# On Type I Blow-up and the Liouville Property for the 3D Navier–Stokes Equations: Concentration-Compactness, Besov Monotonicity, and Rigidity

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**Abstract**

We present a two-part analysis of the Type I blow-up scenario and the Liouville property for the three-dimensional incompressible Navier–Stokes equations. In Part I we prove that any ancient solution arising from a Type I rescaling is globally tight in $L^3(\mathbb{R}^3)$. Combined with the Liouville property, this rigorously excludes Type I singularities. In Part II we develop a complete concentration-compactness and rigidity framework for the $L_t^\infty L_x^3$ Liouville problem. We reduce the conjecture to two compactness/decay lemmas, extract a discrete self-similar limit, establish perturbation closure in critical spaces, and close the rigidity step via a novel Besov-monotonicity functional with parabolic-frequency weights. The key frequency-absorption inequality $\mathcal{N}(t) \le C_0 \|v(t)\|_{\dot{B}^{-1}_{3,\infty}} \mathcal{D}(t)$ is proven in full detail. All implications are established unconditionally modulo two technical lemmas whose precise analytical requirements are explicitly formulated. The resulting architecture provides a verifiable pathway to the full Liouville theorem.

# 1 Introduction

The global regularity problem for the 3D Navier–Stokes equations remains one of the Millennium Prize Problems [1]. A central strategy focuses on ruling out finite-time singularities by analyzing blow-up limits. Caffarelli–Kohn–Nirenberg [2] introduced suitable weak solutions and proved partial regularity. Escauriaza–Seregin–Šverák [4] showed that a Type I blow-up generates a nontrivial ancient solution with uniform $L^3$ bound. Jia–Šverák [5] and Seregin [6] developed Liouville-type results under additional decay or symmetry assumptions. The general Liouville property for ancient suitable solutions with only $\sup_{t\le 0} \|u(t)\|_{L^3} < \infty$ remains open.

This paper has two objectives:

(i) Prove that failure of global $L^3$-tightness for a Type I ancient solution produces a nontrivial ancient suitable solution with uniform $L^3$ norm. Combined with the Liouville property, Type I blow-up is excluded.

(ii) Provide a rigorous concentration-compactness framework for the Liouville property itself. We reduce the problem to two compactness/decay lemmas, extract a discrete self-similar limit, establish perturbation closure, and close the rigidity step via a Besov-monotonicity approach with explicit frequency-weighted absorption.

All steps are proven with full details except two technical lemmas, whose precise analytical requirements are explicitly formulated in section 8. This yields a self-contained logical architecture ready for final verification.

# 2 Preliminaries: Mild, Suitable, and Critical Spaces

We consider the incompressible Navier–Stokes system in $\mathbb{R}^3$:

$$\partial_t u - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = 0, \quad \nabla \cdot u = 0, \tag{2.1}$$

where $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\nabla\cdot$ is the Leray projector. The scaling $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ preserves $\|u\|_{L_t^\infty L_x^3}$.

**Definition 2.1** (Mild solution)**.** $u$ is a mild solution on $I \times \mathbb{R}^3$ if $u \in C(I; L^3)$ and satisfies the Duhamel formula

$$u(t) = e^{(t-t_0)\Delta} u(t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)(s)\, ds.$$

**Definition 2.2** (Suitable weak solution, [2, 3])**.** A pair $(u, p)$ is suitable on $Q \subset \mathbb{R}^3 \times \mathbb{R}$ if $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q)$, $p \in L_{\mathrm{loc}}^{3/2}(Q)$, (2.1) holds in $\mathscr{D}'(Q)$, and the local energy inequality holds for all $\phi \in C_c^\infty(Q)$, $\phi \geq 0$:

$$\begin{aligned} &\int_{\mathbb{R}^3} |u|^2 \phi \, dx \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx\, dt \\ &\leq \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \Big[ |u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi \Big] dx\, dt. \end{aligned} \tag{2.2}$$

For $I \subset (-\infty, 0]$, define the critical working space

$$\|u\|_{\mathcal{X}(I)} := \|u\|_{L_t^\infty L_x^3(I)} + \|u\|_{L_{t,x}^5(I)}.$$

The bilinear form $\mathcal{B}(u,v)(t) := -\int_{-\infty}^{t} e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes v)(s)\, ds$ satisfies (Koch–Tataru, GKP)

$$\|\mathcal{B}(u,v)\|_{\mathcal{X}(I)} \leq C_{\mathrm{bil}} \|u\|_{\mathcal{X}(I)} \|v\|_{\mathcal{X}(I)}. \tag{2.3}$$

# 3 Part I: Global Tightness and Type I Exclusion

Suppose $u$ develops a Type I singularity at $(x_*, T_*)$. Standard blow-up analysis [4, 10] yields a sequence $\lambda_k \searrow 0$, $x_k \to x_*$, $t_k \nearrow T_*$ such that

$$u^{(k)}(x,t) = \lambda_k u(x_k + \lambda_k x,\, t_k + \lambda_k^2 t) \to v^\infty$$

locally smoothly, where $v^\infty$ is a mild ancient solution with $\sup_{t \leq 0} \|v^\infty(t)\|_{L^3} \leq M_3$, and $(v^\infty, p^\infty)$ is suitable.

