Derivation of an Octonionic Master Equation from Physical Minimal Principles

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Abstract

We formulate a consistent octonionic foundational model in which gravity, matter dynamics, and non-associative coupling arise from a common variational principle. The starting point is a set of physical minimal principles: the requirement of real observables, local dynamics, covariant formulation, positivity of a norm, minimal algebraic nontriviality, and the requirement that genuine triple couplings may be fundamental. From these principles it follows step by step that the relevant internal algebra must be a normed division algebra, and that the only finite-dimensional non-associative possibility is the algebra of octonions. Since non-associativity becomes visible only through the associator of three arguments, the minimal dynamical realization is a triplet of octonion-valued fields. On this basis the full covariant action is constructed, and its variation yields the coupled master equation: the Einstein equation together with three octonionic field equations sourced by the associator. Every term, coupling, and parameter is explicitly introduced and physically interpreted.

1 Introduction

Modern theoretical physics rests on two extraordinarily successful but structurally distinct fundamental theories: general relativity and quantum mechanics. General relativity describes gravity geometrically through a dynamical spacetime metric. Quantum mechanics uses linear state spaces, operators, and probability amplitudes. A central problem of foundational physics is to trace both structures back to a common origin.

The approach pursued here assumes that this common root is not primarily an additional spacetime dimension, a gauge-group extension scheme, or a perturbative quantization of the metric, but rather a deeper algebraic structure. The guiding idea is the following: if the fundamental structure of physics is pre-geometric, then spacetime itself should emerge from an underlying algebra and its dynamics. Under this hypothesis, the central question becomes which algebra is compatible with physical minimal requirements.

The thesis of this report is that a step-by-step application of physical minimal principles naturally leads to the octonions, and that the dynamical master equation of the model follows from a single covariant action principle.

2 Physical Minimal Principles

We do not begin with a prescribed equation, but with general requirements that any fundamental model should satisfy.

2.1 Principle of Reality

Physical observables must ultimately be real. Even if internal degrees of freedom are complex or more generally algebraic, energy, action, norms, probabilities, and expectation values must take values in the real numbers.

It follows that the fundamental algebra must at least carry a well-defined real norm or a real positive bilinear product.

2.2 Principle of Local Dynamics

A fundamental field model should be formulated locally. The dynamics at a spacetime point x may depend only on the fields and on a finite number of their derivatives at the same point. Therefore the action must have the form

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}, \tag{1}$$

where $\mathcal{L}$ is a local Lagrangian density.

2.3 Principle of General Covariance

Since gravity is to be understood geometrically, the theory must be invariant under general coordinate transformations. This requires the use of a Lorentzian metric $g_{\mu\nu}$, the Ricci scalar R, and covariant derivatives ∇_μ.

2.4 Principle of Norm Positivity

In order for kinetic energies, action contributions, and transition weights to be physically meaningful, the internal algebra must carry a positive norm. Without positivity, ghost degrees of freedom or uncontrolled instabilities arise immediately.

2.5 Principle of Minimal Algebraic Nontriviality

If one seeks a common origin of gravitational and quantum phenomena, a purely commutative and associative algebra is too poor. In a completely associative structure there is no intrinsic algebraic mechanism that can generate genuine triple couplings, fundamental branching, or nonlinear projection phenomena from within the algebra itself. We therefore require the minimal possible deviation from associativity while preserving the norm structure.

2.6 Principle of Minimal Genuine Multiple Coupling

If non-associativity is fundamental, it must become dynamically visible. By definition, non-associativity appears in the comparison between $(xy)z$ and $x(yz)$, and therefore in a triple coupling. This argues against a single-field model as the minimal nontrivial realization and points instead toward a triplet of fields.

3 From Minimal Principles to Normed Division Algebras

The requirements above have strong algebraic consequences.

3.1 Why a Normed Algebra?

A positive norm $\|x\|^2$ allows the definition of a real inner product

$$\langle x, y\rangle := \mathrm{Re}(x\bar{y}), \tag{2}$$

provided that a conjugation map $x \mapsto \bar{x}$ exists. For physical consistency it is additionally desirable that the norm be compatible with multiplication:

$$\|xy\|^2 = \|x\|^2\|y\|^2. \tag{3}$$

This condition ensures that products of degrees of freedom do not destroy norm positivity.

3.2 Hurwitz Constraint

The classical Hurwitz theorem states that finite-dimensional real normed division algebras exist only in dimensions

$$1,\ 2,\ 4,\ 8. \tag{4}$$

They are

$$\mathbb{R}, \qquad \mathbb{C}, \qquad \mathbb{H}, \qquad \mathbb{O}. \tag{5}$$

3.3 Why Not $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$?

The real numbers are commutative and associative and carry no internal structure beyond scalar factors.

The complex numbers provide phase and interference, but remain associative.

The quaternions are noncommutative, but still associative. They therefore allow richer internal structures, but no genuine non-associativity.

The octonions are the first and only finite-dimensional real normed division algebra that is non-associative. Hence they are the minimal algebraic structure in which the genuine triple nontriviality required in Section 2 can occur.

4 The Octonions as the Minimal Fundamental Algebra

The octonions $\mathbb{O}$ form an eight-dimensional real vector space with basis

$$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}. \tag{6}$$

A general octonion has the form

$$X = x^0 1 + \sum_{a=1}^{7} x^a e_a, \qquad x^A \in \mathbb{R}. \tag{7}$$

4.1 Conjugation

Octonionic conjugation is defined by

$$\bar{X} = x^0 1 - \sum_{a=1}^{7} x^a e_a. \tag{8}$$

4.2 Real Part

The real part is

$$\mathrm{Re}(X) = x^0. \tag{9}$$

4.3 Norm

The octonionic norm is

$$\|X\|^2 = X\bar{X} = \bar{X}X = (x^0)^2 + \sum_{a=1}^{7}(x^a)^2. \tag{10}$$

It is positive definite.

4.4 Inner Product

The real inner product is

$$\langle X, Y\rangle := \mathrm{Re}(X\bar{Y}). \tag{11}$$

In components this is simply the Euclidean scalar product on $\mathbb{R}^8$:

$$\langle X, Y\rangle = x^0y^0 + \sum_{a=1}^{7} x^a y^a. \tag{12}$$

4.5 Non-Associativity and Associator

The fundamental new object is the associator

$$A(X, Y, Z) := (XY)Z - X(YZ). \tag{13}$$

It measures exactly the deviation from associativity. For octonions one has in general

$$A(X, Y, Z) \neq 0. \tag{14}$$

However, the octonions are alternative, meaning that every subalgebra generated by two elements is associative. Non-associativity therefore appears genuinely only for three independent arguments. This is precisely why a triplet field model is minimal.

5 Why Three Fields Are Necessary

Suppose first that one uses only a single octonion-valued field $\Psi(x)$. A seemingly natural associator term would be

$$A(\Psi, \Psi, \Psi). \tag{15}$$

Because of octonionic alternativity, this expression vanishes identically. More precisely, the associator is totally alternating. Whenever two arguments coincide, it vanishes:

$$A(X, X, Z) = 0, \qquad A(X, Y, Y) = 0. \tag{16}$$

In particular,

$$A(\Psi, \Psi, \Psi) = 0. \tag{17}$$

A two-field ansatz also remains algebraically too poor, because at least two arguments coincide or the fields can effectively move inside associative subalgebras in which non-associativity is not generically excited.

Therefore, minimal genuine non-associative dynamics requires three independent octonion-valued fields:

$$\Psi_1(x), \Psi_2(x), \Psi_3(x) \in \mathbb{O}. \tag{18}$$

Thus the field multiplicity is already fixed by a purely algebraic minimal principle.

6 Geometric Framework

We couple these three fields to a dynamical spacetime metric $g_{\mu\nu}$ on a four-dimensional manifold M.

6.1 Spacetime Metric

The field $g_{\mu\nu}$ is the Lorentzian spacetime metric. It defines the causal structure, the covariant derivative ∇_μ, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, and the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{19}$$

6.2 Invariant Volume Factor

The action is integrated with the measure

$$d^4x \sqrt{-g}, \tag{20}$$

where

$$g = \det(g_{\mu\nu}). \tag{21}$$

7 Construction of the Minimal Action

We now derive the action step by step.

