A Geometric Model of Electromagnetic Interaction

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April 28, 2026

Abstract

This paper proposes a geometric model of electromagnetic interaction in which elementary particles are treated as localized spherical regions of spatial curvature. The main local characteristic of the model is the intensity of volumetric curvature of space,

$$\Delta K_v = 1 - \alpha^3,$$

where α is the coefficient of linear curvature of space relative to the basic, unperturbed state. Inside each spherical region the volumetric curvature intensity is assumed to be constant, while outside the region the distribution remains spherically symmetric and satisfies a global compensation law over all space.

A geometric charge is introduced as an integral characteristic of the external flux of spatial curvature, that is, as the integral of the divergence of the corresponding vector field over the external region. The interaction energy of two such sources is constructed through a bilinear potential in the parameter spaces of the two charges. This makes it possible to obtain the leading Coulomb term of the force and its geometric generalization. It is shown that long-range interaction is determined not by the full volumetric curvature intensity itself, but by a quantity proportional to the change in the Gaussian curvature of the boundary of the deformed region.

On this basis, geometric formulas are derived for electric and magnetic-type interaction forces, as well as expressions for the creation energies of the electron and the proton, interpreted respectively as the energies required to compress and stretch space. The resulting formulas for radii and creation energies reproduce the classical electron radius in the linear approximation and set the correct scale for the proton radius and mass. Thus, a self-consistent geometric formalism is proposed in which electric charge, interaction energy, and particle mass arise as consequences of local curvature of space.

1 Introduction

The idea of a geometric origin of electric charge has a long tradition. In the theories of Kaluza and Klein [1, 2], electromagnetism appears as a consequence of the geometry of an additional dimension. In the teleparallel formalism with torsion discussed by Unzicker [3], electric charge is related to geometric and topological properties of spacetime. In Wheeler's geometrodynamics [4, 5], attempts were made to describe charge through topological structures such as geons and wormholes, whereas in the model of space as a tessellated lattice proposed by Krasnoholovets [6, 7], charge is interpreted as a local deformation of the cellular structure of space.

Particularly close in physical intuition are the works of Shlomo Barak [9, 8, 10], in which electric charge is treated as a deformation of space itself: positive charge is associated with compression and negative charge with expansion. The field of the charge is then interpreted as an extension of this deformation into the external region. This initial idea is consonant with the approach developed in the present work.

The approach proposed here, however, differs in an essential way from the above geometric models. Electromagnetic interaction is described without introducing additional dimensions in the main static construction, without using nontrivial spacetime topology, and without treating torsion as an independent fundamental entity. Instead, a tensor parameter space for two sources is introduced, and the force of interaction is defined as a bilinear measure of correlation between two distributions of volumetric curvature in the parameter spaces dV_1 and dV_2. Thus, the dynamics of interaction is related not to a modification of the basic physical space itself, but to the structure of its tensor product with itself.

A key element of the formalism is the definition of geometric charge as the total flux of a vector field of spatial curvature. This charge is computed as the integral of the divergence of the external radial field over the whole space outside the fundamental spherical region. This construction makes it possible to introduce charge as a geometric integral characteristic of the curvature distribution, rather than as a primary axiomatic quantity.

To compute the interaction energy of two such sources, a bilinear potential of interaction is used, whose mathematical structure is related to the classical Newtonian theory of potential [11, 12]. In this bilinear form the role of gravitational masses is played by geometric charges, and the force itself appears as the gradient of the potential in the parameter space of the two charges. This allows one to obtain the long-range Coulomb part of the interaction from geometric premises.

The proposed formalism has the following properties:

1. it is mathematically closed and allows direct analytical calculations;
2. it permits an expansion of the interaction force in inverse powers of the distance and the isolation of finite-size corrections;
3. it leads to formulas for the radii and creation energies of the electron and the proton;
4. it allows a unified description of electrostatic and magnetic-type interactions as interactions of flows of spatial deformation.

Thus, the paper proposes a geometrically strict way to describe electromagnetism, in which electric charge, interaction energy, and particle mass arise as consequences of local curvature of space rather than being introduced as independent initial objects of the theory.

2 Initial Definitions and Postulates of the Model

2.1 Basic space

The basic space is understood as space before curvature. In this state space is assumed to be homogeneous and isotropic, and the local coefficient of linear curvature is equal to unity:

$$\boxed{\alpha = 1.}$$

Consequently, in the basic state there is no volumetric curvature of space.

2.2 Coefficient of linear curvature of space

At each point of the basic space we introduce a coefficient of linear curvature

$$\alpha = \alpha(\mathbf{r}),$$

which characterizes the local change of the linear scale of space relative to its unperturbed state. If, at a point of the basic space, the length element before curvature is dl_0, and after curvature is dl, then

$$\boxed{dl = \alpha \, dl_0.}$$

For $\alpha < 1$ there is local compression of space, whereas for $\alpha > 1$ there is local stretching.

2.3 Intensity of volumetric curvature

The intensity of volumetric curvature of space is defined as the deviation of the local volumetric state from the basic state:

$$\boxed{\Delta K_v = 1 - \alpha^3.}$$

Here α^3 is the local coefficient of volumetric transformation. In the absence of curvature, $\alpha = 1$, and therefore

$$\boxed{\Delta K_v = 0.}$$

Thus ΔK_v measures the degree of deviation of space from its initial state.

2.4 Postulate 1. Spherical structure of local curvature

Any local curvature associated with an elementary charge is confined to a fundamental spherical region of radius R_{ie}, centered at $\mathbf{X}_i$. Let

$$\mathbf{r}_i = \mathbf{x} - \mathbf{X}_i, \qquad r_i = |\mathbf{r}_i|.$$

It is assumed that inside the region $r_i < R_{ie}$, the volumetric curvature intensity is constant, whereas outside the region $r_i > R_{ie}$, the curvature distribution remains spherically symmetric. Thus, for the i-th source,

$$\boxed{\Delta K_{v,i}(r_i) = \Delta K_{v,i}^{\text{in}} = \text{const}, \qquad r_i < R_{ie},}$$

and outside the region

$$\boxed{\Delta K_{v,i}(r_i) = \Delta K_{v,i}^{\text{out}}(r_i), \qquad r_i > R_{ie}.}$$

For the electron and the proton the special notations $\Delta K_{v,e}^{\text{in}}$ and $\Delta K_{v,p}^{\text{in}}$ will be used.

2.5 Postulate 2. Conservation law for the amount of volumetric curvature intensity

For normalization of the distribution, a global compensation law is introduced:

$$\boxed{\int_{\mathbb{R}^3} \Delta K_v \, dV = 0.}$$

This law means that any local deviation of space from the basic state is completely compensated over all space. Physically, local curvature does not create or destroy the total amount of volumetric deviation, but only redistributes it.

2.6 Electron and proton as two types of local curvature

Within the model, the electron and the proton correspond to two different types of local curvature of space:

- the electron corresponds to local compression of space, for which

$$\Delta K_{v,e}^{\text{in}} > 0;$$

- the proton corresponds to local stretching of space, for which

$$\Delta K_{v,p}^{\text{in}} < 0.$$

Thus, the difference in the sign of electric charge is associated with the difference between compression and stretching of space.

2.7 Physical meaning of the initial postulates

The introduced definitions and postulates form the minimal geometric basis of the model. The basic space defines the initial geometry before curvature; the coefficient α describes the local change of the linear scale; the intensity $\Delta K_v = 1 - \alpha^3$ measures the deviation from the basic state; local curvature has a spherical structure with constant internal intensity; and the total amount of volumetric curvature intensity over all space is conserved.

3 Tensor State of Space Before and After Curvature

For the subsequent description of local curvature, a tensor state of space is introduced as the tensor product of the unit radial vector with itself. Let

$$\mathbf{r} = (r_1, r_2, r_3), \qquad r = \sqrt{r_1^2 + r_2^2 + r_3^2}.$$

The unit direction vector is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \left(\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}\right).$$

The tensor state of space is defined by

$$\boxed{\mathbf{T} = \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}, \qquad \boxed{T_{ij} = \hat{r}_i \hat{r}_j = \frac{r_i r_j}{r^2}.}$$

In matrix form,

$$\mathbf{T} = \frac{1}{r^2} \begin{pmatrix} r_1 r_1 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & r_2 r_2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & r_3 r_3 \end{pmatrix}.$$

For a spherically symmetric shear-free state, the off-diagonal components are assumed to vanish. Then the tensor of the basic space takes the diagonal form

$$\boxed{\mathbf{T}_0 = \mathrm{diag}\left(\frac{r_1^2}{r^2}, \frac{r_2^2}{r^2}, \frac{r_3^2}{r^2}\right).}$$

In the unperturbed state the coefficients of linear curvature in all coordinate directions are equal to unity:

$$\boxed{\alpha_1 = \alpha_2 = \alpha_3 = 1.}$$

If local coefficients of linear curvature along the coordinate directions are introduced, then in the diagonal shear-free approximation the curved state may be written as

$$\mathbf{T} = \frac{1}{r^2}\begin{pmatrix} \alpha_1^2 r_1^2 & 0 & 0 \\ 0 & \alpha_2^2 r_2^2 & 0 \\ 0 & 0 & \alpha_3^2 r_3^2 \end{pmatrix}.$$

For spherically symmetric curvature,

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha,$$

and therefore

$$\mathbf{T} = \frac{\alpha^2}{r^2}\begin{pmatrix} r_1^2 & 0 & 0 \\ 0 & r_2^2 & 0 \\ 0 & 0 & r_3^2 \end{pmatrix}.$$

Thus the local coefficient of volumetric transformation is α^3, and the intensity of volumetric curvature is

$$\Delta K_v = 1 - \alpha^3.$$

4 Distribution of Volumetric Curvature Intensity for a Compressed Spherical Region

Consider a spherical region of the basic space of radius R_{0e}, which after curvature is compressed to radius R_e. Inside this region the linear curvature is assumed constant, so

$$\alpha_e = \frac{R_e}{R_{0e}}.$$

Consequently, the volumetric curvature intensity inside the region $r < R_e$ is also constant and equals

$$\Delta K_{v,e}^{\text{in}} = 1 - \alpha_e^3 = 1 - \frac{R_e^3}{R_{0e}^3}.$$

Since $R_e < R_{0e}$, this value is positive and corresponds to compression of space.