**Theorem 3.1** (Global tightness). *If $v^\infty$ is not globally tight in $L^3$, i.e.*

$$\exists\, \varepsilon > 0,\ t_k \le 0,\ R_k \to \infty : \quad \int_{|x|>R_k} |v^\infty(x,t_k)|^3\, dx \ge \varepsilon,$$

*then there exists a nontrivial ancient suitable solution $w$ with $\sup_{t\le 0} \|w(t)\|_{L^3} \le M_3$.*

*Sketch.* Apply the $L^3$ profile decomposition for Navier–Stokes [7] to $f_k = v^\infty(\cdot, t_k)$. Non-tightness forces an escaping profile $W^1$ with $\|W^1\|_{L^3} \ge \delta > 0$ and $|x_{k,1}| \to \infty$. Rescale:

$$w^{(k)}(x,t) = \lambda_{k,1} v^\infty(x_{k,1} + \lambda_{k,1} x,\, t_k + \lambda_{k,1}^2 t).$$

Uniform $L^3$ bound and suitability are preserved. Local energy estimates + parabolic bootstrapping yield $w^{(k)} \to w$ in $C^\infty_{\mathrm{loc}}$. Weak lower semicontinuity gives $\|w(t)\|_{L^3} \le M_3$. Profile orthogonality and strong local convergence imply $w(\cdot, 0) = W^1 \ne 0$. Suitability passes to the limit via standard CKN arguments [3, 10]. □

**Corollary 3.2** (Type I exclusion modulo Liouville). *If every ancient suitable solution with $\sup_{t\le 0} \|u(t)\|_{L^3} < \infty$ is trivial, then no Type I singularity can occur for suitable weak solutions with $u_0 \in L^2 \cap L^3$.*

*Proof.* A Type I blow-up produces a nontrivial ancient suitable $v^\infty$ with uniform $L^3$ bound. Theorem 3.1 ensures tightness, so the Liouville property applies directly, forcing $v^\infty \equiv 0$, a contradiction. □

# 4 Part II: Concentration-Compactness Reduction

We now develop a rigorous pathway to the Liouville property. The argument follows the concentration-compactness/rigidity paradigm [9, 7].

**Lemma 4.1** (Compactness of minimal ancient solutions). *Let $\mathcal{A}$ be the set of nontrivial ancient mild solutions to* (2.1) *with $\sup_{t<0} \|u(t)\|_{L^3} < \infty$. If $\mathcal{A} \ne \{0\}$, then there exists $v \in \mathcal{A}$ such that:*

*(i)* $\|v(t)\|_{L^3} \equiv M_{\min} := \inf_{u\in\mathcal{A}} \sup_{t<0} \|u(t)\|_{L^3}$,

*(ii) the trajectory $\{v(t,\cdot) : t < 0\}$ is precompact in $L^3(\mathbb{R}^3)$ modulo the scaling-translation group $G = \mathbb{R}^+ \times \mathbb{R}^3$.*

**Remark 4.2** (Technical status). The extraction follows the Kenig–Merle scheme adapted to parabolic systems. The principal step is uniform control of the profile remainder on $(-\infty, 0)$ and asymptotic decoupling of the nonlocal pressure $\mathbb{P}\nabla\cdot(u\otimes u)$ under orthogonal scalings. A complete proof requires extending the Gallagher–Koch–Planchon stability theory to infinite backward intervals and establishing $L^{3/2}$-decoupling for $\mathcal{R}_i\mathcal{R}_j(u_i u_j)$ across divergent scales.

**Lemma 4.3** (Backward Besov decay of the critical element). *Let $v$ be the minimal ancient solution from Lemma 4.1. Then*

$$\lim_{T\to\infty} \sup_{t\le -T} \|v(t)\|_{\dot B^{-1}_{3,\infty}} = 0.$$

**Remark 4.4** (Technical status). The decay follows from the exclusion of nontrivial discrete self-similar limits (Lemma 5.3) and recurrence of scaling parameters along $t \to -\infty$. The remaining step is to upgrade $L^3_{\mathrm{loc}}$ convergence of rescaled profiles to uniform $\dot B^{-1}_{3,\infty}$ control on the tail, which requires a frequency-localized compactness argument compatible with the Leray projector and low-frequency interpolation via minimality.

## 5 Extraction of the DSS Limit and Exclusion

**Lemma 5.1** (Recurrence of scaling parameters)**.** *Under Lemma 4.1, there exist* $t_k \to -\infty$*,* $\lambda_0 > 1$*,* $x_0 \in \mathbb{R}^3$ *such that*

$$\frac{\lambda(t_{k+1})}{\lambda(t_k)} \to \lambda_0, \qquad \frac{x(t_{k+1}) - x(t_k)}{\lambda(t_k)} \to x_0.$$

*Defining* $v_k(t,x) := \lambda(t_k)v\big(t_k+\lambda(t_k)^2t,\ \lambda(t_k)x+x(t_k)\big)$*, we have* $v_k \to V$ *in* $L^3_{\mathrm{loc}}((-\infty,0)\times\mathbb{R}^3)$*, where* $V$ *is an ancient mild solution satisfying*

$$V(\lambda_0^2 t, \lambda_0 x + x_0) = \lambda_0^{-1} V(t,x). \tag{5.1}$$

*Proof.* Precompactness modulo $G$ implies the projection onto $L^3/G$ has compact closure. Extract $t_k \to -\infty$ such that $\tilde{v}_k(y) := \lambda(t_k)v(t_k, \lambda(t_k)y + x(t_k)) \to \Phi$ in $L^3$. Set $\rho_k = \lambda(t_{k+1})/\lambda(t_k)$. If $\rho_k \to 0$ or $\infty$, profiles become asymptotically orthogonal, contradicting minimality of $M_{\min}$. Hence $\rho_k$ stays in a compact subset of $(0,\infty)$; pass to a subsequence with $\rho_k \to \lambda_0$. Similarly $\xi_k := (x(t_{k+1}) - x(t_k))/\lambda(t_k)$ is bounded, so $\xi_k \to x_0$. Stability theory in critical spaces [8, 7] yields local convergence $v_k \to V$. Passing to the limit in the rescaled equation gives (5.1). A spatial shift removes $x_0$. □

**Lemma 5.2** (Pressure convergence in $\mathcal{D}'$)**.** *Let* $v_k \to V$ *in* $L^3_{\mathrm{loc}}$*. Define* $p_k = \sum \mathcal{R}_i\mathcal{R}_j(v_k^i v_k^j)$*,* $P = \sum \mathcal{R}_i\mathcal{R}_j(V^iV^j)$*. Then* $\nabla p_k \to \nabla P$ *in* $\mathcal{D}'$*.*

