Ergodicity of the Stochastic Electrodynamic Electron-Vacuum System: A Proof via the Caldeira–Leggett Model

Fusao Ishii

Tokyo Institute of Technology (Alumni)

Abstract

Papers 1 and 2 of this series derived the single-particle and two-particle Schrödinger equations from classical stochastic electrodynamics (SED), resting on the foundational assumption that the coupled electron-vacuum system is ergodic. The present paper converts that assumption into a theorem. We write the SED Hamiltonian explicitly and show that it is isomorphic to the Caldeira–Leggett model of a particle coupled to a harmonic oscillator bath, with spectral density $J(\omega) \propto \omega^3$ (super-Ohmic) and a physical ultraviolet cutoff at the Compton frequency $\omega_c = mc^2/\hbar$. With the cutoff, the system is finite-dimensional. We apply the Ford–Kac–Mazur theorem to establish that the velocity autocorrelation function of the electron decays to zero, proving that the system is *mixing*. Since mixing implies ergodicity, the stationary measure is unique and time averages equal phase-space averages almost everywhere (Birkhoff–von Neumann). As a corollary, the zero-point energy per field mode is rigorously derived as $\varepsilon(\omega) = \hbar\omega/2$, completing the foundation of Papers 1 and 2 without additional postulates. The remaining unproved assumptions of the three-paper programme—Nelson's stochastic mechanics, the fermionic antisymmetry condition, and the identification of the initial cross-correlation with the vacuum two-point function—are identified and discussed.

Contents

1 Introduction

1.1 The Foundational Gap in Papers 1 and 2

Paper 1 [1] derived the single-particle Schrödinger equation from the classical Coulomb field, the Boltzmann ergodic theorem, energy conservation, and Nelson's stochastic mechanics. Paper 2 [2] extended this framework to two-particle systems, providing a physical derivation of quantum entanglement. Both papers rested on the following foundational assumption, stated explicitly in Paper 2:

> *The coupled charged-particle/electromagnetic-vacuum system is ergodic and admits a stationary canonical invariant measure. Under this assumption, the Birkhoff–von Neumann ergodic theorem guarantees that time averages equal phase-space averages almost everywhere, from which the zero-point energy per mode $\varepsilon(\omega) = \hbar\omega/2$ follows.*

The present paper proves this assumption.

1.2 The Physical Picture

An electron in the electromagnetic vacuum continuously emits and absorbs virtual photons through its Coulomb field. At stationarity, emission and absorption balance mode by mode—detailed balance holds. This guarantees the existence of a stationary invariant measure. The question left open by Papers 1 and 2 is whether this stationary measure is *ergodic*: whether the electron-vacuum dynamics explores all accessible states, with no invariant subsets that trap trajectories.

We answer this question affirmatively by mapping the SED electron-vacuum system to the Caldeira–Leggett model [3] and applying the Ford–Kac–Mazur theorem [4] to establish mixing, from which ergodicity follows.

1.3 The Main Theorem

Theorem 1.1 (Ergodicity of the SED electron-vacuum system)**.** *Consider an electron of mass m and charge e coupled to the electromagnetic vacuum with spectral density*

$$J(\omega) = \frac{e^2\omega^3}{6\pi\epsilon_0 c^3}, \qquad 0 \le \omega \le \omega_c, \tag{1}$$

where $\omega_c = mc^2/\hbar$ is the Compton frequency. The coupled electron-vacuum system, described by the SED Hamiltonian (13)*, admits a unique stationary invariant measure μ. With respect to μ, the system is mixing and therefore ergodic: for any square-integrable observables f and g,*

$$\lim_{t\to\infty} \langle f(x(t))\, g(x(0))\rangle_\mu = \langle f\rangle_\mu \langle g\rangle_\mu. \tag{2}$$

Consequently, time averages equal phase-space averages μ-almost everywhere, and the zero-point energy per field mode satisfies $\varepsilon(\omega) = \hbar\omega/2$.

The proof occupies Sections 3 through 6.

1.4 Structure of the Paper

Section 2 reviews the Caldeira–Leggett model and the Ford–Kac–Mazur theorem. Section 3 writes the SED Hamiltonian explicitly. Section 4 proves the isomorphism between the SED system and the Caldeira–Leggett model. Section 5 proves mixing via Ford–Kac–Mazur. Section 6 derives ergodicity and the zero-point energy as corollaries. Section 7 discusses implications for Papers 1 and 2 and the remaining open assumptions.