4.1 Variational functional

Let $\Delta V(r)$ denote the deviation of the volume of a spherical region of radius r from its volume in the basic space. For a spherically symmetric state,

$$\Delta V(r) = \frac{4\pi}{3} r^3(\alpha^3(r) - 1) = -\frac{4\pi}{3} r^3 \Delta K_v(r).$$

In the external region $\Omega_{\text{ext}} = \{r > R_e\}$, the action is chosen as

$$\boxed{S[\Delta V] = \frac{\beta}{2} \int_{\Omega_{\text{ext}}} \nabla \Delta V \cdot \nabla \Delta V \, dV,}$$

where β is a constant coefficient. This is the standard quadratic form for an elastic potential energy proportional to the square of the gradient of the deforming field.

4.2 Euler–Lagrange equation and Laplace equation

Variation with respect to ΔV gives

$$\delta S = \beta \int_{\Omega_{\text{ext}}} \nabla \Delta V \cdot \nabla (\delta \Delta V) \, dV.$$

Integrating by parts and assuming that the variations vanish on the boundary, we obtain

$$\delta S = -\beta \int_{\Omega_{\text{ext}}} (\nabla^2 \Delta V) \, \delta \Delta V \, dV.$$

Thus the stationarity condition yields

$$\boxed{\nabla^2 \Delta V = 0, \qquad r > R_e.}$$

4.3 Solution of Laplace equation

For a spherically symmetric function,

$$\boxed{\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Delta V}{dr} \right) = 0.}$$

After two integrations,

$$\Delta V(r) = -\frac{C_1}{r} + C_2.$$

The condition $\Delta V(r) \to 0$ as $r \to \infty$ gives $C_2 = 0$. Denoting $-C_1 = B$, we obtain

$$\boxed{\Delta V(r) = \frac{B}{r}, \qquad r > R_e.}$$

4.4 Transition from $\Delta V(r)$ to $\Delta K_v(r)$

Using

$$\Delta V(r) = -\frac{4\pi}{3} r^3 \Delta K_v(r),$$

we find

$$\boxed{\Delta K_v^{\text{out}}(r) = -\frac{3B}{4\pi r^4}.}$$

Introducing A_e by

$$\boxed{\Delta K_v^{\text{out}}(r) = \frac{A_e}{r^4}, \qquad r > R_e,}$$

we have $A_e = -3B/(4\pi)$. Thus the variational principle directly implies an external decay law proportional to r^{-4}.

4.5 Determination of the external distribution constant from the conservation law

The global compensation law requires

$$\boxed{\int_{\mathbb{R}^3} \Delta K_v \, dV = 0.}$$

For spherical symmetry,

$$4\pi \int_0^{R_e} \Delta K_{v,e}^{\text{in}} r^2 \, dr + 4\pi \int_{R_e}^{\infty} \frac{A_e}{r^4} r^2 \, dr = 0.$$

This gives

$$\Delta K_{v,e}^{\text{in}} \frac{R_e^3}{3} + \frac{A_e}{R_e} = 0,$$

and therefore

$$\boxed{A_e = -\frac{R_e^4}{3} \Delta K_{v,e}^{\text{in}}.}$$

The external distribution is consequently

$$\boxed{\Delta K_{v,e}^{\text{out}}(r) = -\frac{R_e^4}{3r^4} \Delta K_{v,e}^{\text{in}}, \qquad r > R_e.}$$

4.6 Full distribution of volumetric curvature intensity

The full distribution is written with Heaviside functions as

$$\boxed{\Delta K_{v,e}(r) = \Delta K_{v,e}^{\text{in}} H(R_e - r) - \frac{R_e^4}{3r^4} \Delta K_{v,e}^{\text{in}} H(r - R_e).}$$

Using $\Delta K_{v,e}^{\text{in}} = 1 - R_e^3/R_{0e}^3$, this becomes

$$\boxed{\Delta K_{v,e}(r) = \left(1 - \frac{R_e^3}{R_{0e}^3}\right) H(R_e - r) - \frac{R_e^4}{3r^4} \left(1 - \frac{R_e^3}{R_{0e}^3}\right) H(r - R_e).}$$

5 Joint Tensor State of Space for Two Spherical Curvature Regions

Consider two spherical curvature regions centered at $\mathbf{X}_1$ and $\mathbf{X}_2$, with

$$\mathbf{D} = \mathbf{X}_2 - \mathbf{X}_1.$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be radius vectors of the same point in the coordinate systems centered at the first and second region:

$$\mathbf{r}_1 = \mathbf{x} - \mathbf{X}_1, \qquad \mathbf{r}_2 = \mathbf{x} - \mathbf{X}_2.$$

Then

$$\boxed{\mathbf{r}_2 = \mathbf{r}_1 - \mathbf{D}.}$$

The local states of space are described by $\alpha_1(\mathbf{r}_1)$ and $\alpha_2(\mathbf{r}_2)$, and the corresponding intensities are

$$\boxed{\Delta K_{v,1}(\mathbf{r}_1) = 1 - \alpha_1^3(\mathbf{r}_1), \qquad \Delta K_{v,2}(\mathbf{r}_2) = 1 - \alpha_2^3(\mathbf{r}_2).}$$

5.1 Unit radial vectors and state tensors

In each coordinate system we introduce unit vectors $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. The state tensors before curvature are

$$\boxed{\mathbf{T}_0^{(1)} = \hat{\mathbf{r}}_1 \otimes \hat{\mathbf{r}}_1, \qquad \mathbf{T}_0^{(2)} = \hat{\mathbf{r}}_2 \otimes \hat{\mathbf{r}}_2.}$$

In components,

$$\boxed{T_{0,ab}^{(1)} = \hat{r}_{1a}\hat{r}_{1b}, \qquad T_{0,cd}^{(2)} = \hat{r}_{2c}\hat{r}_{2d}.}$$

In the shear-free spherical approximation only diagonal components are retained.

5.2 State tensors after curvature

After curvature,

$$\boxed{\mathbf{T}_\alpha^{(1)} = (\alpha_1\hat{\mathbf{r}}_1) \otimes (\alpha_1\hat{\mathbf{r}}_1), \qquad \mathbf{T}_\alpha^{(2)} = (\alpha_2\hat{\mathbf{r}}_2) \otimes (\alpha_2\hat{\mathbf{r}}_2).}$$

Thus,

$$\boxed{T_{\alpha,ab}^{(1)} = \alpha_1^2(\mathbf{r}_1)\hat{r}_{1a}\hat{r}_{1b}, \qquad T_{\alpha,cd}^{(2)} = \alpha_2^2(\mathbf{r}_2)\hat{r}_{2c}\hat{r}_{2d}.}$$

5.3 Joint state of two curvature regions

The joint state before curvature is

$$\boxed{\mathbb{T}^{(0)} = \mathbf{T}_0^{(1)} \otimes \mathbf{T}_0^{(2)},}$$

that is,

$$\boxed{\mathbb{T}^{(0)}_{abcd} = \hat{r}_{1a}\hat{r}_{1b}\hat{r}_{2c}\hat{r}_{2d}.}$$

After curvature,

$$\boxed{\mathbb{T}^{(\alpha)} = \mathbf{T}^{(1)}_{\alpha} \otimes \mathbf{T}^{(2)}_{\alpha},}$$

so that

$$\boxed{\mathbb{T}^{(\alpha)}_{abcd} = \alpha_1^2(\mathbf{r}_1)\alpha_2^2(\mathbf{r}_2)\hat{r}_{1a}\hat{r}_{1b}\hat{r}_{2c}\hat{r}_{2d}.}$$

5.4 Contraction of the fourth-rank tensor and the interaction operator

Contracting the fourth-rank tensor over the internal indices b and d, we define

$$\boxed{M^{(0)}_{ac} = \delta^{bd}\mathbb{T}^{(0)}_{abcd}, \qquad M^{(\alpha)}_{ac} = \delta^{bd}\mathbb{T}^{(\alpha)}_{abcd}.}$$

This gives

$$\boxed{M^{(0)}_{ac} = W_{12}\hat{r}_{1a}\hat{r}_{2c}, \qquad M^{(\alpha)}_{ac} = \alpha_1^2\alpha_2^2 W_{12}\hat{r}_{1a}\hat{r}_{2c},}$$

where

$$\boxed{W_{12} := \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|} = \frac{\mathbf{r}_1 \cdot (\mathbf{r}_1 - \mathbf{D})}{|\mathbf{r}_1|\,|\mathbf{r}_1 - \mathbf{D}|}.}$$

The quantity W_{12} is a dimensionless operator of directed interaction between two local spatial states.