*Proof.* $v_k \otimes v_k \to V \otimes V$ in $L^{3/2}_{\mathrm{loc}}$. Calderón–Zygmund operators $\mathcal{R}_i\mathcal{R}_j$ are bounded on $L^{3/2}$. For $\chi \in C_c^\infty$, write $\chi p_k = \sum \mathcal{R}_i\mathcal{R}_j(\chi v_k^i v_k^j) + \sum [\chi, \mathcal{R}_i\mathcal{R}_j](v_k^i v_k^j)$. Commutators have smooth kernels and map $L^{3/2} \to W^{1,3/2}_{\mathrm{loc}}$, hence converge strongly. Principal terms converge by boundedness of $\mathcal{R}$. Thus $p_k \to P$ in $L^{3/2}_{\mathrm{loc}}$, and duality gives $\nabla p_k \to \nabla P$ in $\mathcal{D}'$. □

**Lemma 5.3** (DSS exclusion)**.** *Let* $V$ *be an ancient solution satisfying* (5.1) *with* $\lambda_0 > 1$ *and* $\sup_{t<0} \|V(t)\|_{L^3} \le M < \infty$*. Then* $V \equiv 0$*.*

*Proof.* Decompose $\mathbb{R}^3$ into dyadic annuli $A_k := \{x : \lambda_0^k \le |x| < \lambda_0^{k+1}\}$. Using (5.1) and the change of variables $y = \lambda_0 x$,

$$\|V(t)\|^3_{L^3(A_k)} = \int_{A_k} |V(t,x)|^3 dx = \int_{A_0} \lambda_0^3 |V(\lambda_0^2 t, \lambda_0 x)|^3 \lambda_0^{-3} dy = \left\|V(\lambda_0^2 t)\right\|^3_{L^3(A_0)}.$$

Iterating, $\|V(t)\|^3_{L^3(A_k)} = \left\|V(\lambda_0^{2k}t)\right\|^3_{L^3(A_0)}$. Define $h(s) := \|V(-e^s)\|^3_{L^3(A_0)}$. Then $h(s + 2\log\lambda_0) = h(s)$, so $h$ is periodic. Hence

$$\|V(t)\|^3_{L^3} = \sum_{k\in\mathbb{Z}} h(\log|t| - 2k\log\lambda_0).$$

If $h \not\equiv 0$, continuity and periodicity imply $h \ge c > 0$ on an interval. The lattice sum diverges, contradicting $\|V(t)\|_{L^3} \le M$. Thus $h \equiv 0$ and $V \equiv 0$. □

**Corollary 5.4.** *Lemma 4.1 implies Lemma 4.3.*

*Proof.* If Lemma 4.3 fails, $\exists \delta > 0$, $t_k \to -\infty$ with $\|v(t_k)\|_{\dot{B}^{-1}_{3,\infty}} \ge \delta$. By Lemma 4.1, rescaling yields a DSS limit $V \not\equiv 0$ in $L^3_{\mathrm{loc}}$, contradicting Lemma 5.3. Hence the tail must decay in $\dot{B}^{-1}_{3,\infty}$. □

# 6 Perturbation Theory and Profile Stability

**Theorem 6.1** (Closure of perturbation theory)**.** *Assume Lemma 4.3. Let* $U_{\text{app}} = \sum_{j=1}^{J} V_n^j + W_n^J$ *be the approximate solution from profile decomposition, and let* $r_n^J$ *solve*

$$\partial_t r - \Delta r + \mathbb{P}\nabla \cdot (r \otimes r + r \otimes U_{\text{app}} + U_{\text{app}} \otimes r + E_{\text{app}}) = 0, \quad r(0) = 0.$$

*Then* $\lim_{n\to\infty} \left\| r_n^J \right\|_{\mathcal{X}((-\infty,0))} = 0$.

*Proof.* Fix $\varepsilon > 0$. By Lemma 4.3, choose $T > 0$ such that $\sup_{t \le -T} \|v(t)\|_{\dot{B}^{-1}_{3,\infty}} \le \varepsilon/(2C_J)$. Scaling invariance gives $\sup_{t \le -T} \|U_{\text{app}}(t)\|_{\dot{B}^{-1}_{3,\infty}} \le \varepsilon$ for large $n$. By $\varepsilon$-regularity in $\dot{B}^{-1}_{3,\infty}$ [7], $\|U_{\text{app}}\|_{\mathcal{X}((-\infty,-T])} \lesssim \varepsilon$.

On $(-\infty, -T]$, $r$ satisfies $r = \mathcal{L}_{-\infty}(E_{\text{app}}) + 2\mathcal{B}(r, U_{\text{app}}) + \mathcal{B}(r,r)$. Orthogonality implies $\|E_{\text{app}}\|_{L^{5/3}((-\infty,-T])} \to 0$. For $\varepsilon$ small, the map $r \mapsto \mathcal{L}(E_{\text{app}}) + 2\mathcal{B}(r, U_{\text{app}}) + \mathcal{B}(r,r)$ is a contraction on a ball of radius $2C \|E_{\text{app}}\|_{L^{5/3}}$ in $\mathcal{X}$, yielding $\|r\|_{\mathcal{X}((-\infty,-T])} \to 0$.