2 Background: Caldeira–Leggett and Ford–Kac–Mazur

2.1 The Caldeira–Leggett Model

The Caldeira–Leggett (CL) model [3] describes a particle of mass m coupled to a bath of N independent harmonic oscillators:

$$H_{\mathrm{CL}} = \frac{p^2}{2m} + V(x) + \sum_{k=1}^{N} \left[\frac{\pi_k^2}{2m_k} + \frac{m_k\omega_k^2}{2} \left(q_k - \frac{c_k x}{m_k\omega_k^2} \right)^2 \right], \tag{3}$$

where (x, p) are the particle's position and momentum, (q_k, π_k) are the coordinate and momentum of the k-th oscillator, ω_k is its frequency, and c_k is its coupling to the particle. Expanding the squared term:

$$H_{\mathrm{CL}} = \frac{p^2}{2m} + V(x) + \sum_{k=1}^{N} \left[\frac{\pi_k^2}{2m_k} + \frac{m_k\omega_k^2 q_k^2}{2} - c_k x q_k + \frac{c_k^2 x^2}{2m_k\omega_k^2} \right]. \tag{4}$$

The last term, $c_k^2 x^2/2m_k\omega_k^2$, is the *counter-term*: it cancels the bath-induced renormalisation of the particle's potential, ensuring that $V(x)$ remains the true physical potential.

The equations of motion for the particle derived from (4) are:

$$m\ddot{x} = -V'(x) + \sum_{k=1}^{N} c_k \left(q_k - \frac{c_k x}{m_k\omega_k^2} \right). \tag{5}$$

Eliminating the bath coordinates (which evolve as driven harmonic oscillators), one obtains the generalised Langevin equation:

$$m\ddot{x}(t) = -V'(x(t)) - \int_0^t \gamma(t-s)\dot{x}(s)\,\mathrm{d}s + \xi(t), \tag{6}$$

where the memory kernel and noise are:

$$\gamma(t) = \frac{1}{\pi}\int_0^{\omega_c} \frac{J(\omega)}{\omega}\cos(\omega t)\,\mathrm{d}\omega, \tag{7}$$

$$\langle \xi(t)\rangle = 0, \qquad \langle \xi(t)\xi(s)\rangle = \gamma(t-s)\,k_B T. \tag{8}$$

The spectral density of the bath is defined by:

$$J(\omega) \equiv \frac{\pi}{2}\sum_{k=1}^{N} \frac{c_k^2}{m_k\omega_k}\,\delta(\omega-\omega_k). \tag{9}$$

In the continuum limit $N \to \infty$ with mode spacing $\delta\omega \to 0$, $J(\omega)$ becomes a smooth function characterising the bath.

2.2 The Ford–Kac–Mazur Theorem

Ford, Kac, and Mazur (1965) [4] proved the following theorem for the Caldeira–Leggett model:

Theorem 2.1 (Ford–Kac–Mazur [4])**.** *Let the spectral density* $J(\omega)$ *be absolutely continuous (no discrete components) on* $[0,\omega_c]$*. Then the velocity autocorrelation function of the particle satisfies:*

$$C(t) \equiv \langle v(t)v(0)\rangle_\mu \to 0 \quad \textit{as} \quad t\to\infty. \tag{10}$$

The decay of $C(t)$ to zero is the definition of *mixing* of the velocity observable. We show in Section 5 that this extends to full mixing of the system.

2.3 Mixing Implies Ergodicity

Theorem 2.2 (Halmos [5])**.** *A measure-preserving dynamical system that is mixing is ergodic.*

Proof. Let A be an invariant measurable set: $\phi^{-t}(A) = A$ for all t. By mixing applied to $f = g = \mathbf{1}_A$:

$$\mu(A\cap\phi^{-t}(A)) \to \mu(A)^2 \quad \text{as} \quad t\to\infty. \tag{11}$$

But since A is invariant, $A\cap\phi^{-t}(A) = A$, so $\mu(A) = \mu(A)^2$, giving $\mu(A)\in\{0,1\}$. Hence no non-trivial invariant sets exist and the system is ergodic. □

3 The SED Hamiltonian

3.1 Classical SED as a Hamiltonian System

In stochastic electrodynamics, the electromagnetic vacuum is treated as a classical stochastic field with zero-point spectral density [8, 11, 12]:

$$S_E(\omega) = \frac{\hbar\omega^3}{6\pi^2\epsilon_0 c^3}. \tag{12}$$

This field can be decomposed into independent plane-wave modes. Each mode $(\mathbf{k}, \lambda)$ is a classical harmonic oscillator with frequency $\omega_k = c|\mathbf{k}|$ and random initial phase.

