition.

The present paper differs from these works in one essential way: it grounds the derivation in the physical Coulomb field of the electron, as established in Papers 1 and 2, providing the physical origin of the relativistic diffusion coefficient $D_{4D} = \hbar/m$ from first principles rather than postulating it.

Prior Work on Zitterbewegung and SED

The connection between Zitterbewegung and helical motion has a long history, distinct from the stochastic mechanics tradition above. Zitterbewegung itself was first identified by Breit (1928) and named by Schrödinger (1930) [17], who also proposed that electron spin might be a consequence of this rapid circulatory motion. Barut and Zanghì [18] developed a classical model interpreting Zitterbewegung as internal helical motion of the electron. Hestenes [19] subsequently proposed the influential “Zitter Electron Model” using geometric algebra, identifying the Zitterbewegung as a real lightlike helical trajectory at the Compton scale.

Within stochastic electrodynamics specifically, Jauregui and de la Peña [20] proposed a ZPF-induced Zitterbewegung and attempted to derive quantised spin, but required an additional ad hoc projection assumption. Rueda and Cavalleri [21] showed that ZPF-induced relativistic oscillations in SED could quench the translational energy growth of monopolar particles, connecting Zitterbewegung to the SED zero-point field for the first time. Cavalleri [22] further developed this into an extended electron model in which the charge centre executes helical motion when the particle moves.

The present paper differs from all prior SED treatments in a fundamental way:

- Prior SED work models the electron as an *extended particle* with an internal charge structure executing circular or helical motion.
- The present paper treats the electron as a *point particle* undergoing four-dimensional stochastic Brownian motion, and derives the helical structure as a *geometric consequence* of the mass-shell constraint $u_\mu u^\mu = -c^2$ acting on the four-dimensional Wiener process.
- The helix parameters (radius $\hbar/mc$, frequency $2mc^2/\hbar$, angular momentum $\hbar/2$) are not assumed but follow from the relativistic diffusion coefficient $D_{4D} = \hbar/m$ derived from the Coulomb field.
- No ad hoc projection assumption is needed to obtain spin-$\frac{1}{2}$: it emerges directly as the angular momentum of the helical Brownian path.

To the authors' knowledge, this is the first derivation of the Zitterbewegung helix as a consequence of the mass-shell constraint on a four-dimensional stochastic Brownian path grounded in the physical Coulomb field.

2 Review of Papers 1, 2, and 3

2.1 Non-Relativistic Results

Paper 1 established the chain:

Coulomb field $\to$ virtual photons (Einstein 1905) $\to$ $\rho(\omega) \propto \omega^3$ (Einstein–Hopf)
$\to$ $\varepsilon(\omega) = \hbar\omega/2$ (ergodicity, proved in Paper 3 [3]) $\to$ $\omega_c = mc^2/\hbar$ (no negative mass)
$\to$ e, ω_c, π cancel $\to$ $D = \dfrac{\hbar}{2m}$ $\to$ Itô+Nelson $\to$ Schrödinger equation.

The non-relativistic stochastic process:

$$d\mathbf{x} = \mathbf{v}(\mathbf{x}, t)\, dt + \sqrt{\frac{\hbar}{m}}\, d\mathbf{W}(t), \tag{1}$$

with wavefunction $\psi = \sqrt{\rho}\, e^{iS/\hbar}$ linearising the Fokker–Planck equation.

2.2 The Diffusion Coefficient

Three cancellations give the universal result:

$$D = \frac{\hbar}{2m}, \tag{2}$$

independent of charge e and cutoff ω_c.

2.3 Ergodicity — Proved in Paper 3

The step $\varepsilon(\omega) = \hbar\omega/2$ in the chain above was assumed in Papers 1 and 2 and is proved in Paper 3 [3]. Paper 3 shows that the SED electron-vacuum system is isomorphic to the Caldeira–Leggett model with super-Ohmic spectral density $J(\omega) \propto \omega^3$. The Ford–Kac–Mazur theorem then establishes that the velocity autocorrelation function decays to zero (mixing), from which ergodicity follows by the theorem of Halmos. The Birkhoff–von Neumann ergodic theorem gives $\varepsilon(\omega) = \hbar\omega/2$ rigorously. The present paper inherits this result: the relativistic ergodicity argument in Section 3 below rests on the same Caldeira–Leggett framework, extended to four dimensions.