7.1 Gravitational Sector

The generally covariant minimal choice is the Einstein-Hilbert term with an optional cosmological constant:

$$\mathcal{L}_{\text{grav}} = \frac{1}{2\kappa}(R - 2\Lambda). \tag{22}$$

Here κ is the gravitational coupling and Λ is the cosmological constant. In natural units one usually has

$$\kappa = 8\pi G. \tag{23}$$

7.2 Kinetic Sector of the Octonionic Fields

For each field Ψ_I with $I = 1, 2, 3$, we choose the minimal covariant quadratic term:

$$\mathcal{L}_{\text{kin}} = \sum_{I=1}^{3} \frac{\alpha_I}{2} g^{\mu\nu} \left\langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \right\rangle . \tag{24}$$

The constants α_I weight the kinetic contributions.

7.3 Local Potential

Additional local self-interactions and mixed interactions are allowed. A general low-order polynomial form is

$$\begin{aligned} V_{\text{loc}}(\Psi_1, \Psi_2, \Psi_3) = {} & \sum_{I=1}^{3} \left[\frac{m_I^2}{2} \langle \Psi_I, \Psi_I \rangle + \frac{\beta_I}{4} \langle \Psi_I, \Psi_I \rangle^2 \right] \\ & + \sum_{I<J} \frac{\gamma_{IJ}}{2} \langle \Psi_I, \Psi_I \rangle \langle \Psi_J, \Psi_J \rangle + \eta \, \text{Re}(\Psi_1 \Psi_2 \Psi_3). \end{aligned} \tag{25}$$

This part is important, but it is not the new core of the theory.

7.4 Non-Associative Core Term

The minimal principles require a genuine triple coupling. The smallest positive, real, and locally covariant form is the norm of the associator:

$$U_A(\Psi_1, \Psi_2, \Psi_3) = \lambda \|A(\Psi_1, \Psi_2, \Psi_3)\|^2, \tag{26}$$

with

$$A(\Psi_1, \Psi_2, \Psi_3) = (\Psi_1 \Psi_2)\Psi_3 - \Psi_1(\Psi_2 \Psi_3). \tag{27}$$

This form is singled out for three reasons. First, A is the direct indicator of non-associativity. Second, $\|A\|^2$ is real and nonnegative. Third, it is the lowest scalar expression that not only mentions non-associativity algebraically, but also builds it energetically into the dynamics.

7.5 Optional Nonminimal Curvature Coupling

One may also allow the term

$$\mathcal{L}_{\text{nmc}} = -\xi R \sum_{I=1}^{3} \langle \Psi_I, \Psi_I \rangle. \tag{28}$$

It is not required for the minimal model, but is structurally allowed.

8 The Full Action

The full action is therefore

$$\begin{aligned} S[g, \Psi_1, \Psi_2, \Psi_3] = \int d^4x \sqrt{-g} \Bigg[& \frac{1}{2\kappa}(R - 2\Lambda) + \sum_{I=1}^{3} \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle \\ & - V_{\text{loc}}(\Psi_1, \Psi_2, \Psi_3) - \lambda \|A(\Psi_1, \Psi_2, \Psi_3)\|^2 - \xi R \sum_{I=1}^{3} \langle \Psi_I, \Psi_I \rangle \Bigg]. \end{aligned} \tag{29}$$

This is the master action of the model.

9 Component Decomposition of the Octonionic Fields

Each field is expanded in the octonionic basis:

$$\Psi_I(x) = \psi_I^0(x) 1 + \sum_{a=1}^{7} \psi_I^a(x) e_a. \tag{30}$$

Thus each Ψ_I contains eight real components. The full model therefore contains

$$3 \times 8 = 24 \tag{31}$$

real field degrees of freedom, in addition to the metric.

10 Variation with Respect to the Metric: Derivation of the Einstein Equation

We vary the action with respect to $g^{\mu\nu}$.

10.1 Variation of the Gravitational Sector

The standard variation of the Einstein-Hilbert term yields

$$\delta\left[\sqrt{-g}\,\frac{1}{2\kappa}(R-2\Lambda)\right]=\sqrt{-g}\,\frac{1}{2\kappa}(G_{\mu\nu}+\Lambda g_{\mu\nu})\delta g^{\mu\nu}, \tag{32}$$

up to boundary terms.

10.2 Definition of the Energy-Momentum Tensor

The matter sector defines the total energy-momentum tensor by

$$T^{\text{tot}}_{\mu\nu}:=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{33}$$

It follows that the Einstein equation is

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa T^{\text{tot}}_{\mu\nu}. \tag{34}$$

10.3 Kinetic Contribution

The variation of the kinetic term gives

$$T^{\text{kin}}_{\mu\nu}=\sum_{I=1}^{3}\alpha_I\left[\langle\nabla_\mu\Psi_I,\nabla_\nu\Psi_I\rangle-\frac{1}{2}g_{\mu\nu}\langle\nabla_\rho\Psi_I,\nabla^\rho\Psi_I\rangle\right]. \tag{35}$$

10.4 Local Potential Contribution

Since V_{loc} contains no metric derivatives,

$$T^{\text{loc}}_{\mu\nu}=-g_{\mu\nu}V_{\text{loc}}. \tag{36}$$

10.5 Associator Contribution

The quantity $\|A\|^2$ is also a scalar potential term without derivatives. Therefore

$$T^{A}_{\mu\nu}=-g_{\mu\nu}\lambda\|A(\Psi_1,\Psi_2,\Psi_3)\|^2. \tag{37}$$

10.6 Nonminimal Curvature Coupling

The term

$$-\xi R\sum_I\langle\Psi_I,\Psi_I\rangle \tag{38}$$

produces additional standard curvature-coupling terms. In the minimal case $\xi=0$, they vanish completely.

Thus, in the minimal model,

$$\begin{aligned}T^{\text{tot}}_{\mu\nu}=&\sum_{I=1}^{3}\alpha_I\left[\langle\nabla_\mu\Psi_I,\nabla_\nu\Psi_I\rangle-\frac{1}{2}g_{\mu\nu}\langle\nabla_\rho\Psi_I,\nabla^\rho\Psi_I\rangle\right]\\&-g_{\mu\nu}V_{\text{loc}}-g_{\mu\nu}\lambda\|A\|^2.\end{aligned} \tag{39}$$

11 Variation with Respect to the Fields: Derivation of the Octonionic Field Equations

We now vary the action with respect to each of the three fields Ψ_I.

11.1 Variation of the Kinetic Term

For one field Ψ_I, the kinetic term gives after partial integration

$$\delta S_{\mathrm{kin},I} = -\int d^4x \, \sqrt{-g} \, \alpha_I \, \langle \delta\Psi_I, \Box_g \Psi_I \rangle \,, \tag{40}$$

where

$$\Box_g \Psi_I := \nabla_\mu \nabla^\mu \Psi_I \tag{41}$$

is the covariant d'Alembert operator.

11.2 Variation of the Local Potential

One has

$$\delta V_{\mathrm{loc}} = \sum_{I=1}^{3} \left\langle \frac{\partial V_{\mathrm{loc}}}{\partial \Psi_I}, \delta\Psi_I \right\rangle . \tag{42}$$

Here $\partial V_{\mathrm{loc}}/\partial\Psi_I$ denotes the field derivative of the potential with respect to Ψ_I.

11.3 Variation of the Associator Term

This is the central step.

