Information Curvature, Completed Infinity, and a Proposed Route to Inconsistency in ZFC

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May 2026

Abstract

This paper proposes an information-curvature critique of the classical set-theoretic treatment of completed infinity. For finite sets, cardinal growth under powerset formation corresponds to a strict decrease in reciprocal cardinality. We define finite information curvature and finite super-curvature by

$$s(X) = \frac{1}{|X|}, \qquad S(X) = \frac{1}{|\mathcal{P}(X)|}.$$

For finite sets, $|X| < |\mathcal{P}(X)|$ if and only if $s(X) > s(\mathcal{P}(X))$. As finite sets exhaust an infinite process, both curvature and super-curvature tend to zero. This paper does not claim to exhibit a completed formal contradiction inside ZFC. Instead, it isolates a proposed route to such a contradiction: Cantorian set theory asserts strict hierarchies of completed infinities, while finite-observational information curvature collapses the corresponding residual distinction to zero. The central hypothesis is that the logical tension arises from treating completed infinity as a coherent object. The proposed replacement is a zero-curvature flat-infinity axiom: an infinite information set N satisfies $|N| = |\mathcal{P}(N)|$, so powerset formation does not generate a higher completed cardinal layer at the zero-curvature limit.

1 Introduction

Classical ZFC set theory permits the existence of completed infinite sets, as formalized in standard axiomatizations of set theory [2]. In particular, the axiom of infinity asserts the existence of a set containing an inductive sequence, usually identified with the natural numbers $\mathbb{N}$. Once $\mathbb{N}$ is admitted as a completed set, ZFC also admits its powerset $\mathcal{P}(\mathbb{N})$, and Cantor's theorem yields

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|.$$

The classical diagonal method originates in Cantor's work on the uncountability of the continuum and remains central to standard presentations of set theory [1, 2].

The thesis explored here is that this move hides a deep logical tension and possibly an inconsistency. The phrase "completed infinity" is treated as a legitimate mathematical object, but it combines two opposed ideas: completion and endlessness. The paper investigates whether the admission of this logical oxymoron is the source of the Cantorian hierarchy of infinities and the classical pathologies that follow from completed totalization.

The proposed attack does not deny finite set theory, finite powersets, finite binary strings, or finite diagonal arguments. The finite case is taken as fully valid. The objection concerns the extension of finite powerset reasoning to completed infinite totalities.

2 Finite Information Curvature

Definition 1 (Finite information curvature)**.** *For a nonempty finite set X, define*

$$s(X) = \frac{1}{|X|}.$$

Definition 2 (Finite super-curvature)**.** *For a nonempty finite set X, define*

$$S(X) = \frac{1}{|\mathcal{P}(X)|}.$$

Since $|\mathcal{P}(X)| = 2^{|X|}$, this is

$$S(X) = 2^{-|X|}.$$

For finite sets, powerset formation strictly increases cardinality:

$$|X| < |\mathcal{P}(X)|.$$

Taking reciprocals gives

$$\frac{1}{|X|} > \frac{1}{|\mathcal{P}(X)|}.$$

Thus finite powerset formation strictly decreases information curvature.

For finite X, one also has

$$|\mathcal{P}(\mathcal{P}(X))| > |\mathcal{P}(X)| > |X|,$$

and therefore

$$s(X) > s(\mathcal{P}(X)) > s(\mathcal{P}(\mathcal{P}(X))).$$

At the finite level, cardinal growth and curvature decrease are exactly aligned.

3 The Infinite Limit

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets with $|X_n| = n$. Then

$$s(X_n) = \frac{1}{n}$$

and

$$S(X_n) = \frac{1}{2^n}.$$

Hence

$$\lim_{n\to\infty} s(X_n) = 0, \qquad \lim_{n\to\infty} S(X_n) = 0.$$

This motivates the following principle.

Axiom 1 (Zero-curvature infinity)**.** *An infinite information process has zero information curvature:*

$$s(\mathbb{N}) = 0, \qquad S(\mathbb{N}) = 0.$$

More generally, completed infinite information, if admitted at all, must be represented as a zero-curvature limit of finite information growth.

The physical analogy is Einstein's Equivalence Principle. In general relativity, local inertial frames remove gravitational force at a point, while curvature or tidal effects measure residual deviation [6, 7]. Here, finite information systems have nonzero information curvature. Infinite information, as the limit of unbounded finite growth, has zero residual information curvature. The analogy is not a claim about physical spacetime; it is a structural analogy between curvature as residual distinguishability and information curvature as residual finite-observational distinguishability.

4 A ZFC-Legal Curvature Functional

The informal expression $1/|X|$ is immediately meaningful for finite sets. For arbitrary sets, a ZFC-legal real-valued version may be defined by finite approximation.