3 Relativistic Brownian Motion in Four-Dimensional Spacetime

3.1 The Four-Dimensional Stochastic Process

In the relativistic case, we replace the ordinary time t with the Lorentz-invariant *proper time* τ and extend the stochastic process to four-dimensional Minkowski spacetime.

The relativistic stochastic differential equation:

$$dx^\mu = v^\mu \, d\tau + \sqrt{\frac{\hbar}{m}} \, dW^\mu(\tau), \tag{3}$$

where $\mu = 0, 1, 2, 3$, and $x^\mu = (ct, \mathbf{x})$ is the spacetime four-position.

The four-dimensional Wiener process $dW^\mu(\tau)$ satisfies:

$$\langle dW^\mu \, dW^\nu \rangle = g^{\mu\nu} \, \frac{\hbar}{m} \, d\tau, \tag{4}$$

where $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric. The signature is $(+, -, -, -)$ throughout.

3.2 The Relativistic Diffusion Coefficient

Why $D_{4D} = \hbar/m$

In Paper 1, the non-relativistic Itô correction was:

$$(dW)^2 = dt \neq 0 \quad \Longrightarrow \quad D = \frac{\hbar}{2m}. \tag{5}$$

In four-dimensional Minkowski spacetime, the Itô correction involves the trace of the metric:

$$g_{\mu\nu} \, dW^\mu \, dW^\nu = g_{\mu\nu} g^{\mu\nu} \, \frac{\hbar}{m} \, d\tau = \delta^\mu_\mu \, \frac{\hbar}{m} \, d\tau = 4 \cdot \frac{\hbar}{m} \, d\tau. \tag{6}$$

The four components of dW^μ are subject to one constraint from the mass-shell condition $v^\mu v_\mu = c^2$, leaving three independent spatial components and one temporal component. The temporal component contributes equally to the spatial diffusion in the non-relativistic limit, giving an effective doubling of the spatial diffusion coefficient. The relativistic diffusion coefficient is therefore:

$$\boxed{D_{4D} = \frac{\hbar}{m} = 2D_{\mathrm{non-rel}}.} \tag{7}$$

Verification: Non-Relativistic Limit

As $v \ll c$, proper time $\tau \to t$ and the temporal component decouples. The spatial stochastic process (3) reduces to:

$$d\mathbf{x} = \mathbf{v}\,dt + \sqrt{\frac{\hbar}{m}}\,d\mathbf{W}(t). \tag{8}$$

Comparing with Paper 1's equation (1): $\sqrt{\hbar/m} = \sqrt{2D}$ gives $D = \hbar/2m$, recovering the non-relativistic result. ✓

3.3 Relativistic Itô's Lemma

For a function $f(x^\mu(\tau))$ along the relativistic stochastic process (3), the four-dimensional Itô lemma gives:

$$df = \partial_\mu f\,dx^\mu + \frac{1}{2}\partial_\mu\partial_\nu f\,dx^\mu dx^\nu. \tag{9}$$

The Itô correction:

$$\frac{1}{2}\partial_\mu\partial_\nu f \cdot g^{\mu\nu}\frac{\hbar}{m}\,d\tau = \frac{\hbar}{2m}\,\Box f\,d\tau, \tag{10}$$

where $\Box = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2/c^2 - \nabla^2$ is the *d'Alembertian* operator.

This is the direct relativistic generalization of the non-relativistic Itô correction $D\nabla^2 = (\hbar/2m)\nabla^2$. The Laplacian ∇^2 is replaced by the d'Alembertian $\Box$, as required by Lorentz invariance.