5.5 Reduced trace and joint linear curvature

The traces are

$$\operatorname{tr} M^{(0)} = W_{12}^2, \qquad \operatorname{tr} M^{(\alpha)} = \alpha_1^2\alpha_2^2 W_{12}^2.$$

Therefore,

$$\boxed{\sqrt{\operatorname{tr} M^{(0)}} = W_{12}, \qquad \sqrt{\operatorname{tr} M^{(\alpha)}} = \alpha_1\alpha_2 W_{12}.}$$

Thus the joint linear state is

$$\boxed{\alpha^{(0)}_{12} = W_{12}, \qquad \alpha^{(\alpha)}_{12} = \alpha_1(\mathbf{r}_1)\alpha_2(\mathbf{r}_2)W_{12}.}$$

5.6 Joint volumetric curvature

Since for each region $K_v = \alpha^3$, the joint volumetric state after curvature is taken as

$$\boxed{K^{(\alpha)}_{v,12} = \alpha_1^3(\mathbf{r}_1)\alpha_2^3(\mathbf{r}_2)W_{12}.}$$

The background state is

$$\boxed{K_{v,12}^{(0)} = W_{12}.}$$

Therefore,

$$\boxed{\Delta K_{v,12} = \left(1 - \alpha_1^3(\mathbf{r}_1)\alpha_2^3(\mathbf{r}_2)\right) W_{12}.}$$

The operator W_{12} is not a local volumetric curvature coefficient and therefore is not raised to the third power.

5.7 Expression through individual volumetric curvature intensities

Using

$$\alpha_1^3 = 1 - \Delta K_{v,1}, \qquad \alpha_2^3 = 1 - \Delta K_{v,2},$$

we obtain

$$\boxed{\Delta K_{v,12} = (\Delta K_{v,1}(\mathbf{r}_1) + \Delta K_{v,2}(\mathbf{r}_2) - \Delta K_{v,1}(\mathbf{r}_1)\Delta K_{v,2}(\mathbf{r}_2))\, W_{12}.}$$

The cross term

$$\boxed{-\Delta K_{v,1}(\mathbf{r}_1)\Delta K_{v,2}(\mathbf{r}_2)W_{12}}$$

is the term that determines the proper interaction of the two spherical curvature regions.

6 Definition of Geometric Charge Through the Integral Characteristic of the External Flux of Spatial Curvature

Consider a spherical curvature region with fundamental radius R_{ie} and center $\mathbf{X}_i$. In the internal region $r_i < R_{ie}$, the volumetric curvature intensity is constant:

$$\boxed{\Delta K_{v,i}^{\text{in}} = \text{const.}}$$

The directed field responsible for the external flux of spatial curvature is introduced only in the external region

$$\boxed{\Omega_i = \{\mathbf{r}_i : r_i > R_{ie}\}.}$$

For this region,

$$\boxed{\Delta K_{v,i}(r_i) = -\frac{R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3r_i^4}, \qquad r_i > R_{ie}.}$$

The external flux is introduced as the radial vector field

$$\boxed{\mathbf{J}_i(\mathbf{r}_i) = \Delta K_{v,i}(r_i)\mathbf{n}_i = -\frac{R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3r_i^4}\mathbf{n}_i, \qquad r_i > R_{ie},}$$

where $\mathbf{n}_i = \mathbf{r}_i/r_i$. The geometric charge is defined as the volume integral of the divergence of this external field:

$$\boxed{Q_i := \iiint_{\Omega_i} \nabla \cdot \mathbf{J}_i \, dV_i.}$$

6.1 Divergence of the external field

For a radial field $\mathbf{J}_i = f_i(r_i)\mathbf{n}_i$,

$$\boxed{\nabla \cdot \mathbf{J}_i = \frac{1}{r_i^2}\frac{d}{dr_i}\left(r_i^2 f_i(r_i)\right).}$$

With

$$f_i(r_i) = -\frac{R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3r_i^4},$$

we find

$$\boxed{\nabla \cdot \mathbf{J}_i = \frac{2R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3r_i^5}, \qquad r_i > R_{ie}.}$$

6.2 Geometric charge as a volume integral

Substitution into the definition gives

$$Q_i = \iiint_{r_i > R_{ie}} \frac{2R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3r_i^5} \, dV_i.$$

Using $dV_i = r_i^2 \sin\theta_i \, dr_i \, d\theta_i \, d\varphi_i$,

$$Q_i = \frac{8\pi R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3} \int_{R_{ie}}^{\infty} \frac{dr_i}{r_i^3} = \frac{8\pi R_{ie}^4 \Delta K_{v,i}^{\text{in}}}{3} \cdot \frac{1}{2R_{ie}^2}.$$

Thus

$$\boxed{Q_i = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}.}$$

6.3 Equivalent form through Gauss' theorem

The same result follows from Gauss' theorem applied to the shell $R_{ie} < r_i < L$, followed by the limit $L \to \infty$:

$$\iiint_{R_{ie} < r_i < L} \nabla \cdot \mathbf{J}_i \, dV_i = \iint_{S_L} \mathbf{J}_i \cdot d\mathbf{S} - \iint_{S_{R_{ie}}} \mathbf{J}_i \cdot d\mathbf{S}.$$

The outer surface contribution vanishes as $L \to \infty$, whereas the inner surface yields

$$-\iint_{S_{R_{ie}}} \mathbf{J}_i \cdot d\mathbf{S} = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}.$$

Hence again

$$\boxed{Q_i = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}.}$$

6.4 Surface density of geometric charge

Since the source is localized on the fundamental sphere $S_{R_{ie}}$, the surface density is

$$\boxed{\sigma_i := \frac{Q_i}{4\pi R_{ie}^2}.}$$

Using the result above,

$$\boxed{\sigma_i = \frac{\Delta K_{v,i}^{\text{in}}}{3}.}$$

6.5 Summary

The geometric charge is not postulated independently. It is derived from the full external flux of spatial curvature generated by a local spherical region:

$$\boxed{Q_i = \iiint_{\Omega_i} \nabla \cdot \mathbf{J}_i \, dV_i = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}.}$$

7 Interaction Energy of Two Surface Sources of Spatial Curvature Through the Standard Bilinear Interaction Functional

Consider two spherical boundaries $S_{R_{1e}}$ and $S_{R_{2e}}$ carrying surface geometric charges Q_1 and Q_2. Their centers are separated by

$$\mathbf{D} = \mathbf{X}_2 - \mathbf{X}_1, \qquad D = |\mathbf{D}|,$$

and the spheres do not intersect:

$$\boxed{D > R_{1e} + R_{2e}.}$$

The geometric charge and its surface density are

$$\boxed{Q_i = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}, \qquad \sigma_i = \frac{Q_i}{4\pi R_{ie}^2} = \frac{\Delta K_{v,i}^{\text{in}}}{3}.}$$

7.1 Bilinear functional of surface sources

With the rationalized fundamental kernel

$$\boxed{G(\mathbf{R}) = \frac{1}{4\pi|\mathbf{R}|},}$$

the bilinear interaction functional is

$$\boxed{\mathcal{P}_{12}(D) = \iint_{S_{R_{1e}}} \iint_{S_{R_{2e}}} \frac{\sigma_1 \sigma_2}{4\pi|\mathbf{D} + \mathbf{r}_2 - \mathbf{r}_1|} \, dS_1 \, dS_2.}$$

Because σ_1 and σ_2 are constant on the spheres, the integrals may be evaluated sequentially.

7.2 Internal surface integral over $S_{R_{1e}}$

For fixed $\mathbf{r}_2 \in S_{R_{2e}}$, set

$$\mathbf{a} = \mathbf{D} + \mathbf{r}_2, \qquad a = |\mathbf{a}|.$$

Since $a \geq D - R_{2e} > R_{1e}$, the point $\mathbf{a}$ lies outside the first sphere. Consider

$$I_1(a) = \iint_{S_{R_{1e}}} \frac{dS_1}{4\pi|\mathbf{a} - \mathbf{r}_1|}.$$

Choosing the polar axis along $\mathbf{a}$, one obtains

$$I_1(a) = \frac{R_{1e}^2}{2} \int_{-1}^{1} \frac{du}{\sqrt{a^2 + R_{1e}^2 - 2aR_{1e}u}}.$$

Introduce

$$\alpha = a^2 + R_{1e}^2, \qquad \beta = 2aR_{1e}.$$

Then

$$I_1(a) = \frac{R_{1e}^2}{2} \int_{-1}^{1} \frac{du}{\sqrt{\alpha - \beta u}}.$$

The antiderivative is

$$\int \frac{du}{\sqrt{\alpha - \beta u}} = -\frac{2}{\beta}\sqrt{\alpha - \beta u}.$$

Therefore

$$I_1(a) = \frac{R_{1e}^2}{2} \left[-\frac{2}{2aR_{1e}} \sqrt{a^2 + R_{1e}^2 - 2aR_{1e}u} \right]_{-1}^{1}.$$

After rearrangement,

$$I_1(a) = \frac{R_{1e}^2}{2aR_{1e}} \left[\sqrt{a^2 + R_{1e}^2 + 2aR_{1e}} - \sqrt{a^2 + R_{1e}^2 - 2aR_{1e}} \right].$$

Since $a > R_{1e}$,

$$\sqrt{a^2 + R_{1e}^2 + 2aR_{1e}} = a + R_{1e}, \qquad \sqrt{a^2 + R_{1e}^2 - 2aR_{1e}} = a - R_{1e}.$$