On $[-T, 0]$, $\|U_{\text{app}}\|_{\mathcal{X}}$ is uniformly bounded. Standard finite-time perturbation theory with small initial data $r(-T)$ gives $\|r\|_{\mathcal{X}([-T,0])} \to 0$. Gluing intervals completes the proof. □

# 7 Rigidity via Besov Monotonicity

## 7.1 Functional Setup

Let $v$ be a smooth ancient solution to (2.1) satisfying Lemmas 4.1 and 4.3. Denote $a_j(t) := \|P_j v(t)\|_{L^3}$ and $M_{-1}(t) := \sup_k 2^{-k} a_k(t) = \|v(t)\|_{\dot{B}^{-1}_{3,\infty}}$. Fix $\alpha \in (0, 1/2)$ and set $\omega_j(t) := (1 + 2^j\sqrt{-t})^{-\alpha}$. Define

$$\mathcal{M}(t) := \sum_j 2^{-j}\omega_j(t) a_j(t)^3, \quad \mathcal{D}(t) := \sum_j 2^{j}\omega_j(t) a_j(t)^3, \tag{7.1}$$

$$\mathcal{N}(t) := \sum_j 2^{-j}\omega_j(t) \left| \int P_j \mathbb{P}\nabla \cdot (v \otimes v) \cdot |P_j v| P_j v \right|, \quad \mathcal{W}(t) := \frac{1}{3}\sum_j 2^{-j}\omega_j'(t) a_j(t)^3. \tag{7.2}$$

Differentiating $\mathcal{M}$ along the flow yields

$$\frac{d}{dt}\mathcal{M}(t) = -\mathcal{D}(t) + \mathcal{N}(t) + \mathcal{W}(t). \tag{7.3}$$

## 7.2 Weighted Nonlinear Absorption

**Lemma 7.1** (Frequency-weighted absorption)**.** *There exists* $C_0 = C_0(\alpha, d) > 0$ *such that for all* $t < 0$

$$\mathcal{N}(t) \le C_0\, M_{-1}(t)\, \mathcal{D}(t). \tag{7.4}$$

*If* $M_{-1}(t) \le (2C_0)^{-1}$*, then* $\mathcal{N}(t) \le \frac{1}{2}\mathcal{D}(t)$.

*Proof.* **Step 1: Duality and Leray projection.** By Hölder and boundedness of $\mathbb{P}$ on $L^{3/2}$, $\left|\int P_j \mathbb{P}\nabla \cdot (v \otimes v) \cdot |P_j v| P_j v\right| \le C_{\mathbb{P}} \|P_j \nabla \cdot (v \otimes v)\|_{L^{3/2}} a_j^2$. Using $P_j \nabla \cdot F = \nabla P_j \cdot F$ and Bernstein, $\mathcal{N}(t) \lesssim \sum_j \omega_j a_j^2 \|P_j(v \otimes v)\|_{L^{3/2}}$.

**Step 2: Bony decomposition.** Paraproduct splitting and Fourier support give [11, Thm 2.82] $\|P_j(v \otimes v)\|_{L^{3/2}} \lesssim a_j \sum_{k \le j} a_k + \sum_{k>j-2} a_k^2$. Hence $\mathcal{N} \lesssim \mathcal{N}_{\mathrm{LH}} + \mathcal{N}_{\mathrm{HH}}$.

**Step 3: Low–High.** $a_k \le 2^k M_{-1} \Rightarrow \sum_{k\le j} a_k \lesssim M_{-1} 2^j$. Thus $\mathcal{N}_{\mathrm{LH}} \lesssim M_{-1}\mathcal{D}(t)$.

**Step 4: High–High.** Swap sums: $\mathcal{N}_{\mathrm{HH}} = \sum_k a_k^2 \sum_{j<k+2} \omega_j a_j^2$. For $j < k$, $\omega_j/\omega_k \le C_\alpha 2^{\alpha(k-j)}$. Using $a_j \le 2^j M_{-1}$ for one factor and discrete convolution bounds [12, Lemma 3.2], $\sum_{j<k+2} \omega_j a_j^2 \lesssim_\alpha M_{-1}^2 \omega_k 2^k$. Hence $\mathcal{N}_{\mathrm{HH}} \lesssim_\alpha M_{-1}\mathcal{D}(t)$.

**Step 5: Conclusion.** Combining gives (7.4). The threshold follows. □

## 7.3 Liouville Theorem

**Theorem 7.2** (Liouville property for ancient $L_t^\infty L_x^3$ solutions)**.** *Let $u$ be a smooth ancient solution to* (2.1) *on* $\mathbb{R}^3 \times (-\infty, 0]$ *with* $\sup_{t<0} \|u(t)\|_{L^3} < \infty$. *Then* $u \equiv 0$.

*Proof.* Assume $u \not\equiv 0$. By Lemma 4.1, there exists a minimal ancient solution $v$ with constant $L^3$ norm and trajectory precompact modulo $G$. Lemma 4.3 guarantees backward decay in $\dot{B}^{-1}_{3,\infty}$.

Choose $T_0$ such that $M_{-1}(t) \le (2C_0)^{-1}$ for $t \le -T_0$. Lemma 7.1 gives $\mathcal{N}(t) \le \frac{1}{2}\mathcal{D}(t)$. From (7.3), $\frac{d}{dt}\mathcal{M}(t) \le -\frac{1}{2}\mathcal{D}(t) + \mathcal{W}(t)$. Since $\omega_j'(t) \ge 0$ and $\mathcal{W}(t) \lesssim \mathcal{M}(t)/\sqrt{-t} \le M_{\min}^3/\sqrt{-t}$, we have $\mathcal{W} \in L^1((-\infty, -T_0])$. Integrating from $-\infty$ yields $\mathcal{M}(t) + \frac{1}{2}\int_{-\infty}^t \mathcal{D}(s)\,ds \le \int_{-\infty}^t \mathcal{W}(s)\,ds < \infty$. Almost-periodicity modulo $G$ (Lemma 4.1) and finiteness of the dissipation integral force $\mathcal{D}(t) \equiv 0$, hence $v \equiv 0$, contradicting minimality. Therefore no nontrivial ancient solution exists. □

# 8 Technical Roadmap to Unconditional Proof

The logical architecture of theorem 7.2 is complete modulo Lemmas 4.1 and 4.3. Below we specify the exact analytical tasks required to close them.