Set

$$A := A(\Psi_1, \Psi_2, \Psi_3) = (\Psi_1\Psi_2)\Psi_3 - \Psi_1(\Psi_2\Psi_3). \tag{43}$$

Then

$$\|A\|^2 = \langle A, A \rangle. \tag{44}$$

It follows immediately that

$$\delta\|A\|^2 = 2\langle \delta A, A \rangle. \tag{45}$$

11.3.1 Variation with Respect to Ψ_1

Keeping Ψ_2 and Ψ_3 fixed and varying only Ψ_1, one finds

$$\delta_{\Psi_1} A = (\delta\Psi_1 \, \Psi_2)\Psi_3 - \delta\Psi_1(\Psi_2\Psi_3). \tag{46}$$

Therefore

$$\delta_{\Psi_1}\|A\|^2 = 2 \, \langle (\delta\Psi_1 \, \Psi_2)\Psi_3 - \delta\Psi_1(\Psi_2\Psi_3), A \rangle \, . \tag{47}$$

Since the inner product is nondegenerate, there exists a uniquely defined adjoint expression D_1 such that

$$\delta_{\Psi_1}\|A\|^2 = 2\langle \delta\Psi_1, D_1 \rangle. \tag{48}$$

11.3.2 Variation with Respect to Ψ_2

Analogously,

$$\delta_{\Psi_2} A = (\Psi_1 \delta\Psi_2)\Psi_3 - \Psi_1(\delta\Psi_2\Psi_3), \tag{49}$$

and hence

$$\delta_{\Psi_2}\|A\|^2 = 2\langle \delta\Psi_2, D_2 \rangle. \tag{50}$$

11.3.3 Variation with Respect to Ψ_3

Finally,

$$\delta_{\Psi_3} A = (\Psi_1\Psi_2)\delta\Psi_3 - \Psi_1(\Psi_2\delta\Psi_3), \tag{51}$$

so that

$$\delta_{\Psi_3} \|A\|^2 = 2\langle \delta\Psi_3, D_3\rangle. \tag{52}$$

The objects D_I are the non-associative backreaction operators of the three fields.

11.4 Resulting Field Equations

From $\delta S/\delta\Psi_I = 0$ one obtains, for $I = 1, 2, 3,$

$$\alpha_I \Box_g \Psi_I + \frac{\partial V_{\text{loc}}}{\partial \Psi_I} + 2\lambda D_I + \xi R \Psi_I = 0. \tag{53}$$

These are the three octonionic matter equations.

12 The Complete Master Equation

We can now write the full system in closed form.

12.1 Gravitational Equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T^{\text{tot}}_{\mu\nu}. \tag{54}$$

12.2 Matter Equations

$$\alpha_1 \Box_g \Psi_1 + \frac{\partial V_{\text{loc}}}{\partial \Psi_1} + 2\lambda D_1 + \xi R \Psi_1 = 0, \tag{55}$$

$$\alpha_2 \Box_g \Psi_2 + \frac{\partial V_{\text{loc}}}{\partial \Psi_2} + 2\lambda D_2 + \xi R \Psi_2 = 0, \tag{56}$$

$$\alpha_3 \Box_g \Psi_3 + \frac{\partial V_{\text{loc}}}{\partial \Psi_3} + 2\lambda D_3 + \xi R \Psi_3 = 0. \tag{57}$$

12.3 Definition of the Non-Associative Source

$$A = (\Psi_1\Psi_2)\Psi_3 - \Psi_1(\Psi_2\Psi_3), \qquad \|A\|^2 = \langle A, A\rangle. \tag{58}$$

This is the complete master equation of the model.

13 Explanation of All Parameters and Quantities

13.1 Geometric Quantities

- M: four-dimensional spacetime manifold.
- x^μ: spacetime coordinates, with $\mu = 0, 1, 2, 3$.
- $g_{\mu\nu}$: Lorentzian spacetime metric.
- $g = \det(g_{\mu\nu})$: determinant of the metric.
- $\sqrt{-g}$: invariant volume factor.

- ∇_μ: covariant derivative associated with $g_{\mu\nu}$.
- $R_{\mu\nu}$: Ricci tensor.
- R: Ricci scalar.
- $G_{\mu\nu}$: Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{59}$$

13.2 Gravitational Parameters

- κ: gravitational coupling. In natural units, usually

$$\kappa = 8\pi G. \tag{60}$$

- G: Newton's gravitational constant.
- Λ: cosmological constant.

13.3 Octonionic Algebra

- $\mathbb{O}$: algebra of octonions.
- $1, e_1, \ldots, e_7$: standard basis of the octonions.
- $\bar{X}$: octonionic conjugate of X.
- $\mathrm{Re}(X)$: real part of X.
- $\langle X, Y \rangle$: real inner product on $\mathbb{O}$,

$$\langle X, Y \rangle = \mathrm{Re}(X\bar{Y}). \tag{61}$$

- $\|X\|^2$: octonionic norm,

$$\|X\|^2 = \langle X, X \rangle. \tag{62}$$

- $A(X, Y, Z)$: associator,

$$A(X, Y, Z) = (XY)Z - X(YZ). \tag{63}$$

13.4 Field Quantities

- $\Psi_I(x)$ with $I = 1, 2, 3$: three fundamental octonion-valued fields.
- $\psi_I^0, \psi_I^1, \ldots, \psi_I^7$: real components of the field Ψ_I.

13.5 Kinetic Parameters

- α_I: kinetic normalization constants of the fields. They determine the strength of the respective kinetic terms. For physical stability one typically chooses

$$\alpha_I > 0. \tag{64}$$

13.6 Potential Parameters

- m_I: mass scales of the fields Ψ_I.
- m_I^2: quadratic mass parameters in the local potential.
- β_I: quartic self-interaction parameters of the individual fields.
- γ_{IJ}: mixed couplings between two different fields.
- η: strength of an additional local cubic mixing term $\mathrm{Re}(\Psi_1\Psi_2\Psi_3)$.

13.7 Non-Associative Core Parameters

- λ: central non-associative coupling constant. It determines how strongly the associator energy enters the dynamics.
- $\lambda = 0$: no genuine non-associative dynamics.
- $\lambda > 0$: positive energetic weighting of non-associativity.
- A: shorthand for the dynamical associator,

$$A = A(\Psi_1, \Psi_2, \Psi_3). \tag{65}$$

- D_I: functional backreaction operators of the associator term on the respective field. Formally, they are the gradients of $\|A\|^2$ with respect to Ψ_I.

13.8 Curvature Coupling

- ξ: nonminimal curvature coupling.
- $\xi = 0$: minimal model.
- $\xi \neq 0$: direct coupling of the field norms to the Ricci scalar.

13.9 Differential Operators

- $\Box_g$: covariant d'Alembert operator,

$$\Box_g = \nabla_\mu \nabla^\mu. \tag{66}$$

14 Physical Meaning of Each Term

The full action contains five structurally distinct sectors.

14.1 Einstein-Hilbert Term

$$\frac{1}{2\kappa}(R - 2\Lambda) \tag{67}$$

describes the dynamics of spacetime geometry.

14.2 Kinetic Field Term

$$\sum_I \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle \tag{68}$$

describes the local propagation of the octonionic degrees of freedom.

14.3 Local Potential

$$V_{\text{loc}} \tag{69}$$

fixes masses, local self-interactions, and bilinear mixings.

14.4 Associator Term

$$\lambda \|A(\Psi_1, \Psi_2, \Psi_3)\|^2 \tag{70}$$

is the genuine core of the model. It encodes non-associativity and is the only term that cannot be reduced to an associative internal theory.

14.5 Nonminimal Curvature Coupling

$$\xi R \sum_I \langle \Psi_I, \Psi_I \rangle \tag{71}$$

couples the internal norm structure directly to spacetime curvature.

15 The Minimal Model in Its Clearest Form

For many analytical applications, the reduced form is most transparent. Set

$$\Lambda = 0, \qquad \xi = 0, \qquad \eta = 0, \qquad \gamma_{IJ} = 0, \qquad \beta_I = 0, \tag{72}$$

and additionally

$$\alpha_1 = \alpha_2 = \alpha_3 = 1, \qquad m_1 = m_2 = m_3 = m. \tag{73}$$

Then the action becomes

$$\begin{aligned} S = \int d^4x \sqrt{-g} \Bigg[\frac{R}{2\kappa} &+ \frac{1}{2} \sum_{I=1}^{3} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle \\ &- \frac{m^2}{2} \sum_{I=1}^{3} \langle \Psi_I, \Psi_I \rangle - \lambda \|A(\Psi_1, \Psi_2, \Psi_3)\|^2 \Bigg]. \end{aligned} \tag{74}$$

The field equations are then

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{75}$$

and

$$\Box_g \Psi_1 + m^2 \Psi_1 + 2\lambda D_1 = 0, \tag{76}$$

$$\Box_g \Psi_2 + m^2 \Psi_2 + 2\lambda D_2 = 0, \tag{77}$$

$$\Box_g \Psi_3 + m^2 \Psi_3 + 2\lambda D_3 = 0. \tag{78}$$

This form is the clearest representation of the master equation.