The total SED Hamiltonian for an electron of mass m and charge e coupled to the vacuum field is:

$$H_{\text{SED}} = \frac{p^2}{2m} + V(x) + \sum_{\mathbf{k},\lambda} \left[\frac{\Pi_{\mathbf{k}\lambda}^2}{2} + \frac{\omega_k^2 Q_{\mathbf{k}\lambda}^2}{2} \right] - e\,\mathbf{x} \cdot \mathbf{E}_{\text{vac}}(\mathbf{x}, t), \tag{13}$$

where $(Q_{\mathbf{k}\lambda}, \Pi_{\mathbf{k}\lambda})$ are the normal-mode coordinates and momenta of the vacuum field, and $\mathbf{E}_{\text{vac}} = -\partial_t \mathbf{A}$ is expressed in terms of the mode coordinates.

3.2 Mode Decomposition

Expanding the vector potential in plane waves:

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \left[Q_{\mathbf{k}\lambda}(t)\, \hat{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \text{c.c.} \right], \tag{14}$$

where V is the quantisation volume, $\hat{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}$ are polarisation vectors, and $Q_{\mathbf{k}\lambda}$ are classical oscillator coordinates. In the dipole approximation ($e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \approx 1$, valid for $|\mathbf{x}| \ll c/\omega_k$), the coupling becomes:

$$-e\,\mathbf{x} \cdot \mathbf{E}_{\text{vac}} = \sum_{\mathbf{k},\lambda} c_{\mathbf{k}\lambda}\, x\, Q_{\mathbf{k}\lambda}, \tag{15}$$

with coupling constants:

$$c_{\mathbf{k}\lambda} = e\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\, (\hat{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda})_x. \tag{16}$$

3.3 The Counter-Term

To match the standard Caldeira–Leggett form (4), we add the counter-term:

$$H_{\text{CT}} = \sum_{\mathbf{k},\lambda} \frac{c_{\mathbf{k}\lambda}^2}{2\omega_k^2}\, x^2. \tag{17}$$

This term is physically required: without it, the bath oscillators shift the effective potential $V(x)$, renormalising the electron's rest energy. The counter-term cancels this renormalisation, ensuring

$V(x)$ remains the true Coulomb potential. The full SED Hamiltonian with counter-term is:

$$\tilde{H}_{\text{SED}} = \frac{p^2}{2m} + V(x) + \sum_{\mathbf{k},\lambda} \left[\frac{\Pi_{\mathbf{k}\lambda}^2}{2} + \frac{\omega_k^2 Q_{\mathbf{k}\lambda}^2}{2} - c_{\mathbf{k}\lambda}\, x\, Q_{\mathbf{k}\lambda} + \frac{c_{\mathbf{k}\lambda}^2}{2\omega_k^2}\, x^2 \right]. \tag{18}$$

3.4 The Compton Cutoff and Finite Dimensionality

We impose the ultraviolet cutoff at the Compton frequency:

$$\omega_k \le \omega_c = \frac{mc^2}{\hbar}. \tag{19}$$

This cutoff is not a mathematical regularisation but a physical necessity established in Paper 1 [1]: field modes above ω_c would contribute negative kinetic energy to the electron, violating energy conservation.

With the cutoff, the number of field modes is finite:

$$N = \sum_{\mathbf{k},\lambda:\, \omega_k \le \omega_c} 1 \sim \frac{V\omega_c^3}{3\pi^2 c^3} \quad \text{(per polarisation)}. \tag{20}$$

For any finite quantisation volume V, N is finite. The SED system is therefore a *finite-dimensional* Hamiltonian system with $2(N+1)$ degrees of freedom, to which the standard theory of ergodic Hamiltonian systems applies.

4 Isomorphism with the Caldeira–Leggett Model

Theorem 4.1 (SED–CL isomorphism)**.** *The SED Hamiltonian* (18) *with cutoff* (19) *is isomorphic to the Caldeira–Leggett Hamiltonian* (4) *with bath spectral density:*

$$J(\omega) = \frac{e^2\omega^3}{6\pi\epsilon_0 c^3}, \qquad 0 \le \omega \le \omega_c. \tag{21}$$

Proof. Label the modes $k = 1, \dots, N$ with frequencies ω_k. Set bath oscillator masses $m_k = 1$ for all k. The coupling constants are:

$$c_k = c_{\mathbf{k}\lambda} = e\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\, (\hat{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda})_x. \tag{22}$$

Then (18) matches (4) term by term with $m_k = 1$. It remains to verify that the spectral density (9) of these coupling constants reproduces (21).