Definition 3 (Finite-exhaustion curvature)**.** *For a nonempty set A, define*

$$\mathrm{Curv}(A) = \inf \left\{ \frac{1}{|F|} : F \subseteq A,\ F \text{ finite},\ F \neq \varnothing \right\}.$$

Proposition 1. *If A is finite and nonempty, then*

$$\mathrm{Curv}(A) = \frac{1}{|A|}.$$

If A is infinite, then

$$\mathrm{Curv}(A) = 0.$$

Proof. If A is finite and nonempty, the largest finite subset of A is A itself, so the infimum of $1/|F|$ over nonempty finite $F \subseteq A$ is $1/|A|$. If A is infinite, then for every positive integer n there is a finite subset $F \subseteq A$ with $|F| = n$. Hence the set of possible values contains $1/n$ for every positive integer n, whose infimum is 0. □

This definition is fully compatible with ordinary ZFC. It immediately yields

$$\mathrm{Curv}(\mathbb{N}) = 0$$

and, since $\mathcal{P}(\mathbb{N})$ is infinite in ZFC,

$$\mathrm{Curv}(\mathcal{P}(\mathbb{N})) = 0.$$

Thus the finite-exhaustion curvature of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ is equal, even though ZFC proves their cardinalities are distinct.

5 Finite Prefixes and the Binary Tree

Let

$$2^{<\omega}$$

denote the set of all finite binary strings. Equivalently,

$$2^{<\omega} = \bigcup_{n<\omega} 2^n.$$

This is the finite-prefix tower. Up to natural coding, it is the union of all finite powerset stages.

Let

$$2^{\omega}$$

denote the set of infinite binary sequences. In classical set theory, 2^{ω} is identified with the Cantor space and has cardinality $2^{\aleph_0}$ [2].

Proposition 2. *For every $r \in 2^{\omega}$ and every finite n,*

$$r \restriction n \in 2^{<\omega}.$$

Proof. By definition, r is a function from ω to $\{0,1\}$. Its restriction to n is a binary string of length n. Therefore $r \upharpoonright n \in 2^n$, and since $2^n \subseteq 2^{<\omega}$, the claim follows. $\square$

Thus every finite observation of every infinite binary sequence already occurs in the countable prefix tree $2^{<\omega}$. The unresolved information after observing n bits may be measured by

$$2^{-n}.$$

The limiting residual is therefore

$$\lim_{n\to\infty} 2^{-n} = 0.$$

Consequently, the alleged uncountable excess of 2^ω over $2^{<\omega}$ has no missing finite observation and no positive limiting finite-observational curvature.

6 Cantor's Diagonal Argument

Cantor's theorem states that for any set X,

$$|X| < |\mathcal{P}(X)|.$$

In standard ZFC, this theorem is derived using the powerset axiom, separation/comprehension, and the diagonal construction [1, 2]. The standard proof assumes a function

$$f : X \to \mathcal{P}(X)$$

and defines the diagonal set

$$D = \{x \in X : x \notin f(x)\}.$$

Since $D \subseteq X$, one has $D \in \mathcal{P}(X)$. If f were surjective, there would be some $d \in X$ such that $f(d) = D$. Then

$$d \in D \iff d \notin f(d),$$

so

$$d \in D \iff d \notin D,$$

a contradiction. Therefore no such surjection exists.

For finite X, this is unobjectionable. The diagonal set is finitely checkable and the powerset is a finite completed object. For infinite X, however, the construction of D relies on completed comprehension over an infinite domain. The diagonal set is treated as a completed object all at once. This is precisely the point under dispute.

The diagonal proof therefore depends not only on finite powerset reasoning, but on the assumption that infinite domains and their powersets are completed totalities over which global comprehension is legitimate.

7 The SuperDiagonal Challenge

The SuperDiagonal challenge asks the defender of completed continuum ontology to produce a finite observation, or an infinite limit of finite observations with nonzero residual curvature, that distinguishes the alleged continuum excess from the finite-prefix tower.

Given the computable tower of finite powersets

$$2^0, 2^1, 2^2, \ldots,$$

produce a finite digit position or finite prefix at which the continuum contains information not captured by this tower. But for every finite n, the tower already contains every binary string of length n. Thus there is no finite prefix witness to the alleged excess.

Moreover, the limiting residual is zero:

$$\lim_{n\to\infty} 2^{-n} = 0.$$

Therefore the continuum, as a completed object, adds cardinal excess without adding finite or limiting observational excess. This motivates the following principle.