16 Why This System Can Be Called a Master Equation

An equation or system of equations deserves the name master equation if it contains the different effective sectors of physics as limits, projections, or reductions. This is fulfilled here in the following sense.

First, the metric sector directly contains Einsteinian dynamics.

Second, the octonionic sector contains an internal structure from which complex or quaternionic linear dynamics may arise by projection onto associative subalgebras.

Third, the associator generates a fundamental triple coupling, providing natural candidates for entanglement, decoherence, effective collapse dynamics, and cosmological additional energy components.

Fourth, the entire system is derived from a single action and is therefore formally closed.

In this sense, the equation derived here is not merely another field equation, but the central starting equation of the entire octonionic model.

17 Conclusion

Starting from simple physical minimal principles, we have shown step by step how one arrives at a consistent octonionic master action. The chain of reasoning is as follows:

1. Physics requires real observables, positive norms, and local dynamics.
2. These requirements lead to normed real division algebras.
3. Hurwitz's theorem reduces the possibilities to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.
4. The demand for genuine non-associativity singles out $\mathbb{O}$.
5. Non-associativity becomes visible only for three arguments; hence a triplet field model is minimal.
6. General covariance requires an action on curved spacetime.
7. The minimal new non-associative scalar is the associator norm $\|A\|^2$.
8. Variation of the total action yields a closed system consisting of the Einstein equation and three octonionic field equations.

Thus a mathematically clean derivation of the full master equation has been obtained. It is precise enough to serve as a basis for further derivations, including FLRW reduction, emergent Schrödinger dynamics, entanglement structures, decoherence, and cosmological applications.

18 Compact Final Form of the Master Equation

The most compact form of the master equation is

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \qquad \frac{\delta S}{\delta \Psi_I} = 0, \qquad I = 1, 2, 3, \tag{79}$$

with

$$\begin{aligned} S[g, \Psi_1, \Psi_2, \Psi_3] = \int d^4x \, \sqrt{-g} \, \Bigg[\frac{1}{2\kappa}(R - 2\Lambda) &+ \sum_{I=1}^{3} \frac{\alpha_I}{2} g^{\mu\nu} \left\langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \right\rangle \\ &- V_{\text{loc}}(\Psi_1, \Psi_2, \Psi_3) - \lambda \|(\Psi_1 \Psi_2)\Psi_3 - \Psi_1(\Psi_2 \Psi_3)\|^2 \\ &- \xi R \sum_{I=1}^{3} \langle \Psi_I, \Psi_I \rangle \Bigg]. \end{aligned} \tag{80}$$

This is the full master equation of the octonionic model in action-functional form.

A Derivation Chain from the Octonionic Master Action

This appendix summarizes how gravity, gauge theory, quantum mechanics, cosmology, probability, and the classical limit arise from the same octonionic variational structure. No new model is introduced; notation, fields, signs, and couplings are those of the master action.

A.1 Master Action

The starting point is

$$S[g,\Psi_I] = \int d^4x\,\sqrt{-g}\left[\frac{1}{2\kappa}(R-2\Lambda) + \sum_{I=1}^{3}\frac{\alpha_I}{2}g^{\mu\nu}\langle\nabla_\mu\Psi_I,\nabla_\nu\Psi_I\rangle \right.$$
$$\left. - V_{\mathrm{loc}}(\Psi_1,\Psi_2,\Psi_3) - \lambda^2\|A(\Psi_1,\Psi_2,\Psi_3)\|^2 - \xi R\sum_{I=1}^{3}\langle\Psi_I,\Psi_I\rangle\right], \tag{81}$$

with

$$A(\Psi_1,\Psi_2,\Psi_3) = (\Psi_1\Psi_2)\Psi_3 - \Psi_1(\Psi_2\Psi_3). \tag{82}$$

The minimal model is obtained for $\xi = 0$. Gauge and fermionic sectors appear after associative projection and may be included through

$$\mathcal{L}_{\mathrm{tot}} = \mathcal{L}_\Psi + \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{Dirac}} - V(\Psi) - \lambda^2\|A\|^2. \tag{83}$$

A.2 Metric Variation

Using

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}, \qquad \delta(\sqrt{-g}R) = \sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\Theta^\mu\right), \tag{84}$$

and discarding the boundary term, the Einstein–Hilbert variation gives

$$\delta S_{\mathrm{EH}} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,G_{\mu\nu}\delta g^{\mu\nu}. \tag{85}$$

The stress tensor is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\,\mathcal{L}_{\mathrm{tot}}\right). \tag{86}$$

For the octonionic scalar sector,

$$\mathcal{L}_\Psi = \sum_I \frac{\alpha_I}{2}g^{\rho\sigma}\langle\nabla_\rho\Psi_I,\nabla_\sigma\Psi_I\rangle, \tag{87}$$

one obtains

$$T^{(\Psi)}_{\mu\nu} = \sum_I \alpha_I\left[\langle\nabla_\mu\Psi_I,\nabla_\nu\Psi_I\rangle - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\langle\nabla_\rho\Psi_I,\nabla_\sigma\Psi_I\rangle\right]. \tag{88}$$

The potential and associator terms contribute

$$T^{(V)}_{\mu\nu} = -g_{\mu\nu}V(\Psi), \qquad T^{(A)}_{\mu\nu} = -g_{\mu\nu}\lambda^2\|A\|^2. \tag{89}$$

After projection, the gauge and Dirac sectors yield the standard expressions

$$T^{(\mathrm{gauge})}_{\mu\nu} = F^a_{\mu\rho}F_\nu{}^\rho{}_a - \frac{1}{4}g_{\mu\nu}F^a_{\rho\sigma}F^{\rho\sigma}_a, \tag{90}$$

and

$$T_{\mu\nu}^{(\text{Dirac})} = \frac{i}{2}\left[\bar{\psi}\gamma_{(\mu}D_{\nu)}\psi - (D_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right]. \tag{91}$$

Thus

$$T_{\mu\nu} = T_{\mu\nu}^{(\Psi)} + T_{\mu\nu}^{(V)} + T_{\mu\nu}^{(A)} + T_{\mu\nu}^{(\text{gauge})} + T_{\mu\nu}^{(\text{Dirac})}. \tag{92}$$

Stationarity for arbitrary $\delta g^{\mu\nu}$ gives

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}.} \tag{93}$$

The Bianchi identity implies

$$\nabla^{\mu}T_{\mu\nu} = 0, \tag{94}$$

consistent with the matter, gauge, and fermion equations. The associator behaves as an effective fluid,

$$T_{\mu\nu}^{(A)} = -\rho_A g_{\mu\nu}, \qquad \rho_A = \lambda^2\|A\|^2, \tag{95}$$

and therefore acts as a dynamical cosmological contribution. In the associative limit $A = 0$, standard Einstein gravity coupled to projected matter is recovered.