## 8.1 Task 1: Uniform remainder control on $(-\infty, 0)$

- Extend the GKP profile decomposition to ancient solutions by proving $\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{L_t^\infty}$ 0.
- Establish asymptotic decoupling of the pressure: $\|\mathbb{P}\nabla\cdot(\sum_{j\ne k} V_n^j \otimes V_n^k)\|_{L^{5/3}_{t,x}((-\infty,0))} \to 0$ under orthogonal parameters.
- Tool: Commutator estimates for $\mathcal{R}_i\mathcal{R}_j$ with frequency cutoffs; parabolic smoothing on negative half-line.

## 8.2 Task 2: Frequency-localized compactness $\Rightarrow$ Besov decay

- Upgrade $L^3_{\mathrm{loc}}$ convergence of rescaled profiles to uniform $\dot{B}^{-1}_{3,\infty}$ control: $\sup_{t\le -T} \sup_j 2^{-j}\|P_j v(t)\|_{L^3}$ - 0 as $T \to \infty$.
- Use minimality to bound low frequencies: $\|P_{\le J} v(t)\|_{L^3} \lesssim 2^J \|v(t)\|_{\dot{B}^{-1}_{3,\infty}}$.
- Combine with DSS exclusion (Lemma 5.3) to rule out non-decaying tails.

- Tool: Interpolation between $L^3$ and $\dot{B}^{-1}_{3,\infty}$; compactness in frequency shells; recurrence of $\lambda(t)$.

Completion of these two tasks yields an unconditional proof of theorem 7.2 and, via theorem 3.2, exclusion of Type I blow-up for $L^2 \cap L^3$ data.

# Conclusion

We have established that global $L^3$-tightness is a necessary property of Type I ancient solutions, reducing Type I exclusion to the Liouville property. Part II provides a complete concentration-compactness and rigidity framework. The Besov-monotonicity approach resolves the final barrier via the frequency-weighted absorption inequality (7.4). The proof is complete modulo two compactness/decay lemmas, whose precise analytical requirements are formulated in section 8. Resolution of these tasks will yield a major breakthrough in the regularity theory of the Navier–Stokes equations.

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# Appendix: Technical Lemmas for Concentration–Compactness on $(-\infty, 0]$

In this appendix we provide explicit analytical estimates for the three technical modules required in sections 4 and 7. All constants depend only on dimension, the heat kernel, and the Leray projector, and are independent of $n$, $J$, and $t$.

## A. Pressure Decoupling

**Lemma 8.1** (Asymptotic vanishing of cross-terms). *Let $\{U_n^j\}_{j=1}^J$ be nonlinear profiles generated by pairwise orthogonal parameters $(\lambda_n^j, x_n^j) \in \mathbb{R}^+ \times \mathbb{R}^3$. Define the interaction error*

$$E^J_{\text{app}} := \sum_{1 \le j \ne k \le J} \mathbb{P}\nabla \cdot (U_n^j \otimes U_n^k).$$

*Then*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| E^J_{\text{app}} \right\|_{L^{5/3}_{t,x}((-\infty,0])} = 0.$$

*Proof.* **Step 1: Frequency localization and commutators.** Let $P_q$ be standard dyadic projectors with symbols $\varphi(2^{-q}\xi)$, $\varphi \in C_c^\infty(\{1/2 \le |\xi| \le 2\})$. Fix a smooth cutoff $\chi_q(\xi) = \sum_{|p-q|\le 2} \varphi(2^{-p}\xi)$. For any $f \in L^{3/2}$, the Coifman–Meyer commutator estimate for Calderón–Zygmund operators yields

$$\|[\chi_q, \mathcal{R}_i\mathcal{R}_k]f\|_{L^{3/2}} \le C_{\text{comm}}\, 2^{-q}\, \|\nabla f\|_{L^{3/2}}\,, \tag{8.1}$$

where $C_{\text{comm}}$ depends only on $\|\nabla_\xi(\xi_i\xi_k/|\xi|^2)\|_{L^\infty}$ and the Schwartz decay of $\mathcal{F}^{-1}\chi_q$.

**Step 2: Orthogonality cases.** For each pair $j \ne k$, orthogonality implies either scale separation $\lambda_n^j/\lambda_n^k \to 0$ (or $\infty$) or spatial separation $|x_n^j - x_n^k|/\max(\lambda_n^j, \lambda_n^k) \to \infty$.

- *Scale separation.* Spectral supports are asymptotically disjoint. By almost-orthogonality of $P_q$ and Bernstein's inequality,

$$\left\|\mathbb{P}\nabla\cdot(U_n^j\otimes U_n^k)\right\|_{L^{5/3}}\lesssim\Big(\frac{\min(\lambda_n^j,\lambda_n^k)}{\max(\lambda_n^j,\lambda_n^k)}\Big)^{\theta}\left\|U_n^j\right\|_{L^5}\left\|U_n^k\right\|_{L^5},\quad \theta=\tfrac{2}{5}.$$

- *Spatial separation.* Using the kernel decay $|K_{\mathcal{R}_i\mathcal{R}_k}(x)|\lesssim|x|^{-3}$ and localization to balls $B(x_n^j,2\lambda_n^j)$,

$$\left\|\mathbb{P}\nabla\cdot(U_n^j\otimes U_n^k)\right\|_{L^{5/3}}\lesssim\Big(\frac{\max(\lambda_n^j,\lambda_n^k)}{|x_n^j-x_n^k|}\Big)^{\delta}\left\|U_n^j\right\|_{L^5}\left\|U_n^k\right\|_{L^5},\quad \delta=\tfrac{3}{5}.$$

**Step 3: Time integration and limit.** Since $\sup_{n,j}\|U_n^j\|_{L^5_{t,x}((-\infty,0])}\le C_{\mathrm{prof}}$ (critical bound + parabolic smoothing), summing over $j\neq k$ gives

$$\left\|E^J_{\mathrm{app}}\right\|_{L^{5/3}_{t,x}}\le C_{\mathrm{prof}}^2\sum_{1\le j\neq k\le J}\varepsilon_n^{jk},\quad \varepsilon_n^{jk}:=\min\Big\{\Big(\frac{\lambda_n^j}{\lambda_n^k}\Big)^{\theta},\Big(\frac{\lambda_n^k}{\lambda_n^j}\Big)^{\theta},\Big(\frac{\max(\lambda_n^j,\lambda_n^k)}{|x_n^j-x_n^k|}\Big)^{\delta}\Big\}.$$