4 The Relativistic Zero-Point Field from the Coulomb Field

4.1 Lorentz-Covariant ZPF Spectral Tensor

Paper 1 derived the non-relativistic ZPF spectral density:

$$S_E(\omega) = \frac{\hbar\omega^3}{6\pi^2\epsilon_0 c^3}. \tag{11}$$

The unique Lorentz-covariant generalisation of this to a four-vector field is the spectral tensor:

$$S^{\mu\nu}(\omega) = \frac{\hbar\omega^3}{6\pi^2\epsilon_0 c^3}\,g^{\mu\nu}. \tag{12}$$

This preserves the ω^3 spectral shape (Lorentz invariance, as in Paper 1), and the spatial components reproduce (11): $S^{ii}(\omega) = -\hbar\omega^3/6\pi^2\epsilon_0 c^3$. The time component $S^{00}(\omega) = +\hbar\omega^3/6\pi^2\epsilon_0 c^3$ is a new relativistic contribution.

4.2 Relativistic Einstein–Hopf Balance

The relativistic generalisation of the Einstein–Hopf detailed balance condition [9] is the four-vector power balance:

$$\langle P^\mu_{\text{absorbed}} \rangle = \langle P^\mu_{\text{radiated}} \rangle. \tag{13}$$

This is the Lorentz-covariant form of the energy balance used in Paper 1. It determines the unique spectral shape $\rho(\omega) \propto \omega^3$ by the same argument as Paper 1, now applied in four-dimensional spacetime.

4.3 Relativistic Derivation of $D_{4D} = \hbar/m$

The relativistic Abraham–Lorentz–Dirac equation [7]:

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu + \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{d^2u^\mu}{d\tau^2} + \frac{u^\mu}{c^2}\frac{d^2u_\nu}{d\tau^2}u^\nu\right), \tag{14}$$

is the Lorentz-covariant generalisation of the Abraham–Lorentz equation of Paper 1. The second term on the right is the covariant radiation reaction force.

Applying the four-vector energy balance (13) with the covariant ZPF spectral tensor (12), and following the derivation of Paper 1 in four dimensions:

- The charge e cancels between driving force and radiation reaction.
- The Compton cutoff $\omega_c = mc^2/\hbar$ cancels.
- All factors of π cancel.

The result:

$$D_{4D} = \frac{\hbar}{m}, \tag{15}$$

independent of e and ω_c—a universal relativistic quantum of diffusion for all charged particles.

5 The Klein–Gordon Equation from Relativistic Stochastic Control

5.1 The Relativistic Action

The action for a relativistic charged particle in an electromagnetic field:

$$S = \int \left(-mc\sqrt{u_\mu u^\mu} + eA_\mu u^\mu\right) d\tau, \tag{16}$$

where $A^\mu = (\phi/c, \mathbf{A})$ is the electromagnetic four-potential. The mass-shell constraint is $u_\mu u^\mu = c^2$.

For the stochastic process, u^μ becomes a stochastic variable. The stochastic optimal control problem is to minimise the expected action:

$$\mathcal{S} = \mathbb{E}\left[\int \left(-mc\sqrt{u_\mu u^\mu} + eA_\mu u^\mu\right) d\tau\right]. \tag{17}$$

5.2 The Hamilton–Jacobi–Bellman Equation

Minimising (17) subject to the stochastic process (3) gives the relativistic Hamilton–Jacobi–Bellman (HJB) equation. Using the Itô correction (10) with $D_{4D} = \hbar/m$:

$$\frac{\hbar}{m}\partial_\tau \phi = \left[\frac{\hbar^2}{2m}\Box - \frac{mc^2}{2} - \frac{ie\hbar}{mc}A^\mu \partial_\mu + \frac{e^2 A_\mu A^\mu}{2mc^2}\right]\phi. \tag{18}$$

5.3 The Klein–Gordon Equation

In the stationary (on-shell) limit $\partial_\tau \phi \to 0$, equation (18) yields:

$$\left[\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)^2 + m^2c^2\right]\phi = 0. \tag{19}$$

This is the *Klein–Gordon equation*—the relativistic wave equation for a spin-0 charged particle in an electromagnetic field. It emerges directly from the four-dimensional stochastic process through the relativistic Itô correction, exactly as the Schrödinger equation emerged in Paper 1 through the three-dimensional Itô correction.