A.3 Octonionic Field Variation

The field-dependent action is

$$S_\Psi = \int d^4x\sqrt{-g}\left[\sum_I \frac{\alpha_I}{2} g^{\mu\nu}\langle\nabla_\mu\Psi_I, \nabla_\nu\Psi_I\rangle - V(\Psi) - \lambda^2\|A(\Psi_1, \Psi_2, \Psi_3)\|^2\right]. \tag{96}$$

The kinetic variation gives, after integration by parts,

$$\delta S_\Psi^{\text{kin}} = -\sum_I \alpha_I \int d^4x\sqrt{-g}\,\langle\delta\Psi_I, \nabla^\mu\nabla_\mu\Psi_I\rangle. \tag{97}$$

For the local potential,

$$\delta V = \sum_I \left\langle \frac{\partial V}{\partial \Psi_I}, \delta\Psi_I \right\rangle. \tag{98}$$

The nontrivial term is the associator variation. Since

$$\delta A = A(\delta\Psi_1, \Psi_2, \Psi_3) + A(\Psi_1, \delta\Psi_2, \Psi_3) + A(\Psi_1, \Psi_2, \delta\Psi_3), \tag{99}$$

one has

$$\delta\|A\|^2 = 2\langle A, \delta A\rangle = 2\sum_I \langle \mathcal{D}_I A, \delta\Psi_I\rangle, \tag{100}$$

where $\mathcal{D}_I A$ denotes the adjoint associator derivative with respect to Ψ_I. Schematically,

$$\mathcal{D}_1 A \sim (\Psi_2\Psi_3)^\dagger A - \Psi_2^\dagger(\Psi_3^\dagger A), \tag{101}$$

$$\mathcal{D}_2 A \sim (\Psi_1\Psi_3)^\dagger A - \Psi_1^\dagger(\Psi_3^\dagger A), \tag{102}$$

$$\mathcal{D}_3 A \sim (\Psi_1\Psi_2)^\dagger A - \Psi_1^\dagger(\Psi_2^\dagger A). \tag{103}$$

Collecting all terms gives

$$\delta S_\Psi = \int d^4x\sqrt{-g}\sum_I \left\langle \delta\Psi_I, -\alpha_I\nabla^\mu\nabla_\mu\Psi_I - \frac{\partial V}{\partial\Psi_I} - 2\lambda^2\mathcal{D}_I A \right\rangle. \tag{104}$$

Hence

$$\boxed{\alpha_I \nabla^\mu \nabla_\mu \Psi_I + \frac{\partial V}{\partial \Psi_I} + 2\lambda^2 \mathcal{D}_I A = 0.} \tag{105}$$

If the fields lie in an associative subalgebra, then $A = 0$ and $\mathcal{D}_I A = 0$, reducing the equations to ordinary scalar field equations. For a small deviation

$$\Psi_I = \psi_I + \epsilon \chi_I, \qquad \psi_I \in \mathcal{A}, \tag{106}$$

the associator force is $\mathcal{O}(\epsilon)$ and produces controlled corrections to the projected dynamics.

A.4 Compact Master System

With

$$\Phi = (g_{\mu\nu}, \Psi_I, A_\mu, \psi), \tag{107}$$

the projected master action can be written as

$$S[\Phi] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + \mathcal{L}_\Psi + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Dirac}} - V(\Psi) - \lambda^2 \|A\|^2 \right], \tag{108}$$

with

$$\mathcal{L}_\Psi = \sum_I \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle, \tag{109}$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \tag{110}$$

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{111}$$

The corresponding equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{112}$$

$$\alpha_I \nabla^\mu \nabla_\mu \Psi_I + \frac{\partial V}{\partial \Psi_I} + 2\lambda^2 \mathcal{D}_I A = 0, \tag{113}$$

$$D_\mu F^{\mu\nu} = J^\nu, \tag{114}$$

$$(i\gamma^\mu D_\mu - m)\psi = 0. \tag{115}$$

They admit the schematic unified form

$$\boxed{\mathcal{D}\Phi + \mathcal{N}(\Phi) + \mathcal{A}(\Phi) = 0,} \tag{116}$$

where $\mathcal{D}$ contains kinetic operators, $\mathcal{N}$ standard nonlinearities, and

$$\mathcal{A}(\Phi) \sim \lambda^2 \frac{\delta}{\delta \Phi} \|A(\Psi_1, \Psi_2, \Psi_3)\|^2 \tag{117}$$

contains all non-associative corrections. Thus deviations from standard physics scale as

$$\Delta \mathcal{O} \sim \lambda^2 \langle \|A\|^2 \rangle. \tag{118}$$

A.5 Associative Projection Principle

The octonions are non-associative,

$$A(x, y, z) = (xy)z - x(yz), \tag{119}$$

but contain associative subalgebras

$$\mathbb{R}, \qquad \mathbb{C}, \qquad \mathbb{H} \subset \mathbb{O}. \tag{120}$$

For every associative subalgebra $\mathcal{A} \subset \mathbb{O}$,

$$A(x, y, z) = 0 \qquad \forall x, y, z \in \mathcal{A}. \tag{121}$$

Define a norm-compatible projection

$$\Pi_{\mathcal{A}} : \mathbb{O} \to \mathcal{A}, \qquad \Psi \mapsto \psi = \Pi_{\mathcal{A}}(\Psi), \qquad \|\psi\|^2 \leq \|\Psi\|^2. \tag{122}$$

Then the projected sector satisfies

$$A(\psi, \psi, \psi) = 0, \tag{123}$$

and the action reduces to

$$S_{\text{eff}}[\psi] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + \mathcal{L}_{\text{assoc}}(\psi) \right]. \tag{124}$$

The hierarchy is

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C} \longrightarrow \mathbb{R}, \tag{125}$$

corresponding to

$$\text{non-associativity} \to \text{associativity} \to \text{commutativity} \to \text{classical observables}. \tag{126}$$

The associator term dynamically selects such sectors:

$$S_A = \lambda^2 \|A(\Psi_1, \Psi_2, \Psi_3)\|^2, \qquad A \approx 0 \tag{127}$$

is favored at low energy. For $\Psi = \psi + \delta\Psi$, $\psi \in \mathcal{A}$,

$$A(\Psi, \Psi, \Psi) = A(\psi, \psi, \delta\Psi) + \mathcal{O}((\delta\Psi)^2), \tag{128}$$

so departures from $\mathcal{A}$ are suppressed by

$$\delta S \sim \lambda^2 \|A(\psi, \psi, \delta\Psi)\|^2. \tag{129}$$

Thus observed physics is the associative projection of a non-associative fundamental theory:

$$\text{physics} = \Pi_{\mathcal{A}}(\text{octonionic dynamics}). \tag{130}$$

A.6 Electromagnetic Sector

Electromagnetism arises by restricting the quaternionic gauge sector to a complex subalgebra,

$$\mathbb{C} \subset \mathbb{H} \subset \mathbb{O}. \tag{131}$$

The commutator vanishes,

$$[T, T] = 0, \tag{132}$$

and the gauge group reduces to $U(1)$. For a charged field ψ,

$$D_\mu \psi = (\partial_\mu + iqA_\mu)\psi. \tag{133}$$

The curvature follows from

$$[D_\mu, D_\nu]\psi = iq(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi, \tag{134}$$

hence

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{135}$$

The Maxwell action

$$S_{\text{EM}} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{136}$$

varies as

$$\delta S_{\text{EM}} = \int d^4x\, (\partial_\mu F^{\mu\nu})\delta A_\nu, \tag{137}$$

which yields

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu.} \tag{138}$$

The Bianchi identity

$$\partial_{[\lambda}F_{\mu\nu]} = 0 \tag{139}$$

gives the homogeneous Maxwell equations. Matter currents are

$$J^\mu = iq\left(\psi^* D^\mu \psi - \psi(D^\mu\psi)^*\right) \tag{140}$$

for a scalar field and

$$J^\mu = q\bar{\psi}\gamma^\mu\psi \tag{141}$$

for fermions. Gauge invariance,

$$\psi \to e^{iq\alpha(x)}\psi, \qquad A_\mu \to A_\mu - \partial_\mu\alpha, \tag{142}$$

is inherited from phase rotations in $\mathbb{C} \subset \mathbb{O}$. Residual non-associativity may induce

$$\partial_\mu F^{\mu\nu} = J^\nu + \epsilon\mathcal{A}^\nu(\Psi), \tag{143}$$

allowing small nonlinear, birefringent, or high-field corrections.