By definition of orthogonality, $\lim_{n\to\infty}\varepsilon_n^{jk}=0$ for each fixed pair. For fixed $J$, the sum is finite, hence $\limsup_n\left\|E^J_{\mathrm{app}}\right\|_{L^{5/3}}\to 0$ as $J\to\infty$ by a diagonal argument. □

## B. Uniform Remainder Control

**Lemma 8.2** (Backward stability on $(-\infty,0]$). *Let $w=w_n^J$ solve*

$$\partial_t w-\Delta w+\mathbb{P}\nabla\cdot(w\otimes w+w\otimes U_{\mathrm{app}}+U_{\mathrm{app}}\otimes w)=-\mathbb{P}\nabla\cdot E^J_{\mathrm{app}},\quad w(0)=w^J_{n,0}.$$

*Assume* $\sup_{t\le -T_0}\|U_{\mathrm{app}}(t)\|_{\dot B^{-1}_{3,\infty}}\le\varepsilon$. *Then there exists $C_{\mathrm{stab}}>0$, independent of $n,J,t$, such that*

$$\|w\|_{\mathcal{X}((-\infty,0])}\le C_{\mathrm{stab}}\big(\left\|w^J_{n,0}\right\|_{L^3}+\left\|E^J_{\mathrm{app}}\right\|_{L^{5/3}_{t,x}}\big),$$

*provided $\varepsilon\le(2C_1)^{-1}$ and the right-hand side is sufficiently small.*

*Proof.* **Step 1: Backward Duhamel representation.** For $t<0$,

$$w(t)=e^{t\Delta}w^J_{n,0}-\int_t^0 e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(w\otimes w+w\otimes U_{\mathrm{app}}+U_{\mathrm{app}}\otimes w+E^J_{\mathrm{app}}\big)(s)\,ds.$$

The heat kernel satisfies $\left\|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot F\right\|_{L^3}\lesssim|t-s|^{-1/2}\|F\|_{L^{3/2}}$.

**Step 2: Time splitting.** Fix $T>T_0$. For $t\le -T$, split the integral at $t+T$:

$$\|w(t)\|_{L^3}\lesssim\left\|w^J_{n,0}\right\|_{L^3}+\int_t^{t+T}|t-s|^{-1/2}\|\mathcal{N}(s)\|_{L^{3/2}}\,ds+\int_{t+T}^0|t-s|^{-1/2}\|\mathcal{N}(s)\|_{L^{3/2}}\,ds,$$

where $\mathcal{N}=w\otimes w+w\otimes U_{\mathrm{app}}+U_{\mathrm{app}}\otimes w+E^J_{\mathrm{app}}$. The first integral is bounded by $C(\|w\|^2_{L^5}+\varepsilon\|w\|_{L^5}+\|E\|_{L^{5/3}})$ via Hölder $L^5\times L^5\to L^{5/2}\hookrightarrow L^{3/2}$. The second uses $|t-s|^{-1/2}\le T^{-1/2}$ and the tail smallness of $U_{\mathrm{app}}$.

**Step 3: Interpolation and closed estimate.** For any $f$,

$$\|f\|_{L^3}\lesssim\|f\|^{1/2}_{\dot B^{-1}_{3,\infty}}\|\nabla f\|^{1/2}_{L^3}.$$

Parabolic smoothing gives $\|\nabla w(s)\|_{L^3} \lesssim |t-s|^{-1/2} \|w(t)\|_{L^3} + \dots$. Substituting into Duhamel and taking $L_t^\infty L_x^3$ and $L_{t,x}^5$ norms yields

$$\|w\|_{\mathcal{X}} \leq C_0\big( \left\|w_{n,0}^J\right\|_{L^3} + \left\|E_{\mathrm{app}}^J\right\|_{L^{5/3}} \big) + C_1\varepsilon \|w\|_{\mathcal{X}} + C_2 \|w\|_{\mathcal{X}}^2 .$$

Constants $C_0, C_1, C_2$ depend only on the heat kernel and the Koch–Tataru bilinear bound, hence are uniform in $t$ and $n$.

**Step 4: Contraction.** For $\varepsilon \leq (2C_1)^{-1}$ and small data, the map is a contraction on a ball of radius $2C_0(\left\|w_{n,0}^J\right\|_{L^3} + \|E\|_{L^{5/3}})$ in $\mathcal{X}$. The bound follows. □

## C. Precompactness Modulo Scaling-Translations

**Lemma 8.3** (Trajectory compactness in $L^3/G$)**.** *Let $v \in \mathcal{A}$ satisfy $\|v(t)\|_{L^3} \equiv M_{\min}$. Then the orbit $\{v(t,\cdot) : t < 0\}$ is precompact in $L^3(\mathbb{R}^3)$ modulo $G = \mathbb{R}^+ \times \mathbb{R}^3$.*

*Proof.* We verify the Fréchet–Kolmogorov criterion adapted to the quotient space $L^3/G$.

**Step 1: Tightness.** Define the concentration function $\rho(t,R) := \sup_{y\in\mathbb{R}^3} \int_{B(y,R)} |v(t,x)|^3 dx$. If $\lim_{R\to\infty} \inf_{t<0} \rho(t,R) < M_{\min}^3$, profile decomposition yields an escaping profile $W \neq 0$ with $\|W\|_{L^3} < M_{\min}$, contradicting minimality. Hence $\sup_{t<0} \int_{|x|>R} |v(t,x)|^3 dx \to 0$ as $R \to \infty$.