Summing over polarisations and averaging over directions:

$$\begin{aligned} J(\omega) &= \frac{\pi}{2} \sum_{k:\,\omega_k=\omega} \frac{c_k^2}{\omega_k}\,\delta(\omega-\omega_k) \\ &= \frac{\pi}{2}\cdot\frac{V\omega^2}{3\pi^2c^3}\cdot 2\cdot\frac{e^2\hbar\omega}{2\epsilon_0 V}\cdot\frac{1}{\omega}\cdot\frac{1}{3} \\ &= \frac{e^2\omega^3}{6\pi\epsilon_0c^3}, \end{aligned} \tag{23}$$

where the factor $V\omega^2/3\pi^2c^3$ is the electromagnetic density of states per unit volume, the factor 2 counts polarisations, and the factor $1/3$ arises from the angular average $\langle(\hat{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda})_x^2\rangle = 1/3$. This is exactly (21). □

Remark 4.2. The spectral density (21) is *super-Ohmic*: $J(\omega) \propto \omega^3$ grows as the cube of frequency, in contrast to the Ohmic case $J(\omega) \propto \omega$ studied most extensively in the Caldeira–Leggett literature. The super-Ohmic case corresponds to a bath with weaker low-frequency dissipation and qualitatively different long-time behaviour. In particular, the memory kernel (7) decays faster for super-Ohmic baths, which strengthens the mixing result.

5 Proof of Mixing

5.1 The Stationary Measure

At zero temperature ($T = 0$), the SED system is driven by the zero-point field. The zero-temperature fluctuation-dissipation relation gives the noise correlator:

$$\langle\xi(t)\xi(s)\rangle = \frac{\hbar}{\pi}\int_0^{\omega_c} J(\omega)\cos(\omega(t-s))\,\mathrm{d}\omega. \tag{24}$$

This is the SED zero-point field correlation, consistent with (12). The stationary measure μ of the coupled system is the unique Gaussian measure on phase space with covariance determined by (24). Existence and uniqueness of μ follow from the positive definiteness of $J(\omega)$ on $[0, \omega_c]$ and the fact that the system is linearly damped (radiation reaction provides dissipation).

5.2 Absolute Continuity of the Spectral Density

Lemma 5.1. *The spectral density* $J(\omega) = e^2\omega^3/6\pi\epsilon_0c^3$ *is absolutely continuous on* $[0, \omega_c]$.

Proof. $J(\omega)$ is a smooth function on $[0, \omega_c]$ with no point masses. It is therefore absolutely continuous with respect to Lebesgue measure. □

This is the key condition required by the Ford–Kac–Mazur theorem. Physically, it means the bath has a continuum of frequencies with no isolated resonances that could trap energy indefinitely.

5.3 Decay of the Velocity Autocorrelation

Theorem 5.2 (Mixing of SED electron velocity)**.** *For the SED electron-vacuum system with spectral density* (21)*, the velocity autocorrelation function satisfies:*

$$C(t) = \langle v(t)v(0)\rangle_\mu \to 0 \quad \textit{as} \quad t \to \infty. \tag{25}$$

Proof. By the SED–CL isomorphism (Theorem 4.1), the SED system is a Caldeira–Leggett model with absolutely continuous spectral density (Lemma 5.1). By the Ford–Kac–Mazur theorem [4] (Theorem 2.1), $C(t) \to 0$ as $t \to \infty$.

For completeness we give the explicit form of $C(t)$. The velocity autocorrelation is given by the Laplace transform relation [4]:

$$\hat{C}(z) = \frac{k_B T/m}{z + \hat{\gamma}(z)/m}, \tag{26}$$

where $\hat{\gamma}(z) = \int_0^\infty \gamma(t)e^{-zt}\,\mathrm{d}t$ is the Laplace transform of the memory kernel. For the SED spectral density (21):

$$\hat{\gamma}(z) = \frac{1}{\pi}\int_0^{\omega_c} \frac{J(\omega)}{\omega} \cdot \frac{\omega}{\omega^2 + z^2}\,\mathrm{d}\omega = \frac{e^2}{6\pi^2\epsilon_0 c^3}\int_0^{\omega_c} \frac{\omega^3}{\omega^2 + z^2}\,\mathrm{d}\omega. \tag{27}$$

This integral is finite for all $z > 0$ due to the cutoff ω_c. The denominator $z + \hat{\gamma}(z)/m$ has no zeros on the positive real axis (it is strictly positive), so $\hat{C}(z)$ is analytic in the right half plane and $C(t) \to 0$ by the Riemann–Lebesgue lemma applied to the inverse Laplace transform. □

Remark 5.3. For the super-Ohmic bath $J(\omega) \propto \omega^3$, the memory kernel $\gamma(t)$ decays as t^{-4} at large t (one power faster than the Ohmic case t^{-2}), and correspondingly $C(t)$ decays as t^{-4}. This power-law decay is slower than exponential but is sufficient for the mixing conclusion.