A.7 Yang–Mills Sector

A quaternionic projection selects

$$\mathbb{H} \subset \mathbb{O}, \tag{144}$$

eliminating the associator while preserving non-commutativity. With imaginary generators e_a, $a = 1, 2, 3$,

$$e_a e_b = -\delta_{ab} + f_{ab}{}^c e_c, \qquad [e_a, e_b] = 2f_{ab}{}^c e_c, \tag{145}$$

isomorphic to $\mathfrak{su}(2)$. Introducing

$$A_\mu = A_\mu^a T_a, \qquad [T_a, T_b] = f_{ab}{}^c T_c, \tag{146}$$

the covariant derivative is

$$D_\mu\psi = \partial_\mu\psi + A_\mu\psi. \tag{147}$$

The field strength follows from

$$F_{\mu\nu}\psi = [D_\mu, D_\nu]\psi, \tag{148}$$

namely

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{149}$$

or

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu. \tag{150}$$

The Yang–Mills action

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x \, F^a_{\mu\nu} F^{\mu\nu}_a \tag{151}$$

has variation

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu, \tag{152}$$

so after integration by parts

$$\delta S_{\text{YM}} = \int d^4x \, \delta A^a_\nu (D_\mu F^{\mu\nu})_a. \tag{153}$$

Therefore

$$\boxed{D_\mu F^{\mu\nu} = J^\nu.} \tag{154}$$

Gauge theory thus emerges because associative projection ensures closure of commutators.

A.8 Schrödinger Equation as Nonrelativistic Projection Limit

After projection $\mathbb{O} \to \mathbb{C}$, an octonionic scalar becomes a complex field ψ. The relativistic projected equation is

$$(\Box + m^2)\psi = 0, \qquad \Box = \partial_t^2 - \nabla^2. \tag{155}$$

Factor out the rest energy,

$$\psi(x,t) = e^{-imt} \phi(x,t), \tag{156}$$

so that

$$\partial_t^2 \psi = e^{-imt} \left(\partial_t^2 \phi - 2im\partial_t \phi - m^2 \phi \right). \tag{157}$$

Insertion gives

$$\partial_t^2 \phi - 2im\partial_t \phi - \nabla^2 \phi = 0. \tag{158}$$

In the nonrelativistic regime,

$$|\partial_t^2 \phi| \ll m |\partial_t \phi|, \tag{159}$$

hence

$$i\partial_t \phi = -\frac{1}{2m} \nabla^2 \phi. \tag{160}$$

Including a projected potential gives

$$\boxed{i\partial_t \phi = \left(-\frac{1}{2m} \nabla^2 + V(x) \right) \phi.} \tag{161}$$

The Hilbert inner product,

$$\langle \phi_1, \phi_2 \rangle = \int d^3x \, \phi_1^*(x) \phi_2(x), \tag{162}$$

is inherited from the octonionic norm $\|\Psi\|^2 = \Psi\bar{\Psi}$. Thus quantum linearity arises from complex associative projection, while deviations are expected only when residual associator effects become non-negligible.

A.9 Dirac Equation as Linearized Relativistic Projection

Projection to

$$\mathbb{H} \subset \mathbb{O} \tag{163}$$

gives a quaternionic field, which may be represented as a spinor through

$$\mathbb{H} \simeq \mathbb{C}^2. \tag{164}$$

The projected relativistic equation is

$$(\Box + m^2)\psi = 0. \tag{165}$$

To linearize, seek matrices γ^μ satisfying

$$(i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu + m) = \Box + m^2. \tag{166}$$

This requires the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}. \tag{167}$$

Associativity of $\mathbb{H}$ ensures closure of the projected spinor algebra. The first-order equation is therefore

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0.} \tag{168}$$

Gauge coupling follows by

$$\partial_\mu \to D_\mu = \partial_\mu + A_\mu, \tag{169}$$

so

$$(i\gamma^\mu D_\mu - m)\psi = 0. \tag{170}$$

With

$$\bar{\psi} = \psi^\dagger \gamma^0, \tag{171}$$

one obtains the conserved current

$$j^\mu = \bar{\psi}\gamma^\mu \psi, \qquad \partial_\mu j^\mu = 0. \tag{172}$$

The chain is

$$\mathbb{O} \to \mathbb{H} \to \text{Clifford linearization} \to \text{Dirac equation}. \tag{173}$$

Residual corrections may take the schematic form

$$i\gamma^\mu \partial_\mu \psi - m\psi + \epsilon \mathcal{A}(\Psi, \Psi, \Psi) = 0. \tag{174}$$

A.10 Cosmological Reduction

For a spatially flat FLRW metric,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^{\,2}, \tag{175}$$

and homogeneous fields

$$\Psi_I(x) \to \Psi_I(t), \tag{176}$$

the kinetic term becomes

$$g^{\mu\nu}\langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle = -\langle \dot{\Psi}_I, \dot{\Psi}_I \rangle. \tag{177}$$

The effective energy density and pressure are

$$\rho = \sum_I \frac{\alpha_I}{2}\langle \dot{\Psi}_I, \dot{\Psi}_I \rangle + V(\Psi) + \lambda^2 \|A\|^2, \tag{178}$$

$$p = \sum_I \frac{\alpha_I}{2}\langle \dot{\Psi}_I, \dot{\Psi}_I \rangle - V(\Psi) - \lambda^2 \|A\|^2. \tag{179}$$

The Friedmann equations are

$$H^2 = \frac{\kappa}{3}\rho, \qquad \dot{H} = -\frac{\kappa}{2}(\rho + p), \qquad H = \frac{\dot{a}}{a}. \tag{180}$$

Defining

$$\rho_A := \lambda^2 \|A(\Psi_1, \Psi_2, \Psi_3)\|^2, \tag{181}$$

one obtains

$$H^2 = \frac{\kappa}{3}(\rho_{\text{kin}} + \rho_V + \rho_A). \tag{182}$$

A phenomenological scaling following from the field dynamics is

$$\rho_A(z) = \rho_{A,0}(1+z)^\beta e^{-\gamma z}, \tag{183}$$

hence

$$\boxed{H^2(z) = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_\Lambda + \Omega_A (1+z)^\beta e^{-\gamma z}\right].} \tag{184}$$

For $z \gg 1$ and $\gamma > 0$, ρ_A is suppressed and ΛCDM is recovered; for $z \sim 0$, ρ_A acts as a late-time dark-energy-like contribution. The observable quantity

$$E(z)^2 = \frac{H^2(z)}{H_0^2} \tag{185}$$

enters supernova, BAO, and CMB tests.

A.11 Born Rule from Norm Projection

The octonionic norm is

$$\|\Psi\|^2 = \Psi\bar{\Psi}, \tag{186}$$

with multiplicativity

$$\|\Psi_1 \Psi_2\|^2 = \|\Psi_1\|^2 \|\Psi_2\|^2. \tag{187}$$

Choose a complex subalgebra

$$\mathbb{C} \subset \mathbb{O}, \qquad \Psi = \psi + \chi, \qquad \psi \in \mathbb{C}, \quad \chi \perp \mathbb{C}. \tag{188}$$

Projection gives

$$\|\Psi\|^2 \to |\psi|^2 = \psi^* \psi. \tag{189}$$

For

$$\psi = c_1 \psi_1 + c_2 \psi_2, \tag{190}$$

the norm is

$$|\psi|^2 = |c_1|^2 |\psi_1|^2 + |c_2|^2 |\psi_2|^2 + c_1^* c_2 \psi_1^* \psi_2 + c_2^* c_1 \psi_2^* \psi_1, \tag{191}$$

so interference follows from the quadratic norm. For a measurement basis,

$$\psi = \sum_n c_n \psi_n, \tag{192}$$

the outcome weight is

$$\boxed{P_n = \frac{|c_n|^2}{\sum_k |c_k|^2}.} \tag{193}$$

The quadratic rule is fixed by positivity, norm multiplicativity, phase invariance, and additivity under orthogonal decomposition:

$$P \propto |\psi|^2. \tag{194}$$

Residual components may induce

$$\|\Psi\|^2 = |\psi|^2 + \epsilon \Delta(\psi, \chi), \tag{195}$$

suggesting possible Born-rule deviations when the associator sector is not fully suppressed.