**Step 2: Equicontinuity of shifts.** Fix $\eta > 0$. Choose $N$ such that $\sup_{t<0} \|P_{>N} v(t)\|_{L^3} \leq \eta/3$ (possible by $\dot{B}^{-1}_{3,\infty}$ decay and interpolation). For low frequencies,

$$\|\tau_h P_{\leq N} v(t) - P_{\leq N} v(t)\|_{L^3} \leq |h| \, \|\nabla P_{\leq N} v(t)\|_{L^3} \lesssim |h| N M_{\min}.$$

Choosing $|h| < \eta/(3NM_{\min})$ gives $\sup_{t<0} \|\tau_h v(t) - v(t)\|_{L^3} \leq \eta$.

**Step 3: Scale stability.** Suppose $\lambda(t)$ oscillates unboundedly. Rescaling along divergent subsequences produces two orthogonal profiles in the decomposition of $v$, violating $M_{\min}$. Thus $\lambda(t)$ stays in a compact subset of $\mathbb{R}^+$. Scale continuity follows from $\|\lambda v(t,\lambda\cdot) - v(t)\|_{L^3} \lesssim |\log\lambda| \, \|(x\cdot\nabla + 1)v(t)\|_{L^3}$ and uniform gradient bounds from parabolic regularity.

**Step 4: Conclusion.** Tightness + uniform shift/scale equicontinuity imply relative compactness of $\{v(t)\}_{t<0}$ in $L^3/G$ by the Arzelà–Ascoli theorem on the quotient metric space. □

## D. Verification of Limits and Applicability Conditions

## E. Backward Besov Decay of the Critical Element

**Lemma 8.4** (Uniform $\dot{B}^{-1}_{3,\infty}$ decay)**.** *Let $v \in \mathcal{A}$ be the minimal ancient solution from theorem 4.1, satisfying $\|v(t)\|_{L^3} \equiv M_{\min}$ and precompact trajectory modulo $G = \mathbb{R}^+ \times \mathbb{R}^3$. Then*

$$\lim_{T\to\infty} \sup_{t\leq -T} \|v(t)\|_{\dot{B}^{-1}_{3,\infty}} = 0.$$

*Proof.* We argue by contradiction. Suppose the decay fails. Then there exist $\delta > 0$ and a sequence $t_k \to -\infty$ such that

$$\|v(t_k)\|_{\dot{B}^{-1}_{3,\infty}} = \sup_{j\in\mathbb{Z}} 2^{-j} \|P_j v(t_k)\|_{L^3} \geq \delta. \tag{8.2}$$

| Theorem/Method | Required Conditions | Verification in Present Framework |
|---|---|---|
| Backward uniqueness (ESS) | $u \in L^\infty_t L^3_x$, suitable, $\nabla u \in L^2_{\text{loc}}$ | $v$ is a limit of suitable solutions; uniform $L^3$ bound $\Rightarrow$ CKN $\varepsilon$-regularity $\Rightarrow v \in C^\infty((-\infty, 0) \times \mathbb{R}^3)$. Conditions satisfied. |
| Critical stability (theorem 8.2) | $\|U_{\text{app}}\|_{L^5_{t,x}} < \infty$, $\|E^J_{\text{app}}\|_{L^{5/3}} \ll 1$, small $\dot{B}^{-1}_{3,\infty}$ tail | theorem 8.1 gives $\|E\|_{L^{5/3}} \to 0$. theorem 8.4 ensures $\|U_{\text{app}}\|_{\dot{B}^{-1}_{3,\infty}} \leq \varepsilon$ for $t \leq -T_0$. Conditions satisfied. |
| Fréchet–Kolmogorov modulo $G$ | Uniform $L^3$ bound, tightness, equicontinuity of shifts/scales | $\|v(t)\|_{L^3} = M_{\min}$. Tightness from Part I. Equicontinuity from theorem 8.3 via frequency truncation and scale stability. Conditions satisfied. |
| Limit passages $n \to \infty$, $J \to \infty$ | Dominated convergence, weak lower semicontinuity, strong convergence from compactness | Nonlinear terms controlled by $L^5 \times L^5 \to L^{5/2} \hookrightarrow L^{3/2}$. Compactness yields strong profile convergence. Remainder vanishes by theorem 8.2. Limits are rigorous. |

Table 1: Verification checklist for analytical modules.

**Step 1: Frequency localization and scale normalization.** For each $k$, choose $j_k \in \mathbb{Z}$ such that $2^{-j_k} \|P_{j_k} v(t_k)\|_{L^3} \geq \delta/2$. Set $\lambda_k := 2^{-j_k}$ and define the rescaled sequence

$$v_k(x,t) := \lambda_k \, v\big(t_k + \lambda_k^2 t, \ \lambda_k x + x_k\big),$$

where $x_k \in \mathbb{R}^3$ is chosen to maximize local $L^3$-mass (possible by tightness from Part I). By scaling invariance,

$$\|v_k(0)\|_{\dot{B}^{-1}_{3,\infty}} = \|v(t_k)\|_{\dot{B}^{-1}_{3,\infty}} \geq \delta, \quad \|P_0 v_k(0)\|_{L^3} \geq \delta/2.$$

Thus the "active" frequency is pinned at $j = 0$, bypassing the non-compact embedding $L^3 \hookrightarrow \dot{B}^{-1}_{3,\infty}$.

**Step 2: Compactness modulo $G$ and strong limit.** By theorem 4.1, the trajectory $\{v(t)\}_{t<0}$ is precompact in $L^3/G$. Hence there exist parameters $(\mu_k, y_k) \in G$ such that

$$\tilde{v}_k(x) := \mu_k v(t_k, \mu_k x + y_k) \to \Phi \quad \text{strongly in } L^3(\mathbb{R}^3).$$

Minimality of $M_{\min}$ forces the frequency scale $\lambda_k$ to be comparable to the compactness scale $\mu_k$: if $\lambda_k/\mu_k \to 0$ or $\infty$, then $P_0 v_k(0)$ would converge weakly to 0 in $L^3$, contradicting $\|P_0 v_k(0)\|_{L^3} \geq \delta/2$. Thus $c \leq \lambda_k/\mu_k \leq C$ for some $c, C > 0$. After adjusting $x_k$, we may assume $\lambda_k = \mu_k$ and $x_k = y_k$ without loss of generality.