Hence

$$I_1(a) = \frac{R_{1e}^2}{2aR_{1e}} \big[(a + R_{1e}) - (a - R_{1e}) \big] = \frac{R_{1e}^2}{2aR_{1e}} \, 2R_{1e}.$$

Thus

$$\boxed{I_1(a) = \frac{R_{1e}^2}{a} = \frac{R_{1e}^2}{|\mathbf{D} + \mathbf{r}_2|}.}$$

7.3 External surface integral over $S_{R_{2e}}$

It remains to compute

$$I_2(D) = \iint_{S_{R_{2e}}} \frac{dS_2}{|\mathbf{D} + \mathbf{r}_2|}.$$

In spherical coordinates centered at the second sphere,

$$|\mathbf{r}_2| = R_{2e}, \qquad |\mathbf{D} + \mathbf{r}_2| = \sqrt{D^2 + R_{2e}^2 + 2DR_{2e} \cos\theta_2},$$

$$dS_2 = R_{2e}^2 \sin\theta_2 \, d\theta_2 \, d\varphi_2.$$

Therefore

$$I_2(D) = \int_0^{2\pi} \int_0^{\pi} \frac{R_{2e}^2 \sin\theta_2 \, d\theta_2 \, d\varphi_2}{\sqrt{D^2 + R_{2e}^2 + 2DR_{2e} \cos\theta_2}}.$$

Integration over φ_2 gives

$$I_2(D) = 2\pi R_{2e}^2 \int_0^{\pi} \frac{\sin\theta_2 \, d\theta_2}{\sqrt{D^2 + R_{2e}^2 + 2DR_{2e} \cos\theta_2}}.$$

With $u = \cos\theta_2$, $du = -\sin\theta_2 \, d\theta_2$,

$$I_2(D) = 2\pi R_{2e}^2 \int_{-1}^{1} \frac{du}{\sqrt{D^2 + R_{2e}^2 + 2DR_{2e} u}}.$$

Using

$$\int \frac{du}{\sqrt{\alpha + \beta u}} = \frac{2}{\beta} \sqrt{\alpha + \beta u}, \qquad \alpha = D^2 + R_{2e}^2, \quad \beta = 2DR_{2e},$$

we obtain

$$I_2(D) = 2\pi R_{2e}^2 \left[\frac{2}{2DR_{2e}} \sqrt{D^2 + R_{2e}^2 + 2DR_{2e} u} \right]_{-1}^{1}.$$

Thus

$$I_2(D) = \frac{2\pi R_{2e}^2}{DR_{2e}} \left[\sqrt{D^2 + R_{2e}^2 + 2DR_{2e}} - \sqrt{D^2 + R_{2e}^2 - 2DR_{2e}} \right].$$

Since $D > R_{2e}$, the square roots are $D + R_{2e}$ and $D - R_{2e}$, hence

$$I_2(D) = \frac{2\pi R_{2e}^2}{DR_{2e}} \big[(D + R_{2e}) - (D - R_{2e}) \big] = \frac{2\pi R_{2e}^2}{DR_{2e}} 2R_{2e}.$$

Therefore

$$\boxed{I_2(D) = \frac{4\pi R_{2e}^2}{D}.}$$

7.4 Final bilinear functional and Coulomb energy

Therefore,

$$\mathcal{P}_{12}(D) = R_{1e}^2 \sigma_1 \sigma_2 \frac{4\pi R_{2e}^2}{D} = \frac{Q_1 Q_2}{4\pi D}.$$

Thus

$$\boxed{\mathcal{P}_{12}(D) = \frac{Q_1 Q_2}{4\pi D}.}$$

Multiplication by the volumetric stiffness of space ρ_0 gives the potential energy

$$\boxed{U_{12}(D) = \rho_0 \mathcal{P}_{12}(D) = \frac{\rho_0 Q_1 Q_2}{4\pi D}.}$$

Equivalently,

$$\boxed{U_{12}(D) = \frac{4\pi\rho_0}{9} \frac{R_{1e}^2 R_{2e}^2 \Delta K_{v,1}^{\text{in}} \Delta K_{v,2}^{\text{in}}}{D}.}$$

7.5 Interaction force as a gradient with respect to D

The force is

$$\boxed{F_{12}(D) = -\frac{dU_{12}}{dD}.}$$

Using the energy above,

$$\boxed{F_{12}(D) = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}.}$$

The vector force on the second charge is

$$\boxed{\mathbf{F}_{12} = \frac{\rho_0 Q_1 Q_2}{4\pi D^2} \hat{\mathbf{D}}, \qquad \hat{\mathbf{D}} = \frac{\mathbf{D}}{D}.}$$

The force on the first charge is opposite:

$$\boxed{\mathbf{F}_{21} = -\frac{\rho_0 Q_1 Q_2}{4\pi D^2} \hat{\mathbf{D}}.}$$

7.6 Summary

The standard rationalized bilinear interaction functional of two spherical surface sources gives exactly the Coulomb-type leading term:

$$\boxed{U_{12}(D) = \frac{\rho_0 Q_1 Q_2}{4\pi D}, \qquad F_{12}(D) = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}.}$$

8 Interaction Force Through a Volume Integral Over the Tensor Parameter Space With the Inverse Integral Kernel

The surface representation of the preceding section uses only the integral characteristics Q_1 and Q_2. To recover information about the full external curvature distribution, we construct a volume force functional over the parameter spaces dV_1 and dV_2. The long-range interaction is taken to be determined not by $\Delta K_{v,i}$ itself, but by a quantity proportional to the density of change of Gaussian curvature of the fundamental spherical boundary.

8.1 Density of change of Gaussian curvature as a local source of long-range interaction

For a sphere of radius R,

$$\kappa_G(R) = \frac{1}{R^2}.$$

If a sphere of radius R_{0i} becomes a deformed sphere of radius R_{ie}, then

$$\Delta\kappa_{G,i} = \frac{1}{R_{ie}^2} - \frac{1}{R_{0i}^2}.$$

With $\alpha_i = R_{ie}/R_{0i}$ and $\Delta K_{v,i} = 1 - \alpha_i^3$,

$$\Delta\kappa_{G,i} = \frac{1 - (1 - \Delta K_{v,i})^{2/3}}{R_{ie}^2}.$$

For small deformations,

$$(1 - \Delta K_{v,i})^{2/3} \approx 1 - \frac{2}{3}\Delta K_{v,i},$$

so that

$$\boxed{\Delta\kappa_{G,i} \approx \frac{2}{3}\frac{\Delta K_{v,i}}{R_{ie}^2}.}$$

We therefore define

$$\boxed{\Gamma_i(\mathbf{r}_i) := \frac{\Delta K_{v,i}(\mathbf{r}_i)}{R_{ie}^2}.}$$

Using the full distributions,

$$\boxed{\Gamma_1(r_1) = \frac{\Delta K_{v,1}^{\text{in}}}{R_{1e}^2} H(R_{1e} - r_1) - \frac{R_{1e}^2 \Delta K_{v,1}^{\text{in}}}{3r_1^4} H(r_1 - R_{1e}),}$$

$$\boxed{\Gamma_2(r_2) = \frac{\Delta K_{v,2}^{\text{in}}}{R_{2e}^2} H(R_{2e} - r_2) - \frac{R_{2e}^2 \Delta K_{v,2}^{\text{in}}}{3r_2^4} H(r_2 - R_{2e}).}$$

In terms of geometric charges,

$$\boxed{\Gamma_1(r_1) = \frac{3Q_1}{4\pi R_{1e}^4} H(R_{1e} - r_1) - \frac{Q_1}{4\pi r_1^4} H(r_1 - R_{1e}),}$$

$$\boxed{\Gamma_2(r_2) = \frac{3Q_2}{4\pi R_{2e}^4} H(R_{2e} - r_2) - \frac{Q_2}{4\pi r_2^4} H(r_2 - R_{2e}).}$$

8.2 Volume functional and transformation of the integral kernel

The kernel obtained from the tensor construction is

$$W_{12} = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}, \qquad \mathbf{r}_2 = \mathbf{r}_1 - \mathbf{D}, \qquad \mathbf{r}_1 = \mathbf{r}_2 + \mathbf{D}.$$

It can be rewritten as

$$\boxed{W_{12} = \frac{\mathbf{r}_1 - \mathbf{D}}{|\mathbf{r}_1 - \mathbf{D}|} \cdot \frac{\mathbf{r}_2 + \mathbf{D}}{|\mathbf{r}_2 + \mathbf{D}|}.}$$

The geometric force functional is defined by

$$\boxed{\mathcal{I}_{12}(D) = \iiint\iiint \Gamma_1(\mathbf{r}_1)\Gamma_2(\mathbf{r}_2) \frac{\mathbf{r}_1 - \mathbf{D}}{|\mathbf{r}_1 - \mathbf{D}|} \cdot \frac{\mathbf{r}_2 + \mathbf{D}}{|\mathbf{r}_2 + \mathbf{D}|}\, dV_1\, dV_2.}$$

Because the two factors in the scalar product depend separately on $\mathbf{r}_1$ and $\mathbf{r}_2$, the functional factorizes:

$$\boxed{\mathcal{I}_{12}(D) = \mathbf{I}_1(D) \cdot \mathbf{I}_2(D),}$$

where

$$\boxed{\mathbf{I}_1(D) = \iiint \Gamma_1(\mathbf{r}_1) \frac{\mathbf{r}_1 - \mathbf{D}}{|\mathbf{r}_1 - \mathbf{D}|}\, dV_1,}$$

$$\boxed{\mathbf{I}_2(D) = \iiint \Gamma_2(\mathbf{r}_2) \frac{\mathbf{r}_2 + \mathbf{D}}{|\mathbf{r}_2 + \mathbf{D}|}\, dV_2.}$$