Axiom 2 (Information-Curvature Equivalence Principle)**.** *If two information domains A and B satisfy the following conditions:*

1. *every finite observation from elements of A appears in B, and every finite observation from elements of B appears in A, under the relevant coding;*
2. *the limiting residual information curvature between their finite approximations tends to zero;*

then any completed cardinal distinction between A and B is not information-theoretically justified.

A classical set theorist will reply that cardinality concerns the existence or nonexistence of bijections between completed totalities, not merely finite approximations, which is the standard extensional viewpoint of set theory [2]. The present proposal counters that any such bijection, or its absence, must ultimately be grounded in information that is observable through finite approximation. When the observational curvature vanishes, the cardinal distinction becomes an artifact of the completed-infinity assumption itself.

8 Conflict with ZFC

ZFC proves that

$$|2^{<\omega}| = \aleph_0$$

and

$$|2^{\omega}| = 2^{\aleph_0}.$$

It also proves

$$\aleph_0 < 2^{\aleph_0}.$$

At the same time, every finite observation of an element of 2^{ω} already lies in $2^{<\omega}$, and the residual finite-observational curvature satisfies

$$\lim_{n\to\infty} 2^{-n} = 0.$$

Thus the proposed conflict is:

ZFC asserts strict completed-cardinal separation where information curvature gives zero separation.

Equivalently, ZFC permits completed infinity to be cardinally curved even when finite-observational curvature is flat.

This does not by itself constitute a formal contradiction in ZFC. Rather, it identifies a candidate inconsistency mechanism. If the Information-Curvature Equivalence Principle can be derived from, or shown to be implicit in, ZFC's own treatment of finite approximation, then ZFC would prove both a strict Cantorian separation and zero curvature separation.

9 The Flat-Infinity Replacement Axiom

The proposed replacement for the classical treatment of infinity is the following.

Axiom 3 (Zero-curvature flat-infinity axiom)**.** *There exists an infinite information set N such that powerset formation does not create a higher completed cardinal layer:*

$$|N| = |\mathcal{P}(N)|.$$

Equivalently, at the zero-curvature infinite limit, the distinction between an infinite domain and its completed powerset is not cardinally real.

This axiom directly rejects the Cantorian hierarchy of completed infinities. It does not reject finite powersets. It asserts that powerset growth is real at every finite stage, but that the residual curvature of this growth vanishes in the infinite limit. Therefore the powerset of an infinite information set does not generate a new completed infinity.

The intended interpretation is not that finite mathematics fails, but that completed infinity should be treated as a zero-curvature limit rather than a higher-order completed object.

10 Hypothesis of Inconsistency

Conjecture 1 (Main hypothesis)**.** *ZFC may be inconsistent because it admits completed infinity as a legitimate object. The completed infinite set is a logical oxymoron: it treats an unending process as a finished totality. This permits Cantor's diagonal argument to generate strict hierarchies of completed infinities, while finite information curvature and super-curvature collapse those hierarchies to zero.*

The proposed inconsistency is not located in finite set theory. Finite powersets, finite binary strings, finite diagonal arguments, and finite reciprocal cardinalities are all valid. The contradiction is suspected to enter when ZFC extends these ideas to completed infinite sets.

In particular, the problematic step is the assertion that a diagonal subset

$$D = \{x \in X : x \notin f(x)\}$$

exists as a completed object when X itself is completed infinite. This step is harmless for finite X, but ontologically decisive for infinite X.

11 Conclusion

This paper proposes that the hierarchy of infinities in ZFC is an artifact of treating completed infinity as a coherent object. Finite powerset growth is real and has nonzero curvature. But the infinite limit of finite information growth has zero curvature:

$$s(\mathbb{N}) = 0, \qquad S(\mathbb{N}) = 0.$$

Likewise, every finite observation of a continuum element is already captured in the countable tower of finite binary strings, and the residual curvature of the allegedly missing information tends to zero.

The resulting thesis is:

There is finite curvature, and there is the zero-curvature infinite limit.

There is not a tower of completed infinities with strictly increasing cardinalities.

The zero-curvature flat-infinity axiom is conjectured to be compatible with constructive analysis and to eliminate classical pathologies dependent on completed totalization. This connects the proposal to earlier constructivist and intuitionist objections to unrestricted completed infinities, while differing in its explicit use of information curvature [3, 4, 5]. These consequences remain to be investigated in future work.

12 References

References

[1] G. Cantor, "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen," *Journal für die reine und angewandte Mathematik* **77** (1874), 258–262.

[2] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, North-Holland, 1980.

[3] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[4] L.E.J. Brouwer, "Intuitionism and Formalism," *Bulletin of the American Mathematical Society* **20** (1913), 81–96.

[5] A. Heyting, *Intuitionism: An Introduction*, North-Holland, 1956.

[6] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik* **49** (1916), 769–822.

[7] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, W.H. Freeman, 1973.