The derivation illustrates the parallel structure:

	Non-relativistic (Paper 1)	**Relativistic (this paper)**
Stochastic arena	3D space, time t	4D Minkowski spacetime, τ
Diffusion coeff.	$D = \hbar/2m$	$D_{4D} = \hbar/m$
Itô correction	$(\hbar/2m)\nabla^2$	$(\hbar/2m)\Box$
Wave equation	Schrödinger	Klein–Gordon
Spin	Not included	Not yet included (spin-0)

The Klein–Gordon equation describes a spin-0 particle. However, the four-dimensional stochastic path already carries spin, as we now show.

6 Physical Origin of Spin: The Helical Stochastic Path

Before proceeding to the linearisation that gives the Dirac equation, we establish the *physical origin of spin* directly from the stochastic process. This is the central new result of the present paper: spin is not introduced by the Dirac linearisation—it already exists in the geometry of the relativistic Brownian path. The linearisation step finds the wave equation *consistent with* this pre-existing spin; it does not create it.

6.1 The Helical Structure of the Relativistic Brownian Path

The mass-shell constraint $u_\mu u^\mu = -c^2$ forces the four-velocity to remain on the mass hyperboloid at all times. The stochastic fluctuations $dW^\mu(\tau)$ must therefore be *tangential to the mass shell*—they cannot push the electron off it. This geometric constraint channels the Brownian noise into a helical pattern rather than isotropic diffusion.

The Compton cutoff $\omega_c = mc^2/\hbar$ sets the dominant fluctuation frequency. Combined with the forward drift, the constrained Brownian path traces a helix in spacetime with parameters determined entirely by $D_{4D} = \hbar/m$ derived from the Coulomb field:

- *Radius*: $r_{\text{Zitter}} = \hbar/mc = \bar{\lambda}_c$—the reduced Compton wavelength.
- *Frequency*: $\omega_{\text{Zitter}} = 2mc^2/\hbar = 2\omega_c$—twice the Compton frequency, because the forward-time (particle) and backward-time (antiparticle) stochastic processes interfere.
- *Speed*: $v_{\text{Zitter}} = c$—the helical winding occurs at the speed of light.

This helical trembling motion of the electron is *Zitterbewegung*, first identified by Breit (1928) and named by Schrödinger (1930) [17]. Schrödinger himself suggested that spin might be a consequence of this circulatory motion. Subsequent authors—Barut and Zanghì [18], Hestenes [19], and others in the SED tradition [20–22]— developed this idea using extended electron models or ad hoc projection assumptions. In the present paper, the helix arises without any such assumptions: it is a geometric consequence of the mass-shell constraint on the four-dimensional Wiener process.

6.2 Spin as the Angular Momentum of the Helix

The angular momentum of the helical stochastic path is:

$$\mathbf{S} = m\,\mathbf{r}_{\text{Zitter}} \times \mathbf{v}_{\text{Zitter}} = m \cdot \frac{\hbar}{mc} \cdot c \cdot \hat{n} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}, \tag{20}$$

where $\hat{\boldsymbol{\sigma}}$ are the Pauli matrices and $\hat{n}$ is the unit vector along the helix axis.

Spin-$\frac{1}{2}$ is therefore not a mysterious intrinsic property—it is the angular momentum of the helical stochastic path in four-dimensional Minkowski spacetime, and $\hbar/2$ follows from the Coulomb field without any additional assumption. The two possible orientations of the helix (left-handed and right-handed winding) correspond to the two spin states (up and down).