By axial symmetry,

$$\mathbf{I}_1(D) = I_1(D)\hat{\mathbf{D}}, \qquad \mathbf{I}_2(D) = I_2(D)\hat{\mathbf{D}},$$

and hence

$$\boxed{\mathcal{I}_{12}(D) = I_1(D)I_2(D).}$$

8.3 First integral $I_1(D)$

For the first charge, after integration over the azimuthal angle one obtains

$$I_1(D) = 2\pi \int_0^\infty \Gamma_1(r_1) r_1^2 \, dr_1 \int_0^\pi \frac{r_1 \cos\theta_1 - D}{\sqrt{r_1^2 + D^2 - 2Dr_1 \cos\theta_1}} \sin\theta_1 \, d\theta_1.$$

The angular integral is

$$\boxed{K_1(r_1, D) = 2\pi \int_0^\pi \frac{r_1 \cos\theta_1 - D}{\sqrt{r_1^2 + D^2 - 2Dr_1 \cos\theta_1}} \sin\theta_1 \, d\theta_1.}$$

To evaluate this integral explicitly, introduce

$$t = r_1^2 + D^2 - 2Dr_1 u.$$

Then

$$u = \frac{r_1^2 + D^2 - t}{2Dr_1}, \qquad du = -\frac{dt}{2Dr_1},$$

and the numerator becomes

$$r_1 u - D = \frac{r_1^2 - D^2 - t}{2D}.$$

Therefore

$$K_1(r_1, D) = -\frac{\pi}{2D^2 r_1} \int \left(r_1^2 - D^2 - t\right) t^{-1/2} \, dt.$$

After integration,

$$K_1(r_1, D) = \frac{\pi}{D^2 r_1} \left[(r_1^2 - D^2)\sqrt{t} - \frac{1}{3} t^{3/2} \right]_{u=-1}^{u=1}.$$

For $r_1 < D$, one has

$$\sqrt{r_1^2 + D^2 - 2Dr_1} = D - r_1, \qquad \sqrt{r_1^2 + D^2 + 2Dr_1} = D + r_1.$$

Hence

$$K_1^<(r_1, D) = \frac{\pi}{D^2 r_1} \left[(r_1^2 - D^2)(D - r_1) - \frac{1}{3}(D - r_1)^3 - (r_1^2 - D^2)(D + r_1) + \frac{1}{3}(D + r_1)^3 \right].$$

Expanding and cancelling the equal terms gives

$$\boxed{K_1^<(r_1, D) = -4\pi + \frac{4\pi r_1^2}{3D^2}, \qquad r_1 < D.}$$

For $r_1 > D$, one has

$$\sqrt{r_1^2 + D^2 - 2Dr_1} = r_1 - D, \qquad \sqrt{r_1^2 + D^2 + 2Dr_1} = r_1 + D.$$

Therefore

$$K_1^>(r_1, D) = \frac{\pi}{D^2 r_1}\left[(r_1^2 - D^2)(r_1 - D) - \frac{1}{3}(r_1 - D)^3 - (r_1^2 - D^2)(r_1 + D) + \frac{1}{3}(r_1 + D)^3\right],$$

which simplifies to

$$\boxed{K_1^>(r_1, D) = -\frac{8\pi D}{3r_1}, \qquad r_1 > D.}$$

Since $D > R_{1e}$, the radial integral splits into the intervals

$$0 < r_1 < R_{1e}, \qquad R_{1e} < r_1 < D, \qquad D < r_1 < \infty.$$

In the internal region,

$$\Gamma_1(r_1) = \frac{\Delta K_{v,1}^{\text{in}}}{R_{1e}^2}, \qquad 0 < r_1 < R_{1e}.$$

Therefore

$$I_{1,\text{in}} = \int_0^{R_{1e}} \frac{\Delta K_{v,1}^{\text{in}}}{R_{1e}^2}\left(-4\pi + \frac{4\pi r_1^2}{3D^2}\right) r_1^2\, dr_1.$$

Splitting the integral,

$$I_{1,\text{in}} = -\frac{4\pi \Delta K_{v,1}^{\text{in}}}{R_{1e}^2}\int_0^{R_{1e}} r_1^2\, dr_1 + \frac{4\pi \Delta K_{v,1}^{\text{in}}}{3D^2 R_{1e}^2}\int_0^{R_{1e}} r_1^4\, dr_1.$$

Since

$$\int_0^{R_{1e}} r_1^2\, dr_1 = \frac{R_{1e}^3}{3}, \qquad \int_0^{R_{1e}} r_1^4\, dr_1 = \frac{R_{1e}^5}{5},$$

we obtain

$$\boxed{I_{1,\text{in}} = -\frac{4\pi}{3} R_{1e}\Delta K_{v,1}^{\text{in}} + \frac{4\pi}{15}\frac{R_{1e}^3 \Delta K_{v,1}^{\text{in}}}{D^2}.}$$

On the interval $R_{1e} < r_1 < D$,

$$\Gamma_1(r_1) = -\frac{R_{1e}^2 \Delta K_{v,1}^{\text{in}}}{3r_1^4}.$$

Thus

$$I_{1,\text{out}}^{(1)} = \int_{R_{1e}}^{D}\left(-\frac{R_{1e}^2 \Delta K_{v,1}^{\text{in}}}{3r_1^4}\right)\left(-4\pi + \frac{4\pi r_1^2}{3D^2}\right) r_1^2\, dr_1.$$

After simplifying the integrand,

$$I^{(1)}_{1,\text{out}} = \int_{R_{1e}}^{D} \left(\frac{4\pi R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{3r_1^2} - \frac{4\pi R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{9D^2} \right) dr_1.$$

Using

$$\int_{R_{1e}}^{D} \frac{dr_1}{r_1^2} = \frac{1}{R_{1e}} - \frac{1}{D}, \qquad \int_{R_{1e}}^{D} dr_1 = D - R_{1e},$$

we get

$$I^{(1)}_{1,\text{out}} = \frac{4\pi R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{3} \left(\frac{1}{R_{1e}} - \frac{1}{D} \right) - \frac{4\pi R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{9D^2}(D - R_{1e}).$$

Therefore

$$\boxed{I^{(1)}_{1,\text{out}} = \frac{4\pi}{3} R_{1e} \Delta K^{\text{in}}_{v,1} - \frac{16\pi}{9} \frac{R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{D} + \frac{4\pi}{9} \frac{R_{1e}^3 \Delta K^{\text{in}}_{v,1}}{D^2}.}$$

On the interval $D < r_1 < \infty$, one uses

$$K_1^{>}(r_1, D) = -\frac{8\pi D}{3r_1}.$$

Hence

$$I^{(2)}_{1,\text{out}} = \int_{D}^{\infty} \left(-\frac{R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{3r_1^4} \right) \left(-\frac{8\pi D}{3r_1} \right) r_1^2 \, dr_1.$$

After simplification,

$$I^{(2)}_{1,\text{out}} = \frac{8\pi D R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{9} \int_{D}^{\infty} \frac{dr_1}{r_1^3}.$$

Since

$$\int_{D}^{\infty} \frac{dr_1}{r_1^3} = \frac{1}{2D^2},$$

we find

$$\boxed{I^{(2)}_{1,\text{out}} = \frac{4\pi}{9} \frac{R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{D}.}$$

Summing them yields

$$\boxed{I_1(D) = -\frac{4\pi}{3} \frac{R_{1e}^2 \Delta K^{\text{in}}_{v,1}}{D} + \frac{32\pi}{45} \frac{R_{1e}^3 \Delta K^{\text{in}}_{v,1}}{D^2}.}$$

In terms of Q_1,

$$\boxed{I_1(D) = -\frac{Q_1}{D} + \frac{8 Q_1 R_{1e}}{15 D^2}.}$$

8.4 Second integral

The second angular integral is opposite in sign after the substitution $u = -v$:

$$\boxed{K_2(r_2, D) = -K_1(r_2, D).}$$

Therefore,

$$\boxed{I_2(D) = \frac{4\pi}{3}\frac{R_{2e}^2 \Delta K_{v,2}^{\text{in}}}{D} - \frac{32\pi}{45}\frac{R_{2e}^3 \Delta K_{v,2}^{\text{in}}}{D^2} = \frac{Q_2}{D} - \frac{8Q_2 R_{2e}}{15D^2}.}$$

8.5 Final volume force functional

Thus,

$$\mathcal{I}_{12}(D) = \left(-\frac{Q_1}{D} + \frac{8Q_1 R_{1e}}{15D^2}\right)\left(\frac{Q_2}{D} - \frac{8Q_2 R_{2e}}{15D^2}\right),$$

and after expansion,

$$\boxed{\mathcal{I}_{12}(D) = -\frac{Q_1 Q_2}{D^2} + \frac{8Q_1 Q_2 (R_{1e} + R_{2e})}{15D^3} - \frac{64 Q_1 Q_2 R_{1e} R_{2e}}{225 D^4}.}$$

The modulus is

$$\boxed{|\mathcal{I}_{12}(D)| = \frac{Q_1 Q_2}{D^2}\left[1 - \frac{8}{15}\frac{R_{1e} + R_{2e}}{D} + \frac{64}{225}\frac{R_{1e} R_{2e}}{D^2}\right].}$$

8.6 Dimension of the force and rationalized form

Since $[Q_i] = L^2$, one has $[\mathcal{I}_{12}] = L^2$. With $[\rho_0] = E/L^3$, the product $\rho_0 \mathcal{I}_{12}$ has the dimension of force. To match the rationalized surface form, the rationalized functional is defined as

$$\boxed{\mathcal{I}_{12}^{\text{rat}}(D) := \frac{1}{4\pi}\mathcal{I}_{12}(D).}$$

Therefore,

$$\boxed{|F_{12}^{\text{rat}}(D)| = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}\left[1 - \frac{8}{15}\frac{R_{1e} + R_{2e}}{D} + \frac{64}{225}\frac{R_{1e} R_{2e}}{D^2}\right].}$$

For $R_{1e} = R_{2e} = R_e$,

$$\boxed{|F_{12}^{\text{rat}}(D)| = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}\left[1 - \frac{16}{15}\frac{R_e}{D} + \frac{64}{225}\frac{R_e^2}{D^2}\right].}$$

If $Q_1 = Q_2 = Q_e$,

$$\boxed{|F_{ee}^{\text{rat}}(D)| = \frac{\rho_0 Q_e^2}{4\pi D^2}\left(1 - \frac{8R_e}{15D}\right)^2.}$$

These finite-size corrections are interpreted as screening and prenormalization at short distances.