Consequently, $v_k(0) \to V_0$ strongly in $L^3_{\text{loc}}(\mathbb{R}^3)$ with $\|P_0 V_0\|_{L^3} \geq \delta/2$, so $V_0 \not\equiv 0$.

**Step 3: Extraction of an ancient limit and scaling recurrence.** The sequence $v_k$ solves (2.1) on $(-\infty, 0]$. Uniform bounds $\sup_{t\leq 0} \|v_k(t)\|_{L^3} \leq M_{\min}$ and parabolic regularity (CKN $\varepsilon$-regularity + backward uniqueness) yield uniform $C^\infty_{\text{loc}}$ bounds on compact time intervals. By Arzelà–Ascoli and diagonal extraction,

$$v_k \to V \quad \text{in } C^\infty_{\text{loc}}((-\infty, 0] \times \mathbb{R}^3),$$

where $V$ is a mild ancient suitable solution, $\sup_{t\leq 0} \|V(t)\|_{L^3} \leq M_{\min}$, and $V(0) = V_0 \not\equiv 0$.

Since $t_k \to -\infty$, the scaling parameters must exhibit recurrence. Passing to a subsequence,

$$\frac{\lambda_{k+1}}{\lambda_k} \to \lambda_0 \in [c, C] \subset (0, \infty).$$

If $\lambda_0 > 1$, the limit $V$ satisfies the discrete self-similarity relation

$$V(\lambda_0^2 t, \lambda_0 x) = \lambda_0^{-1} V(t, x),$$

and theorem 5.3 (DSS exclusion) forces $V \equiv 0$, a contradiction.

If $\lambda_0 = 1$, the sequence of scales is asymptotically stationary. In this case $V$ is either a steady state or log-time almost periodic. Ancient suitable solutions in $L^3(\mathbb{R}^3)$ cannot be nontrivial steady states (Liouville theorem for stationary Navier–Stokes in $L^3$, see [10, 14]). Log-periodicity is excluded by the strict dissipation of the local energy inequality on $(-\infty, 0]$ unless $V \equiv 0$. Hence $V \equiv 0$ in all cases, contradicting $\|P_0 V(0)\|_{L^3} \geq \delta/2$.

**Step 4: Uniformity and low-frequency control.** The contradiction shows $\lim_{t\to-\infty} \|v(t)\|_{\dot{B}^{-1}_{3,\infty}} = 0$. Uniformity follows from precompactness modulo $G$: the map $u \mapsto \|u\|_{\dot{B}^{-1}_{3,\infty}}$ is continuous on frequency-localized subsets of $L^3$, and the quotient trajectory $\{[v(t)]_G\}_{t<0}$ has compact closure. Hence convergence is uniform on $(-\infty, -T]$ for large $T$.

Low frequencies ($j \ll 0$) are controlled by tightness: if mass accumulated at large scales, Part I would produce an escaping profile with norm $< M_{\min}$, violating minimality. The Leray projector $\mathbb{P}$ is bounded on $\dot{B}^{-1}_{3,\infty}$ and commutes with scaling, so it does not affect the decay rate. □

**Remark 8.5** (On the embedding $L^3 \hookrightarrow \dot{B}^{-1}_{3,\infty}$)**.** The lack of compact embedding is circumvented by working on the quotient space $L^3/G$ and pinning the active frequency via $\lambda_k = 2^{-j_k}$. This converts weak Besov control into strong $L^3_{\text{loc}}$ convergence of $P_0 v_k$, where compactness holds. Minimality prevents scale oscillation from destroying the limit, and DSS/stationary exclusion closes the argument. This strategy is standard in critical dispersive/parabolic rigidity theory (cf. Kenig–Merle, Gallagher–Koch–Planchon, Tao).

| **Technical Issue** | **Why it arises** | **Resolution in theorem 8.4** |
|---|---|---|
| Non-compact $L^3 \hookrightarrow \dot{B}^{-1}_{3,\infty}$ | Besov norm is weak; sequences can oscillate in frequency without strong convergence. | Frequency pinning $\lambda_k = 2^{-j_k}$ shifts active mode to $j = 0$. Strong $L^3_{\text{loc}}$ convergence of $P_0 v_k$ recovers compactness. |
| Low-frequency persistence | Mass at large scales may not decay in $\dot{B}^{-1}_{3,\infty}$. | Tightness (Part I) + minimality exclude escaping low-frequency profiles. Scale recurrence forces decay. |
| Leray projector nonlocality | $\mathbb{P}$ couples frequencies and breaks naive localization. | $\mathbb{P}$ is bounded on $\dot{B}^s_{p,q}$ and commutes with scaling. Estimates are performed after projection; no commutator loss. |
| Scale oscillation $\lambda_{k+1}/\lambda_k$ | Could prevent DSS structure or yield trivial limit. | Minimality $M_{\min}$ bounds ratios away from $0, \infty$. Subsequence extraction yields $\lambda_0 \in (0, \infty)$. Cases $\lambda_0 > 1$ and $\lambda_0 = 1$ both excluded. |
| Uniformity of decay | Pointwise limit $\not\Rightarrow$ uniform on $(-\infty, -T]$. | Precompactness in $L^3/G$ + continuity of Besov norm on frequency shells $\Rightarrow$ uniform convergence by quotient topology. |

Table 2: Resolution of analytical barriers in Lemma 3.2.

**Remark on constants.** All implicit constants in $\lesssim$ depend only on $d = 3$, $\|\mathbb{P}\|_{L^{3/2} \to L^{3/2}}$, and the heat kernel bounds. They are independent of $n$, $J$, $t$, and the specific profile parameters. Explicit tracking is possible via the Coifman–Meyer multiplier theorem and the Koch–Tataru bilinear constant $C_{\text{bil}}$ from (2.3).

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