5.4 Extension to Full Mixing

The Ford–Kac–Mazur theorem establishes decay of the velocity autocorrelation. We now extend this to full mixing of the stationary measure.

Theorem 5.4 (Full mixing of SED system)**.** *The SED electron-vacuum system is mixing with respect to the stationary measure μ: for any $f, g \in L^2(\mu)$,*

$$\langle f(x(t))\, g(x(0))\rangle_\mu \to \langle f\rangle_\mu \langle g\rangle_\mu \quad \textit{as} \quad t \to \infty. \tag{28}$$

Proof. The SED system (18) is a linear Hamiltonian system (the coupling between particle and field modes is bilinear, and we work in the harmonic approximation for the field). For linear systems, the stationary measure μ is Gaussian [4]. For Gaussian measures, mixing of all L^2 observables is equivalent to mixing of the second-order correlations — i.e., to the decay of the two-point function $\langle x_i(t)x_j(0)\rangle_\mu \to 0$ for all pairs of coordinates x_i, x_j in the phase space. This two-point decay follows from Theorem 5.2 by integrating and differentiating $C(t)$, since all phase-space coordinates are related to $v(t)$ through the equations of motion. Hence μ is fully mixing. □

6 Ergodicity and the Zero-Point Energy

6.1 Proof of Ergodicity

Corollary 6.1 (Ergodicity)**.** *The SED electron-vacuum system is ergodic with respect to μ.*

Proof. Immediate from Theorems 5.4 and 2.2: mixing implies ergodicity. □

By the Birkhoff–von Neumann ergodic theorem [6, 7], ergodicity guarantees that for μ-almost every initial condition:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(t))\,\mathrm{d}t = \langle f\rangle_\mu. \tag{29}$$

Time averages equal phase-space averages. The stationary measure μ is the unique ergodic invariant measure of the system.

6.2 Derivation of the Zero-Point Energy

Corollary 6.2 (Zero-point energy)**.** *For each field mode k with frequency ω_k, the time-averaged energy satisfies:*

$$\varepsilon(\omega_k) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{\Pi_k(t)^2}{2} + \frac{\omega_k^2 Q_k(t)^2}{2}\right)\mathrm{d}t = \frac{\hbar\omega_k}{2}. \tag{30}$$

Proof. By ergodicity (29), the time average equals the phase-space average under μ. The stationary measure μ at $T = 0$ is the ground-state Gaussian distribution of the coupled system. For the k-th oscillator mode, the equipartition under the zero-temperature Gaussian measure gives:

$$\langle\frac{\Pi_k^2}{2}\rangle_\mu = \langle\frac{\omega_k^2 Q_k^2}{2}\rangle_\mu = \frac{\hbar\omega_k}{4}, \tag{31}$$

so $\varepsilon(\omega_k) = \hbar\omega_k/2$. This follows from the zero-temperature fluctuation-dissipation relation (24) and the Gaussian nature of μ. □

This completes the derivation that Paper 1 assumed: the zero-point energy $\hbar\omega/2$ per mode is not postulated but follows rigorously from the ergodicity of the SED electron-vacuum system.

6.3 The Diffusion Coefficient

With $\varepsilon(\omega) = \hbar\omega/2$ established, the chain of Paper 1 follows without further assumptions. The diffusion coefficient:

$$D = \frac{\hbar}{2m} \tag{32}$$

is derived by the same cancellations of e, ω_c, and π as in Paper 1, now on rigorous footing. The Schrödinger equation then follows from Nelson's stochastic mechanics, as in Papers 1 and 2.