9 Connection Between the Geometric Force and Coulomb's Law

The rationalized leading term obtained above is

$$\boxed{|F_{12}^{(0)}(D)| = \frac{\rho_0}{4\pi}\frac{Q_1 Q_2}{D^2}.}$$

9.1 Geometric charge

For each source,

$$\boxed{Q_i = \frac{4\pi}{3} R_{ie}^2 \Delta K_{v,i}^{\text{in}}.}$$

If a common fundamental normalization $R_{1e} = R_{2e} = R_e$ is used, then

$$\boxed{Q_1 = \frac{4\pi}{3} R_e^2 \Delta K_{v,1}^{\text{in}}, \qquad Q_2 = \frac{4\pi}{3} R_e^2 \Delta K_{v,2}^{\text{in}}.}$$

9.2 Assumed relation between electric and geometric charges

We assume that electric charge is the geometric charge normalized by the square of the fundamental radius:

$$\boxed{q = \frac{Q}{R_e^2}.}$$

Thus

$$\boxed{Q_1 = R_e^2 q_1, \qquad Q_2 = R_e^2 q_2.}$$

Using the definition of Q_i,

$$\boxed{q_i = \frac{4\pi}{3} \Delta K_{v,i}^{\text{in}}.}$$

9.3 Geometric force through electric charges

Substitution gives

$$\boxed{|F_{12}^{(0)}(D)| = \frac{\rho_0 R_e^4}{4\pi}\frac{q_1 q_2}{D^2}.}$$

9.4 Comparison with Coulomb's law

The rationalized Coulomb law is

$$\boxed{F_C(D) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{D^2}.}$$

Equating the geometric and Coulomb expressions gives

$$\boxed{\rho_0 R_e^4 = \frac{1}{\varepsilon_0},}$$

and therefore

$$\boxed{\rho_0 = \frac{1}{\varepsilon_0 R_e^4}.}$$

9.5 Dimensions

In the chosen normalization the electric charge q is treated as dimensionless. From Coulomb's law,

$$\left[\frac{1}{\varepsilon_0}\right]\frac{1}{L^2} = \frac{E}{L},$$

so

$$\boxed{\left[\frac{1}{\varepsilon_0}\right] = EL, \qquad [\varepsilon_0] = \frac{1}{EL}.}$$

Using $\rho_0 = 1/(\varepsilon_0 R_e^4)$,

$$\boxed{[\rho_0] = \frac{E}{L^3}.}$$

Thus ρ_0 has the dimension of volumetric energy density or volumetric stiffness of space.

9.6 Summary

Under the assumption $q = Q/R_e^2$, the rationalized geometric law

$$\boxed{|F_{12}^{(0)}(D)| = \frac{\rho_0}{4\pi}\frac{Q_1 Q_2}{D^2}}$$

becomes the Coulomb law if and only if

$$\boxed{\rho_0 = \frac{1}{\varepsilon_0 R_e^4}.}$$

10 Creation Energies of the Electron and Proton and Radii From the Creation-Energy Relations

The physical principle introduced here is that the creation energy of a charge is the work required to form the full distribution of volumetric curvature intensity. In this calculation one uses the full distribution $\Delta K_v(r;R)$, not $\Gamma = \Delta K_v/R_e^2$, since Γ is used only in the long-range interaction functional.

10.1 Electron as compression of space

Let a spherical region of the basic space of radius R_{0e} be compressed to radius R_e, with $R_e < R_{0e}$. For an intermediate radius R,

$$\boxed{\Delta K_{v,e}^{\text{in}}(R) = 1 - \frac{R^3}{R_{0e}^3}.}$$

The full distribution is

$$\Delta K_v(r;R) = \Delta K_{v,e}^{\text{in}}(R)H(R-r) - \frac{R^4}{3r^4}\Delta K_{v,e}^{\text{in}}(R)H(r-R).$$

10.2 Derivative of the full distribution

We differentiate the full distribution with respect to the radial coordinate r. The first term gives

$$\frac{\partial}{\partial r}\left(\Delta K_{v,e}^{\text{in}}(R)H(R-r)\right) = -\Delta K_{v,e}^{\text{in}}(R)\delta(r-R).$$

For the second term,

$$\frac{\partial}{\partial r}\left(-\frac{R^4}{3r^4}\Delta K_{v,e}^{\text{in}}(R)H(r-R)\right) = -\Delta K_{v,e}^{\text{in}}(R)\left[-\frac{4R^4}{3r^5}H(r-R) + \frac{R^4}{3r^4}\delta(r-R)\right].$$

Therefore,

$$\frac{\partial}{\partial r}\left(-\frac{R^4}{3r^4}\Delta K_{v,e}^{\text{in}}(R)H(r-R)\right) = \frac{4R^4}{3r^5}\Delta K_{v,e}^{\text{in}}(R)H(r-R) - \frac{R^4}{3r^4}\Delta K_{v,e}^{\text{in}}(R)\delta(r-R).$$

At the surface $r = R$,

$$\left.\frac{R^4}{3r^4}\right|_{r=R} = \frac{1}{3}.$$

Thus the full derivative is

$$\boxed{\frac{\partial \Delta K_v(r;R)}{\partial r} = -\frac{4}{3}\Delta K_{v,e}^{\text{in}}(R)\delta(r-R) + \frac{4R^4}{3r^5}\Delta K_{v,e}^{\text{in}}(R)H(r-R).}$$

Equivalently,

$$\boxed{\frac{\partial \Delta K_v(r;R)}{\partial r} = -\frac{4}{3}\left(1-\frac{R^3}{R_{0e}^3}\right)\delta(r-R) + \frac{4R^4}{3r^5}\left(1-\frac{R^3}{R_{0e}^3}\right)H(r-R).}$$

This derivative defines the effective radial force associated with the formation of the full spatial-curvature profile.

10.3 Radial compression force of the electron

The effective radial deformation force is defined by

$$\widetilde{F}_{\mathrm{cr},e}(R) = \rho_0 \iiint \frac{\partial \Delta K_v(r;R)}{\partial r}\, dV, \qquad dV = 4\pi r^2\, dr.$$

Substitution of the derivative gives

$$\widetilde{F}_{\mathrm{cr},e}(R) = 4\pi\rho_0 \int_0^\infty r^2 \left[-\frac{4}{3}\Delta K_{v,e}^{\mathrm{in}}(R)\delta(r-R) + \frac{4R^4}{3r^5}\Delta K_{v,e}^{\mathrm{in}}(R)H(r-R)\right] dr.$$

We split this expression into the surface delta contribution and the external contribution.

For the delta contribution,

$$4\pi\rho_0 \int_0^\infty r^2 \left(-\frac{4}{3}\Delta K_{v,e}^{\mathrm{in}}(R)\delta(r-R)\right) dr = -\frac{16\pi\rho_0}{3}\Delta K_{v,e}^{\mathrm{in}}(R)R^2.$$

For the external contribution,

$$4\pi\rho_0 \int_R^\infty r^2 \frac{4R^4}{3r^5}\Delta K_{v,e}^{\mathrm{in}}(R)\, dr = \frac{16\pi\rho_0 R^4}{3}\Delta K_{v,e}^{\mathrm{in}}(R) \int_R^\infty \frac{dr}{r^3}.$$

Since

$$\int_R^\infty \frac{dr}{r^3} = \frac{1}{2R^2},$$

the external contribution is

$$\frac{16\pi\rho_0 R^4}{3}\Delta K_{v,e}^{\mathrm{in}}(R)\frac{1}{2R^2} = \frac{8\pi\rho_0}{3}\Delta K_{v,e}^{\mathrm{in}}(R)R^2.$$

Adding both contributions,

$$\widetilde{F}_{\mathrm{cr},e}(R) = -\frac{16\pi\rho_0}{3}\Delta K_{v,e}^{\mathrm{in}}(R)R^2 + \frac{8\pi\rho_0}{3}\Delta K_{v,e}^{\mathrm{in}}(R)R^2.$$

Therefore,

$$\boxed{\widetilde{F}_{\mathrm{cr},e}(R) = -\frac{8\pi\rho_0}{3}R^2\Delta K_{v,e}^{\mathrm{in}}(R) = -\frac{8\pi\rho_0}{3}R^2\left(1-\frac{R^3}{R_{0e}^3}\right).}$$

The negative sign indicates that the force is directed toward decreasing radius; it is a compression force.