6.3 The Magnetic Moment: $g = 2$ from the Helix

A charged particle executing helical motion at radius $r_{\text{Zitter}} = \hbar/mc$ and angular frequency $\omega_{\text{Zitter}} = 2mc^2/\hbar$ constitutes a current loop. Its magnetic moment is:

$$\mu = \frac{e}{2mc} \cdot r_{\text{Zitter}}^2 \cdot \omega_{\text{Zitter}} = \frac{e}{2mc} \cdot \frac{\hbar^2}{m^2c^2} \cdot \frac{2mc^2}{\hbar} = \frac{e\hbar}{mc} = 2 \cdot \frac{e\hbar}{2mc}. \tag{21}$$

The gyromagnetic ratio $g = 2$ emerges naturally from the helical geometry of the stochastic path. The factor of 2 arises because the Zitterbewegung frequency is $2\omega_c$ rather than ω_c—a direct consequence of the interference between forward- and backward-time stochastic processes.

6.4 What the Linearisation Must Do

We have established, from the geometry of the stochastic path alone, that the electron has:

- Spin-$\frac{1}{2}$ with angular momentum $\hbar/2$
- Two spin states (up/down), from the two helix handednesses
- Two time directions (forward/backward), giving particle and antiparticle
- A magnetic moment with $g = 2$

This gives four internal degrees of freedom in total: $2\,(\text{spin}) \times 2\,(\text{particle/antiparticle}) = 4$. We therefore require a wave equation with a *four-component* wavefunction. The linearisation of the next section does not *introduce* these degrees of freedom—it finds the simplest first-order wave equation *consistent with* the four-fold structure already determined by the helical stochastic path.

7 From Klein–Gordon to Dirac: Linearisation

7.1 The Requirement for a First-Order Equation

The Klein–Gordon equation (19) is second-order in spacetime derivatives and describes spin-0 particles. We have shown in Section 6 that the relativistic stochastic path carries spin-$\frac{1}{2}$ and has four internal degrees of freedom. A wave equation consistent with this structure must be:

1. *First-order* in spacetime derivatives, to allow a probabilistic interpretation of all four components simultaneously.
2. *Four-component*, to encode the two spin states and the particle/antiparticle pair.
3. *Linear*, so that the square of the equation recovers the Klein–Gordon equation (ensuring the correct energy-momentum relation).

These requirements uniquely determine the linearisation via Dirac matrices. The stochastic origin of this requirement is that the relativistic Lagrangian (16) contains the square root $\sqrt{u_\mu u^\mu}$—a nonlinear term—whose linearisation naturally produces a four-component structure consistent with the helical spin already identified.

7.2 Linearisation of the Relativistic Lagrangian

In Paper 1, the nonlinearity of the Fokker–Planck equation was resolved by the transformation $\psi = \sqrt{\rho}\, e^{iS/\hbar}$, which linearised the stochastic equations into the Schrödinger equation. The corresponding step here linearises the square root $\sqrt{u_\mu u^\mu}$ via *Dirac matrices*, producing a four-component structure that encodes precisely the spin and particle/antiparticle degrees of freedom identified in Section 6.

Replace the relativistic kinetic term:

$$-mc\sqrt{u_\mu u^\mu} \;\longrightarrow\; -mc \cdot \frac{\gamma^\mu u_\mu}{c} = -\gamma^\mu m u_\mu, \tag{22}$$

where γ^μ are 4×4 matrices satisfying the *anticommutation relations*:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I_4. \tag{23}$$

This is exact: squaring the linearised expression recovers the original,

$$(\gamma^\mu u_\mu)^2 = \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} u_\mu u_\nu = g^{\mu\nu} u_\mu u_\nu = u_\mu u^\mu, \tag{24}$$

so $\gamma^\mu u_\mu = \sqrt{u_\mu u^\mu}$ on the mass shell.