A.12 Thermodynamic Limit

For N octonionic degrees of freedom,

$$\Psi^{(N)} = (\Psi_1, \ldots, \Psi_N), \qquad \Psi_i \in \mathbb{O}, \tag{196}$$

norm multiplicativity gives

$$\|\Psi^{(N)}\|^2 = \prod_{i=1}^{N} \|\Psi_i\|^2 \to \prod_{i=1}^{N} |\psi_i|^2 \tag{197}$$

after complex projection. Taking the logarithm,

$$\log \|\Psi^{(N)}\|^2 = \sum_{i=1}^{N} \log |\psi_i|^2, \tag{198}$$

turns multiplicative microscopic weights into additive macroscopic quantities. This motivates

$$P[\psi] \propto \prod_{i=1}^{N} |\psi_i|^2, \tag{199}$$

with entropy

$$S = -\sum_{\{\psi\}} P[\psi] \log P[\psi] \propto N. \tag{200}$$

Associator-mediated interactions,

$$A(\Psi_i, \Psi_j, \Psi_k), \tag{201}$$

suppress coherence between macroscopically distinct configurations, so reduced density matrices become effectively diagonal,

$$\rho \to \mathrm{diag}(P_n), \qquad P_n = |c_n|^2. \tag{202}$$

In the limit

$$N \to \infty, \tag{203}$$

the law of large numbers gives

$$\frac{1}{N} \log \|\Psi^{(N)}\|^2 \to \langle \log |\psi|^2 \rangle, \qquad \Delta \sim \frac{1}{\sqrt{N}} \to 0. \tag{204}$$

Thus quantum superpositions become classical mixtures, interference is suppressed, and deterministic macroscopic dynamics emerges. With

$$Z = \sum_{\{\psi\}} e^{-\beta E[\psi]}, \qquad F = -\frac{1}{\beta} \log Z, \tag{205}$$

classical equations follow from minimizing F. The associator contribution

$$E_A \sim \lambda^2 \|A(\Psi_i, \Psi_j, \Psi_k)\|^2 \tag{206}$$

drives decoherence, selects pointer states, and supports entropy increase.

A.13 Summary of the Emergence Chain

The complete structure is

$$\Psi(x) \in \mathbb{O}, \qquad A(x,y,z) = (xy)z - x(yz), \tag{207}$$

with dynamics governed by

$$S[\Psi, g] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + \mathcal{L}_\Psi - V(\Psi) - \lambda^2 \|A(\Psi_1, \Psi_2, \Psi_3)\|^2 \right]. \tag{208}$$

Observable physics arises by

$$\Psi \to \psi = \Pi_{\mathcal{A}}(\Psi), \qquad \mathcal{A} \subset \mathbb{O}, \qquad A(\psi, \psi, \psi) = 0. \tag{209}$$

The hierarchy is

$$\boxed{\mathbb{O} \xrightarrow{\text{projection}} \mathbb{H} \xrightarrow{\text{projection}} \mathbb{C} \xrightarrow{\text{limit}} \mathbb{R}.} \tag{210}$$

Accordingly,

$$\mathbb{H} \to \text{Yang–Mills theory and Dirac fermions}, \tag{211}$$

$$\mathbb{C} \to \text{Schrödinger dynamics, Hilbert space, Born rule}, \tag{212}$$

$$N \to \infty \to \text{decoherence, thermodynamics, classical physics}. \tag{213}$$

Metric variation yields

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{214}$$

while the associator contributes

$$\rho_A = \lambda^2 \|A\|^2. \tag{215}$$

In cosmology,

$$H^2(z) = H_0^2 \left[\Omega_m (1+z)^3 + \Omega_\Lambda + \Omega_A (1+z)^\beta e^{-\gamma z} \right]. \tag{216}$$

All deviations are controlled by

$$\Delta \mathcal{O} \sim \lambda^2 \langle \|A\|^2 \rangle. \tag{217}$$

Thus the logical synthesis is

$$\boxed{\text{All known physics} = \Pi_{\mathcal{A}}\big(\text{octonionic master dynamics}\big) + \text{scaling limits.}} \tag{218}$$

Non-associativity is fundamental, associativity is emergent, and observed physical laws are effective projected descriptions.

B Reproducible Test Strategy for the Octonionic Master Equation

This appendix provides precise, reproducible, and quantitatively defined test protocols for three independent experimental pillars:

1. cosmology, as the primary and statistically dominant test;
2. black-hole imaging, especially Event Horizon Telescope precision observables;
3. quantum experiments, especially decoherence and collapse dynamics.

The goal is strict data-driven falsifiability across widely separated physical scales. A viable signal must appear coherently in more than one sector and must not be explainable by a single instrumental, astrophysical, or environmental systematic.

B.1 Common Null Hypothesis and Octonionic Deformation

The reference null model is the standard associative limit,

$$A(\Psi_1, \Psi_2, \Psi_3) = 0, \tag{219}$$

which reduces the theory to the corresponding standard effective model: ΛCDM in cosmology, Kerr/GR plus plasma astrophysics for black holes, and ordinary environmental decoherence for quantum experiments.

The octonionic signal is controlled by a dimensionless associator parameter ϵ_A or, in the cosmological sector, by a redshift-dependent effective density fraction $\Omega_A f_A(z)$. A useful benchmark form is

$$E^2(z) = \Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_A(1+z)^\beta e^{-\gamma z}, \tag{220}$$

with

$$E(z) = \frac{H(z)}{H_0}. \tag{221}$$

The model is falsified if the posterior is statistically consistent with

$$\Omega_A = 0 \tag{222}$$

and if the addition of Ω_A, β, γ is not favored by information criteria or Bayesian evidence.

B.2 Pillar I: Cosmology as the Primary Test

Cosmology is the statistically dominant pillar because it uses large correlated datasets: CMB, BAO, supernovae, and cosmic chronometers. Current DESI DR2 cosmology products include BAO results based on the first three years of DESI data, with public chains and best-fit products released in 2025. :contentReference[oaicite:0]index=0

B.2.1 Observable Predictions

The primary observables are

$$H(z) = H_0 E(z), \tag{223}$$

$$d_L(z) = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{E(z')}, \tag{224}$$

$$D_M(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}, \tag{225}$$

$$D_H(z) = \frac{c}{H(z)}. \tag{226}$$

The supernova distance modulus is

$$\mu_{\rm th}(z) = 5\log_{10}\left(\frac{d_L(z)}{\rm Mpc}\right) + 25 + M_B. \tag{227}$$

The BAO predictions are

$$\frac{D_M(z)}{r_d}, \qquad \frac{D_H(z)}{r_d}, \qquad \frac{D_V(z)}{r_d}, \tag{228}$$

where

$$D_V(z) = \left[z D_M^2(z) D_H(z)\right]^{1/3}. \tag{229}$$

B.2.2 Likelihood

The combined likelihood is

$$\ln \mathcal{L}_{\text{tot}} = \ln \mathcal{L}_{\text{SN}} + \ln \mathcal{L}_{\text{BAO}} + \ln \mathcal{L}_{\text{CMB}} + \ln \mathcal{L}_{H(z)}. \tag{230}$$

For a data vector $\mathbf{d}$ and theory vector $\mathbf{t}(\theta)$,

$$\chi^2(\theta) = [\mathbf{d} - \mathbf{t}(\theta)]^T C^{-1} [\mathbf{d} - \mathbf{t}(\theta)], \tag{231}$$

and

$$\ln \mathcal{L} = -\frac{1}{2}\chi^2. \tag{232}$$

The parameter vector is

$$\theta = \{H_0, \Omega_m, \Omega_b, \Omega_\Lambda, r_d, M_B, \Omega_A, \beta, \gamma\}. \tag{233}$$

A practical prior choice is

$$\begin{aligned} H_0 &\in [60, 80]\ \mathrm{km\,s^{-1}\,Mpc^{-1}}, && (234)\\ \Omega_m &\in [0.20, 0.40], && (235)\\ \Omega_A &\in [0, 0.15], && (236)\\ \beta &\in [-5, 5], && (237)\\ \gamma &\in [0, 20]. && (238) \end{aligned}$$

B.2.3 Decision Criteria

The octonionic model is supported only if

$$\Delta\chi^2 = \chi^2_{\Lambda\text{CDM}} - \chi^2_{\text{oct}} \tag{239}$$

is large enough to compensate for the additional parameters.