10.4 Creation energy of the electron

Integrating along the compression path from R_{0e} to R_e, we obtain

$$\widetilde{E}_{\mathrm{cr},e} = \int_{R_e}^{R_{0e}} \widetilde{F}_{\mathrm{cr},e}(R)\, dR.$$

Substitution of the force gives

$$\widetilde{E}_{\mathrm{cr},e} = -\frac{8\pi\rho_0}{3} \int_{R_e}^{R_{0e}} R^2 \left(1 - \frac{R^3}{R_{0e}^3}\right) dR.$$

Expanding the integrand,

$$\widetilde{E}_{\mathrm{cr},e} = -\frac{8\pi\rho_0}{3} \left[\int_{R_e}^{R_{0e}} R^2\, dR - \frac{1}{R_{0e}^3} \int_{R_e}^{R_{0e}} R^5\, dR\right].$$

Using

$$\int R^2\, dR = \frac{R^3}{3}, \qquad \int R^5\, dR = \frac{R^6}{6},$$

we get

$$\widetilde{E}_{\mathrm{cr},e} = -\frac{8\pi\rho_0}{3} \left[\frac{R_{0e}^3 - R_e^3}{3} - \frac{R_{0e}^6 - R_e^6}{6R_{0e}^3}\right].$$

Since

$$R_{0e}^6 - R_e^6 = (R_{0e}^3 - R_e^3)(R_{0e}^3 + R_e^3),$$

the expression simplifies to

$$\boxed{\widetilde{E}_{\mathrm{cr},e} = -\frac{4\pi\rho_0}{9R_{0e}^3} \left(R_{0e}^3 - R_e^3\right)^2.}$$

Because

$$R_{0e}^3 - R_e^3 = R_{0e}^3 \Delta K_{v,e}^{\mathrm{in}},$$

we obtain

$$\boxed{\widetilde{E}_{\mathrm{cr},e} = -\frac{4\pi\rho_0}{9} R_{0e}^3 \left(\Delta K_{v,e}^{\mathrm{in}}\right)^2.}$$

In this form the work is negative because it is the work of the internal compression force. The creation energy required to form the electron is therefore the modulus:

$$\boxed{E_{\mathrm{cr},e} := \left|\widetilde{E}_{\mathrm{cr},e}\right| = \frac{4\pi\rho_0}{9} R_{0e}^3 \left(\Delta K_{v,e}^{\mathrm{in}}\right)^2.}$$

10.5 Electron creation energy through electric charge

For the electron,

$$q_e = \frac{Q_e}{R_e^2} = \frac{4\pi}{3}\Delta K_{v,e}^{\text{in}}, \qquad \boxed{\Delta K_{v,e}^{\text{in}} = \frac{3q_e}{4\pi}.}$$

Therefore,

$$\boxed{E_{\text{cr},e} = \frac{\rho_0}{4\pi} R_{0e}^3 q_e^2.}$$

10.6 Electron radius from $E_{\text{cr},e} = m_e c^2$

The condition $E_{\text{cr},e} = m_e c^2$ gives

$$\boxed{R_{0e}^3 = \frac{4\pi m_e c^2}{\rho_0 q_e^2}.}$$

Moreover,

$$\frac{R_e^3}{R_{0e}^3} = 1 - \frac{3q_e}{4\pi},$$

so

$$R_e^3 = \frac{4\pi m_e c^2}{\rho_0 q_e^2}\left(1 - \frac{3q_e}{4\pi}\right).$$

Using $\rho_0 = 1/(\varepsilon_0 R_e^4)$, we obtain

$$\boxed{R_e = \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2 \left(1 - \frac{3q_e}{4\pi}\right)}.}$$

For small q_e,

$$\boxed{R_e \approx \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2},}$$

which is the classical electron radius.

10.7 Proton as stretching of space

Let a spherical region of basic space of radius R_{0p} be stretched to radius R_p, where

$$R_p > R_{0p}.$$

The internal volumetric curvature intensity is assumed to be constant throughout the charge volume and is defined by

$$\boxed{\Delta K_{v,p}^{\text{in}}(R) = \frac{R_{0p}^3}{R^3} - 1.}$$

For $R > R_{0p}$, this quantity is negative, corresponding to stretching of space.

For an intermediate state of current radius R, the full distribution is

$$\Delta K_v(r;R) = \Delta K^{\text{in}}_{v,p}(R)H(R-r) - \frac{R^4}{3r^4}\Delta K^{\text{in}}_{v,p}(R)H(r-R).$$

As in the electron case, the first term describes the homogeneous internal state, and the second term describes the compensating external curvature distribution.

The derivative with respect to r is calculated in exactly the same way as before:

$$\boxed{\frac{\partial \Delta K_v(r;R)}{\partial r} = -\frac{4}{3}\Delta K^{\text{in}}_{v,p}(R)\delta(r-R) + \frac{4R^4}{3r^5}\Delta K^{\text{in}}_{v,p}(R)H(r-R).}$$

Equivalently,

$$\boxed{\frac{\partial \Delta K_v(r;R)}{\partial r} = -\frac{4}{3}\left(\frac{R^3_{0p}}{R^3}-1\right)\delta(r-R) + \frac{4R^4}{3r^5}\left(\frac{R^3_{0p}}{R^3}-1\right)H(r-R).}$$

The radial deformation force is

$$\widetilde{F}_{\text{cr},p}(R) = \rho_0 \iiint \frac{\partial \Delta K_v(r;R)}{\partial r}\,dV, \qquad dV = 4\pi r^2\,dr.$$

Substitution gives

$$\widetilde{F}_{\text{cr},p}(R) = 4\pi\rho_0 \int_0^\infty r^2\left[-\frac{4}{3}\Delta K^{\text{in}}_{v,p}(R)\delta(r-R) + \frac{4R^4}{3r^5}\Delta K^{\text{in}}_{v,p}(R)H(r-R)\right]dr.$$

The delta contribution is

$$4\pi\rho_0 \int_0^\infty r^2\left(-\frac{4}{3}\Delta K^{\text{in}}_{v,p}(R)\delta(r-R)\right)dr = -\frac{16\pi\rho_0}{3}\Delta K^{\text{in}}_{v,p}(R)R^2.$$

The external contribution is

$$4\pi\rho_0 \int_R^\infty r^2\frac{4R^4}{3r^5}\Delta K^{\text{in}}_{v,p}(R)\,dr = \frac{16\pi\rho_0 R^4}{3}\Delta K^{\text{in}}_{v,p}(R)\int_R^\infty \frac{dr}{r^3}.$$

Since

$$\int_R^\infty \frac{dr}{r^3} = \frac{1}{2R^2},$$

this contribution becomes

$$\frac{8\pi\rho_0}{3}\Delta K^{\text{in}}_{v,p}(R)R^2.$$

Thus

$$\widetilde{F}_{\text{cr},p}(R) = -\frac{16\pi\rho_0}{3}\Delta K^{\text{in}}_{v,p}(R)R^2 + \frac{8\pi\rho_0}{3}\Delta K^{\text{in}}_{v,p}(R)R^2.$$

Therefore

$$\boxed{\widetilde{F}_{\text{cr},p}(R) = -\frac{8\pi\rho_0}{3} R^2 \Delta K^{\text{in}}_{v,p}(R).}$$

Substituting the explicit form of $\Delta K^{\text{in}}_{v,p}(R)$,

$$\widetilde{F}_{\text{cr},p}(R) = -\frac{8\pi\rho_0}{3} R^2 \left(\frac{R_{0p}^3}{R^3} - 1\right) = \frac{8\pi\rho_0}{3} R^2 \left(1 - \frac{R_{0p}^3}{R^3}\right).$$

Hence

$$\boxed{\widetilde{F}_{\text{cr},p}(R) = \frac{8\pi\rho_0}{3} R^2 \left(1 - \frac{R_{0p}^3}{R^3}\right).}$$

The positive sign corresponds to work performed during the stretching of space.

10.8 Creation energy of the proton

Integrating over the stretching path from R_{0p} to R_p, one obtains

$$E_{\text{cr},p} = \int_{R_{0p}}^{R_p} \widetilde{F}_{\text{cr},p}(R)\, dR.$$

Substitution of the force gives

$$E_{\text{cr},p} = \frac{8\pi\rho_0}{3} \int_{R_{0p}}^{R_p} R^2 \left(1 - \frac{R_{0p}^3}{R^3}\right) dR.$$

Expanding the integrand,

$$E_{\text{cr},p} = \frac{8\pi\rho_0}{3} \int_{R_{0p}}^{R_p} \left(R^2 - \frac{R_{0p}^3}{R}\right) dR.$$

Using

$$\int R^2\, dR = \frac{R^3}{3}, \qquad \int \frac{dR}{R} = \ln R,$$

we obtain

$$\boxed{E_{\text{cr},p} = \frac{8\pi\rho_0}{3} \left[\frac{R_p^3 - R_{0p}^3}{3} - R_{0p}^3 \ln \frac{R_p}{R_{0p}}\right].}$$

This logarithmic structure distinguishes the stretching mechanism from the compression mechanism of the electron.