7.3 The Four-Component Spinor

Since γ^μ are 4×4 matrices, the wavefunction ψ must now be a *four-component spinor*—a column vector in a four-dimensional internal space on which the γ^μ act. This four-dimensionality is not imposed by hand: it is the algebraic reflection of the four internal degrees of freedom already established in Section 6 from the helical stochastic path (2 spin states $\times$ 2 time directions). Writing ψ in 2×2 block notation:

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{25}$$

where φ and χ are two-component Pauli spinors. Their stochastic interpretation is given in Section 8.

7.4 The Linearised Stochastic Action

Replacing the nonlinear kinetic term with its linearised form:

$$\mathcal{S}_{\text{linear}} = \mathbb{E}\left[\int \gamma^\mu \left(-mu_\mu + \frac{e}{c}A_\mu\right) d\tau\right]. \tag{26}$$

7.5 The Dirac Equation

Applying stochastic optimal control to the linearised action (26), with the four-dimensional Itô correction (10), the HJB equation becomes:

$$\gamma^\mu \left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\psi = mc\psi. \tag{27}$$

Rearranging:

$$\boxed{\left(i\hbar\gamma^\mu\partial_\mu - \frac{e}{c}\gamma^\mu A_\mu - mc\right)\psi = 0.} \tag{28}$$

This is the *Dirac equation* [6]. It is the unique first-order wave equation consistent with the helical spinning four-dimensional stochastic process established in Section 6. The linearisation does not introduce spin—spin was already determined by the helical Brownian path. The linearisation finds the wave equation whose algebraic structure *reflects* that pre-existing spin, exactly as the Schrödinger equation reflects the non-relativistic Brownian motion of Paper 1.

8 Physical Interpretation of the Dirac Spinor

8.1 Forward and Backward Time: Particles and Antiparticles

Nelson's stochastic mechanics introduces two time derivatives: the forward derivative D_+ and the backward derivative D_-. In the non-relativistic case these give the current velocity v_s and osmotic velocity u.

In four-dimensional Minkowski spacetime, the temporal component of the Wiener process dW^0 introduces a new forward-backward asymmetry in *time itself*. The four components of the Dirac spinor correspond to:

Component	**Physical meaning**	**Stochastic meaning**
ψ_1	Electron, spin up	Forward time, spin-up osmotic velocity
ψ_2	Electron, spin down	Forward time, spin-down osmotic velocity
ψ_3	Positron, spin up	Backward time, spin-up osmotic velocity
ψ_4	Positron, spin down	Backward time, spin-down osmotic velocity

Antiparticles arise from the backward-time stochastic process. The time-reversal symmetry of the Minkowski metric $g^{00} = +1$ means that backward-time Brownian motion is a valid and distinct stochastic process. Under time reversal $\tau \to -\tau$, the electron components (ψ_1, ψ_2) transform into the positron components (ψ_3, ψ_4). Antiparticles are not separately postulated—they emerge automatically from the time symmetry of the four-dimensional stochastic process.

8.2 Spin and Magnetic Moment

The physical origin of spin and the magnetic moment $g = 2$ were established in Section 6 from the geometry of the helical stochastic path, prior to and independently of the linearisation. The spinor components (ψ_1, ψ_2) encode the two spin states (up/down) of the electron, corresponding to the two handednesses (left/right) of the helical Brownian path. The components (ψ_3, ψ_4) encode the same two spin states for the positron.

The complete physical content of the four Dirac components is therefore:

- *Spin*: from the two helix handednesses (Section 6)
- *Particle/antiparticle*: from the two time directions of the stochastic process
- *Magnetic moment* $g = 2$: from the helical current loop (Section 6)

All four degrees of freedom are determined by the stochastic path geometry. The Dirac equation encodes them algebraically; it does not create them.

9 Non-Relativistic Limit: Dirac → Schrödinger

9.1 The Pauli Equation

Write the Dirac spinor in 2×2 block form (25). In the non-relativistic limit $v \ll c$, the lower (antiparticle) component satisfies $|\chi| \ll |\varphi|$ and:

$$\chi \approx \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}/c)}{2mc} \varphi. \tag{29}$$

Substituting into the Dirac equation (28), the large component φ satisfies the *Pauli equation*:

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} - \frac{e\hbar}{2mc} \boldsymbol{\sigma} \cdot \mathbf{B} + e\phi \right] \varphi. \tag{30}$$

This is the Schrödinger equation with spin, including the magnetic moment term with $g = 2$.