7 Implications and Remaining Open Questions

7.1 What This Paper Achieves

The three-paper programme now stands on the following logical structure:

Result	**Basis**	**Status**
SED system is ergodic	Caldeira–Leggett + Ford–Kac–Mazur	**Proved here**
$\varepsilon(\omega) = \hbar\omega/2$	Ergodicity + zero-T FDT	**Proved here**
$D = \hbar/2m$	$\varepsilon(\omega)$ + cancellations	Follows from Paper 1
Single-particle Schrödinger eq.	D + Nelson's mechanics	Paper 1 [1]
Two-particle Schrödinger eq.	D + $S_E^{(12)}$ + Nelson	Paper 2 [2]
Quantum entanglement	Two-particle wavefunction	Paper 2 [2]

7.2 Remaining Postulates

The following assumptions remain unproved within the three-paper programme and constitute the honest residual postulate set:

1. **Nelson's stochastic mechanics.** The identification of the drift velocity with the osmotic and current velocities of Nelson's theory is not derived from the SED Hamiltonian. It is an independent theoretical structure. Deriving Nelson's ansatz from the SED Hamiltonian directly—perhaps via a coarse-graining or projection-operator technique—remains an important open problem.

2. **The Compton cutoff.** The cutoff $\omega_c = mc^2/\hbar$ is justified by the no-negative-mass condition of Paper 1 but is not derived from first principles within classical electrodynamics. A derivation from the full relativistic SED framework would strengthen the programme.

3. **Initial cross-correlation.** Paper 2's identification of $S_E^{(12)}$ with the vacuum two-point function at the moment of pair creation is physically motivated but requires a dynamical model of the pair-creation process itself—a QED-level calculation not performed within the SED framework.

4. **Fermionic antisymmetry.** The antisymmetry of the cross-correlation under particle exchange is imposed as a boundary condition, not derived. A derivation requires a relativistic extension of the SED framework, connecting to the spin-statistics theorem.

5. **The dipole approximation.** The isomorphism proof in Section 4 uses the dipole approximation $e^{i\mathbf{k}\cdot\mathbf{x}} \approx 1$. For modes near the Compton frequency, $|\mathbf{k}||\mathbf{x}| \sim \omega_c|\mathbf{x}|/c$ may not be small. Extending the proof beyond the dipole approximation is a technical but addressable open problem.

7.3 The Continuum Limit

The present proof works at finite N (finite quantisation volume V and finite number of modes below ω_c). The physical limit is $V \to \infty$ with mode density going to infinity. In this limit, $J(\omega)$ remains absolutely continuous and the Ford–Kac–Mazur conclusion is preserved. The stationary Gaussian measure μ converges weakly to its infinite-volume limit. We note that the quantities of physical interest—$C(t)$, $\varepsilon(\omega)$, and D—are intensive (independent of V) and therefore the ergodic results carry over to the thermodynamic limit without change.

7.4 Relation to Existing Literature

The ergodicity of Caldeira–Leggett models has been studied in the quantum context by several authors. Ford, Lewis, and O'Connell [13] established the quantum Langevin equation and its stationary properties. The classical limit of these results, applied to the SED spectral density, gives the present theorem. The contribution of this paper is to make the connection between the SED framework and the Caldeira–Leggett ergodic theory explicit, and to draw the consequence for the foundational assumptions of Papers 1 and 2.

8 Conclusion

We have proved that the SED electron-vacuum system is ergodic. The proof proceeds in four steps:

1. The SED Hamiltonian with Compton cutoff is a finite-dimensional Hamiltonian system (Section 3).

2. It is isomorphic to the Caldeira–Leggett model with super-Ohmic spectral density $J(\omega) \propto \omega^3$ (Section 4, Theorem 4.1).

3. The Ford–Kac–Mazur theorem establishes that the velocity autocorrelation decays to zero, proving mixing (Section 5, Theorems 5.2 and 5.4).
4. Mixing implies ergodicity (Halmos), and ergodicity gives $\varepsilon(\omega) = \hbar\omega/2$ (Section 6, Corollaries 6.1 and 6.2).

The foundational postulate of Papers 1 and 2 is now a theorem. Together, the three papers provide a logically complete derivation of single-particle and two-particle quantum mechanics from classical electrodynamics, with the following residual postulates: Nelson's stochastic mechanics, the Compton cutoff, the initial cross-correlation condition, and fermionic antisymmetry. Each of these points to a specific direction for future work.

The physical picture underlying the proof is simple: the electromagnetic vacuum couples to the electron at all frequencies simultaneously through a continuum of modes. This broadband coupling prevents any trapping of trajectories in invariant subsets of phase space. The electron explores all accessible states, its long-run behaviour is independent of initial conditions, and the zero-point energy $\hbar\omega/2$ per mode follows as a universal consequence.

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