Use

$$\begin{aligned} \text{AIC} &= \chi^2_{\min} + 2k, && (240)\\ \text{BIC} &= \chi^2_{\min} + k \ln N, && (241) \end{aligned}$$

where k is the number of fitted parameters and N the number of data points.

A robust detection requires

$$\Delta\text{AIC} < -6, \qquad \Delta\text{BIC} < -6, \tag{242}$$

where

$$\Delta\text{AIC} = \text{AIC}_{\text{oct}} - \text{AIC}_{\Lambda\text{CDM}}, \tag{243}$$

and similarly for BIC.

The primary falsification condition is

$$\Omega_A = 0 \tag{244}$$

within the 95% credible interval, together with no improvement in AIC/BIC.

B.3 Pillar II: Black-Hole Imaging with EHT

The black-hole test probes whether the same associator sector modifies strong-field geometry. The Event Horizon Telescope has shown a persistent M87* shadow between 2017 and 2018 observations, with comparable ring diameter and a changed brightness orientation; later EHT work also connects horizon-scale structure to jet-launching regions. :contentReference[oaicite:1]index=1

B.3.1 Effective Metric Deformation

Use the phenomenological strong-field metric

$$ds^2 = -f_A(r)dt^2 + \frac{dr^2}{f_A(r)} + r^2 d\Omega^2, \tag{245}$$

with

$$f_A(r) = 1 - \frac{2GM}{c^2 r} + \epsilon_A \left(\frac{r_g}{r}\right)^p, \qquad r_g = \frac{GM}{c^2}. \tag{246}$$

The photon-sphere radius follows from

$$r f_A'(r) - 2 f_A(r) = 0. \tag{247}$$

The predicted angular shadow diameter is

$$\theta_{\rm sh} = \frac{2b_{\rm ph}}{D}, \tag{248}$$

where

$$b_{\rm ph} = \frac{r_{\rm ph}}{\sqrt{f_A(r_{\rm ph})}}. \tag{249}$$

B.3.2 EHT Observable Vector

Fit the vector

$$\mathbf{O}_{\rm EHT} = \{\theta_{\rm ring}, d_{\rm ring}, \Delta_{\rm asym}, P_{\rm pol}, \chi_{\rm EVPA}, V(u,v)\}, \tag{250}$$

where $\theta_{\rm ring}$ is the ring diameter, $d_{\rm ring}$ the ring width, $\Delta_{\rm asym}$ the brightness asymmetry, $P_{\rm pol}$ the polarization fraction, $\chi_{\rm EVPA}$ the polarization angle field, and $V(u,v)$ the interferometric visibility data.

The likelihood is

$$\ln \mathcal{L}_{\rm EHT} = -\frac{1}{2}\left[\mathbf{O}_{\rm obs} - \mathbf{O}_{\rm th}(\epsilon_A, p, M, D, a_*, i, \mathcal{P})\right]^T C_{\rm EHT}^{-1} \left[\mathbf{O}_{\rm obs} - \mathbf{O}_{\rm th}(\epsilon_A, p, M, D, a_*, i, \mathcal{P})\right], \tag{251}$$

where $\mathcal{P}$ denotes plasma and emission-model nuisance parameters.

B.3.3 Black-Hole Falsification Criterion

The model is ruled out in the black-hole sector if

$$\epsilon_A = 0 \tag{252}$$

is preferred, or if nonzero ϵ_A improves one source only by absorbing plasma systematics but fails jointly for M87* and Sgr A*.

A genuine octonionic signal must satisfy

$$\epsilon_A^{\rm M87^*} \approx \epsilon_A^{\rm SgrA^*} \tag{253}$$

after scaling by the dimensionless compactness variable r/r_g.

B.4 Pillar III: Quantum Decoherence and Collapse

The quantum pillar tests whether associator dynamics produces a residual intrinsic decoherence or collapse rate beyond environmental decoherence. Recent experimental and theoretical work continues to refine optomechanical and macroscopic-superposition protocols for distinguishing ordinary decoherence from intrinsic collapse dynamics. :contentReference[oaicite:2]index=2

B.4.1 Effective Master Equation

The standard density-matrix evolution is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \mathcal{D}_{\text{env}}[\rho]. \tag{254}$$

The octonionic extension adds

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \mathcal{D}_{\text{env}}[\rho] + \mathcal{D}_A[\rho]. \tag{255}$$

A minimal phenomenological form is

$$\mathcal{D}_A[\rho] = -\Gamma_A(\Delta x, m, \tau_A)[x,[x,\rho]], \tag{256}$$

with

$$\Gamma_A = \Gamma_0 \left(\frac{m}{m_0}\right)^{\alpha} \left(\frac{\Delta x}{x_0}\right)^{2} F_A(\tau_A). \tag{257}$$

The experimentally measured interference visibility is then

$$\mathcal{V}(t) = \mathcal{V}_0 \exp[-\Gamma_{\text{env}} t] \exp[-\Gamma_A t]. \tag{258}$$

B.4.2 Experimental Protocol

A reproducible test requires:

1. prepare a spatial or mechanical superposition with controlled mass m and separation Δx;
2. independently calibrate environmental decoherence:

$$\Gamma_{\text{env}} = \Gamma_{\text{gas}} + \Gamma_{\text{blackbody}} + \Gamma_{\text{vibration}} + \Gamma_{\text{technical}}; \tag{259}$$

3. measure visibility decay $\mathcal{V}(t)$ for several masses, separations, temperatures, and pressures;
4. fit both models:

$$M_0 : \Gamma_A = 0, \tag{260}$$

$$M_A : \Gamma_A \neq 0; \tag{261}$$

5. check whether the residual rate scales with m and Δx as predicted by Γ_A rather than with pressure, temperature, or technical noise.

B.4.3 Quantum Falsification Criterion

The octonionic collapse sector is falsified if

$$\Gamma_A = 0 \tag{262}$$

within experimental uncertainty, or if the residual decoherence scales with environmental variables rather than with the intrinsic variables

$$m, \qquad \Delta x, \qquad \tau_A. \tag{263}$$

A positive signal requires

$$\frac{\partial \Gamma_{\text{res}}}{\partial P} \approx 0, \qquad \frac{\partial \Gamma_{\text{res}}}{\partial T} \approx 0, \qquad \frac{\partial \Gamma_{\text{res}}}{\partial m} \neq 0, \qquad \frac{\partial \Gamma_{\text{res}}}{\partial (\Delta x)} \neq 0. \tag{264}$$

B.5 Cross-Scale Consistency Condition

The decisive feature of the program is not a single anomaly. The model predicts that the same underlying associator sector appears in three effective limits:

$$\Omega_A \neq 0 \quad \text{in cosmology,} \tag{265}$$

$$\epsilon_A \neq 0 \quad \text{in black-hole imaging,} \tag{266}$$

$$\Gamma_A \neq 0 \quad \text{in quantum decoherence.} \tag{267}$$

The strongest consistency test is therefore the joint likelihood

$$\ln \mathcal{L}_{\text{joint}} = \ln \mathcal{L}_{\text{cosmo}} + \ln \mathcal{L}_{\text{EHT}} + \ln \mathcal{L}_{\text{quantum}}. \tag{268}$$

The model survives only if the inferred parameters are compatible with one common associator scale:

$$\Lambda_A^{\text{cosmo}} \sim \Lambda_A^{\text{BH}} \sim \Lambda_A^{\text{quantum}}, \tag{269}$$

after applying the appropriate renormalization and projection maps.

B.6 Final Falsifiability Statement

The octonionic master-equation model is falsifiable in a strict sense. It is excluded if the combined posterior satisfies

$$P(\Omega_A = 0, \epsilon_A = 0, \Gamma_A = 0 \mid \text{data}) \approx 1, \tag{270}$$

or if nonzero signals appear only in one sector and fail the cross-scale consistency condition.

Conversely, a serious empirical indication would require:

1. a statistically significant late-time cosmological deformation beyond ΛCDM;
2. a strong-field shadow or polarization residual not removable by Kerr plasma nuisance parameters;
3. a residual quantum decoherence/collapse rate with intrinsic mass–separation scaling;
4. a common associator scale linking all three.

Only the simultaneous satisfaction of these conditions would constitute evidence for the octonionic master equation.

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