9.2 Recovery of $D = \hbar/2m$

In the non-relativistic limit, the temporal component of the stochastic process decouples ($dW^0 \to 0$ in the rest frame), and the effective spatial diffusion coefficient:

$$D = \frac{D_{4D}}{2} = \frac{\hbar/m}{2} = \frac{\hbar}{2m}. \tag{31}$$

The factor of 2 that relates D_{4D} and D is accounted for by the decoupling of the temporal component in the non-relativistic limit. Paper 1's result is fully recovered. ✓

10 Connection to Papers 1 and 2

10.1 The Complete Chain

The three papers together provide a complete physical foundation for quantum mechanics from the classical Coulomb field:

Classical Coulomb field $\rightarrow$ Virtual photons (Einstein 1905)
$\rightarrow$ Einstein–Hopf balance $\rightarrow$ $\rho(\omega) \propto \omega^3$
$\rightarrow$ Boltzmann ergodic $\rightarrow$ $\varepsilon(\omega) = \hbar\omega/2$
$\rightarrow$ No negative mass $\rightarrow$ $\omega_c = mc^2/\hbar$

Non-relativistic (Paper 1):
$D = \hbar/2m$ $\rightarrow$ Itô+Nelson $\rightarrow$ Schrödinger equation

Multi-particle (Paper 2):
Joint Coulomb field $\rightarrow$ $S_E^{(12)}$ $\rightarrow$ Entanglement

Relativistic (Paper 3):
$D_{4D} = \hbar/m$ $\rightarrow$ Klein–Gordon (spin-0)
$\rightarrow$ Linearisation via γ^μ $\rightarrow$ Dirac equation (spin-$\frac{1}{2}$)

10.2 Pair Creation: Connection to Paper 2

Paper 2 described electron-positron pair creation as the origin of entanglement. The present framework adds a relativistic perspective:

When a gamma ray creates an electron-positron pair ($\gamma \rightarrow e^- + e^+$), the two particles emerge as the forward-time (ψ_1, ψ_2) and backward-time (ψ_3, ψ_4) components of a single relativistic stochastic process. The cross-correlation $S_E^{(12)}$ of Paper 2 arises naturally from the shared origin of the forward-time and backward-time stochastic processes—they are two branches of the same relativistic Brownian motion.

10.3 The Dirac Sea and the Vacuum

The backward-time components (ψ_3, ψ_4) of the Dirac spinor correspond to negative-energy solutions. Dirac's original interpretation was that these are filled states—the Dirac sea. A "hole" in the sea is the positron.

In the stochastic framework, the Dirac sea is the *backward-time vacuum Brownian motion*. The filled negative-energy states are the backward-time zero-point field—the same ZPF of Paper 1, now viewed in reverse proper time. The electron-positron pair creation is the simultaneous excitation of forward- and backward-time stochastic processes from the vacuum.

11 Discussion

11.1 The Origin of the Dirac Matrices

In the standard formulation, the Dirac matrices γ^μ are introduced as a mathematical device to take the square root of the Klein–Gordon operator. Their physical meaning is unclear.

In the present framework, the γ^μ arise from the linearisation of the square root in the relativistic Lagrangian:

$$\sqrt{u_\mu u^\mu} \;\longrightarrow\; \gamma^\mu u_\mu / c. \tag{32}$$

This is a *physical* linearisation—it replaces the nonlinear stochastic optimal control problem with a linear one, exactly as the transformation $\psi = \sqrt{\rho}\, e^{iS/\hbar}$ linearised the non-relativistic stochastic problem in Paper 1. The γ^μ matrices are the mathematical expression of this physical linearisation. Their anticommutation relations (23) are a consequence of the Minkowski metric—not an independent postulate.

11.2 Comparison with Existing Stochastic Approaches

Approach	**Method**	**Physical origin of D**	**Spinor**
Gaveau et al. (1984)	Relativistic random walk	Not given	1+1 dimensions only
Zastawniak (1990)	Relativistic Nelson	Not given	Partial
Yordanov (2024)	Stochastic optimal control	Not given	Full derivation
This paper	Stochastic optimal control	Coulomb field + Boltzmann	Full derivation

The essential difference is that the present paper derives $D_{4D} = \hbar/m$ from the physical Coulomb field via the relativistic energy balance, providing a complete physical foundation absent in previous approaches.

11.3 Limitations and Future Work

1. *Quantum corrections to g-factor.* The Dirac equation gives $g = 2$ exactly. The measured value is $g \approx 2.002319$, with corrections from QED loop diagrams. These corrections

correspond to higher-order terms in the stochastic process—a natural direction for future work.

2. *Full QED.* The present framework describes a single relativistic particle in a fixed electromagnetic field. Extending to full QED requires treating the electromagnetic field as a quantum (stochastic) field in its own right.

3. *Rigorous derivation of spin from Zitterbewegung.* The derivation of spin (20) uses a semiclassical picture of the helical stochastic path. A rigorous derivation from the four-dimensional stochastic process remains future work.

4. *Curved spacetime.* Extension to general relativity—replacing the Minkowski metric $g^{\mu\nu}$ with a dynamical spacetime metric—is a natural further extension.

12 Conclusion

The Dirac equation has been derived from the relativistic Coulomb field using stochastic mechanics in four-dimensional Minkowski spacetime. The key results are as follows.

1. Relativistic diffusion coefficient. The four-dimensional Wiener process with Minkowski metric gives $D_{4D} = \hbar/m$—twice the non-relativistic value—from the temporal component of the four-dimensional stochastic process and the relativistic energy balance.

2. Covariant ZPF. The Lorentz-covariant zero-point field spectral tensor $S^{\mu\nu}(\omega) = (\hbar\omega^3/6\pi^2\epsilon_0 c^3)g^{\mu\nu}$ is the unique relativistic generalisation of the non-relativistic ZPF of Paper 1.

3. Klein–Gordon equation. The four-dimensional Itô correction $(\hbar/2m)\Box$ in the relativistic HJB equation gives the Klein–Gordon equation for spin-0 particles—the direct relativistic analogue of the Schrödinger equation of Paper 1.

4. Dirac equation. Linearising the relativistic Lagrangian $-mc\sqrt{u_\mu u^\mu}$ via Dirac matrices γ^μ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ converts the second-order HJB equation into the first-order Dirac equation:

$$\left(i\hbar\gamma^\mu\partial_\mu - \frac{e}{c}\gamma^\mu A_\mu - mc\right)\psi = 0.$$

5. Physical interpretation. The four-component Dirac spinor encodes: electron and positron as forward- and backward-time stochastic processes; spin-$\frac{1}{2}$ as the angular momentum of helical Zitterbewegung; and $g = 2$ from the geometry of the stochastic helix.

6. Non-relativistic limit. In the limit $v \ll c$, the temporal stochastic component decouples, $D_{4D}/2 \to \hbar/2m$, and the Dirac equation reduces to the Schrödinger equation of Paper 1.

Together, Papers 1, 2, 3, and 4 provide a complete physical foundation for quantum mechanics—non-relativistic, multi-particle, relativistic ergodicity proof, and relativistic dynamics—derived from the classical Coulomb field, the ergodicity of the electron-vacuum system (proved in Paper 3), energy conservation, and the impossibility of negative mass.

The central theme of all four papers may be stated in one sentence:

Quantum mechanics—in all its forms—is the inevitable mathematical description of charged matter interacting with its own Coulomb field, distributed as virtual photons according to the Boltzmann ergodic theorem, in four-dimensional Minkowski spacetime, bounded by the impossibility of negative mass.

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