Now use

$$\Delta K^{\text{in}}_{v,p} = \frac{R_{0p}^3}{R_p^3} - 1,$$

so that

$$R_{0p}^3 = R_p^3 \left(1 + \Delta K^{\text{in}}_{v,p}\right).$$

Substitution into the energy gives

$$E_{\text{cr},p} = \frac{8\pi\rho_0}{3}\left[\frac{R_p^3 - R_p^3(1+\Delta K_{v,p}^{\text{in}})}{3} - R_p^3(1+\Delta K_{v,p}^{\text{in}})\ln\frac{R_p}{R_p(1+\Delta K_{v,p}^{\text{in}})^{1/3}}\right].$$

The first term is

$$-\frac{R_p^3 \Delta K_{v,p}^{\text{in}}}{3},$$

and in the logarithmic term,

$$\ln\frac{R_p}{R_p(1+\Delta K_{v,p}^{\text{in}})^{1/3}} = -\frac{1}{3}\ln(1+\Delta K_{v,p}^{\text{in}}).$$

Therefore,

$$\boxed{E_{\text{cr},p} = \frac{8\pi\rho_0 R_p^3}{9}\left[-\Delta K_{v,p}^{\text{in}} + \left(1+\Delta K_{v,p}^{\text{in}}\right)\ln\left(1+\Delta K_{v,p}^{\text{in}}\right)\right].}$$

This is the exact expression for the proton creation energy through the internal volumetric curvature intensity.

10.9 Proton creation energy through electric charge

For the proton,

$$q_p = \frac{Q_p}{R_e^2} = \frac{4\pi}{3}\frac{R_p^2}{R_e^2}\Delta K_{v,p}^{\text{in}},$$

and hence

$$\boxed{\Delta K_{v,p}^{\text{in}} = \frac{3q_p R_e^2}{4\pi R_p^2}.}$$

Therefore,

$$\boxed{E_{\text{cr},p} = \frac{8\pi\rho_0 R_p^3}{9}\left[-\frac{3q_p R_e^2}{4\pi R_p^2} + \left(1+\frac{3q_p R_e^2}{4\pi R_p^2}\right)\ln\left(1+\frac{3q_p R_e^2}{4\pi R_p^2}\right)\right].}$$

10.10 Proton radius from $E_{\text{cr},p} = m_p c^2$

Introduce

$$x = \frac{3q_p R_e^2}{4\pi R_p^2}, \qquad x < 0, \qquad |x| \ll 1.$$

Using

$$-x + (1+x)\ln(1+x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \cdots,$$

the leading term gives

$$\boxed{E_{\text{cr},p} \approx \frac{\rho_0}{4\pi} q_p^2 \frac{R_e^4}{R_p}.}$$

The condition $E_{\text{cr},p} = m_p c^2$, together with $\rho_0 = 1/(\varepsilon_0 R_e^4)$, yields

$$\boxed{R_p = \frac{q_p^2}{4\pi\varepsilon_0 m_p c^2}.}$$

10.11 Final formulas

The electron and proton formulas are summarized as

$$\boxed{E_{\text{cr},e} = \frac{4\pi\rho_0}{9} R_{0e}^3 \left(\Delta K_{v,e}^{\text{in}}\right)^2 = \frac{\rho_0}{4\pi} R_{0e}^3 q_e^2,}$$

$$\boxed{E_{\text{cr},p} = \frac{8\pi\rho_0}{3} \left[\frac{R_p^3 - R_{0p}^3}{3} - R_{0p}^3 \ln \frac{R_p}{R_{0p}}\right],}$$

$$\boxed{R_e = \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2 \left(1 - \dfrac{3q_e}{4\pi}\right)}, \qquad R_p = \frac{q_p^2}{4\pi\varepsilon_0 m_p c^2}.}$$

11 Numerical Verification of the Radius and Creation-Energy Formulas

The following numerical values are used:

$$R_e^{\text{cl}} = 2.8179403205 \times 10^{-15}\ \text{m},$$

$$m_e = 9.1093837139 \times 10^{-31}\ \text{kg}, \qquad m_p = 1.67262192595 \times 10^{-27}\ \text{kg},$$

$$\varepsilon_0 = 8.8541878188 \times 10^{-12}, \qquad c = 299792458\ \text{m/s},$$

$$\frac{m_p}{m_e} = 1836.152673426, \qquad q_e = 1.602176634 \times 10^{-19}, \qquad q_p = -q_e.$$

The constants are taken from CODATA 2022 / NIST [27, 28, 29].

11.1 Electron radius

Substitution gives

$$\boxed{R_e = 2.817940320459489 \times 10^{-15}\ \text{m}.}$$

Comparison with the classical electron radius gives

$$\boxed{\frac{R_e}{R_e^{\text{cl}}} = 0.9999999999856239.}$$

Furthermore,

$$R_{0e} = \frac{R_e}{\left(1 - \frac{3q_e}{4\pi}\right)^{1/3}},$$

so numerically

$$\boxed{R_{0e} = 2.817940320459489 \times 10^{-15}\ \text{m}.}$$

11.2 Proton radius

For $q_p = -q_e$,

$$\boxed{R_p = 1.5346982640656337 \times 10^{-18}\ \text{m}.}$$

The internal intensity is

$$\Delta K_{v,p}^{\text{in}} = \frac{3q_p R_e^2}{4\pi R_p^2},$$

which gives

$$\boxed{\Delta K_{v,p}^{\text{in}} = -1.289553495699288 \times 10^{-13}.}$$

Then

$$\boxed{R_{0p} = 1.5346982640655678 \times 10^{-18}\ \text{m}.}$$

11.3 Volumetric stiffness of space

Using

$$\rho_0 = \frac{1}{\varepsilon_0 R_e^4},$$

we obtain

$$\boxed{\rho_0 = 1.7911176149703568 \times 10^{69}\ \text{J/m}^3.}$$

11.4 Electron mass from creation energy

The electron creation energy is

$$E_{\text{cr},e} = \frac{\rho_0}{4\pi} R_{0e}^3 q_e^2,$$

which gives

$$\boxed{E_{\text{cr},e} = 8.18710578796845 \times 10^{-14}\ \text{J}.}$$

Therefore,

$$\boxed{m_e^{(\text{calc})} = \frac{E_{\text{cr},e}}{c^2} = 9.1093837139 \times 10^{-31}\ \text{kg}.}$$

The ratio is

$$\boxed{\frac{m_e^{(\text{calc})}}{m_e} = 1.0000000000000000.}$$

11.5 Proton mass from the exact creation energy

The exact proton formula gives

$$\boxed{E_{\mathrm{cr},p} = 1.5061034824428223 \times 10^{-10}\ \mathrm{J}.}$$

Thus

$$\boxed{m_p^{(\mathrm{calc})} = 1.6757661241625563 \times 10^{-27}\ \mathrm{kg}.}$$

The ratio to the tabulated proton mass is

$$\boxed{\frac{m_p^{(\mathrm{calc})}}{m_p} = 1.001879802102182,}$$

and the relative deviation is

$$\boxed{1.879802102181939 \times 10^{-3} \approx 0.188\%.}$$

11.6 Proton mass in the leading approximation

In the leading approximation,

$$E_{\mathrm{cr},p}^{(0)} = \frac{\rho_0}{4\pi} q_p^2 \frac{R_e^4}{R_p},$$

which gives

$$\boxed{E_{\mathrm{cr},p}^{(0)} = 1.503277618016312 \times 10^{-10}\ \mathrm{J}.}$$

Therefore,

$$\boxed{m_p^{(0)} = 1.6726219259499996 \times 10^{-27}\ \mathrm{kg},}$$

and

$$\boxed{\frac{m_p^{(0)}}{m_p} = 0.9999999999999998.}$$

11.7 Mass ratios

For the exact proton formula,

$$\boxed{\frac{m_p^{(\mathrm{calc})}}{m_e^{(\mathrm{calc})}} = 1839.6042770769513.}$$

The tabulated ratio is

$$\boxed{\frac{m_p}{m_e} = 1836.152673426.}$$

Thus

$$\boxed{\frac{(m_p^{(\mathrm{calc})}/m_e^{(\mathrm{calc})})}{(m_p/m_e)} = 1.001879802102182.}$$

In the leading approximation,

$$\boxed{\frac{m_p^{(0)}}{m_e^{(\mathrm{calc})}} = 1836.152673421526.}$$

11.8 Numerical summary

The numerical verification confirms that the electron radius practically coincides with the classical electron radius, the electron mass is reproduced exactly within numerical precision, and the proton mass is reproduced exactly in the leading approximation. The exact logarithmic formula gives a small nonlinear correction of order 10^{-3}, naturally interpreted as a correction beyond the leading approximation.

12 Propagation Speed of Disturbances in the Elastic Medium of Space and the Transition From Volumetric Stiffness to Energy and Momentum Density

To complete the geometric interpretation of electromagnetic interaction, it is necessary to justify how the volumetric stiffness of space ρ_0 may simultaneously be interpreted as the volumetric energy density of space and then as the source of the inertiality of the medium, which determines the density of momentum flux.

12.1 Volumetric stiffness and its energetic meaning

In elasticity theory, the energy density of volumetric deformation is

$$w = \frac{1}{2} B \varepsilon_v^2,$$

where B is the bulk modulus and ε_v is the relative volumetric deformation. Since ε_v is dimensionless,

$$[B] = \frac{E}{L^3}.$$

In the present model,

$$\boxed{[\rho_0] = \frac{E}{L^3}.}$$

It is therefore natural to identify

$$\boxed{B = \rho_0.}$$

12.2 Propagation speed of disturbances

For an isotropic elastic medium, the speed of longitudinal disturbances is

$$\boxed{u = \sqrt{\frac{B}{\rho_m}},}$$

where ρ_m is the mass density. With $B = \rho_0$,

$$\boxed{u = \sqrt{\frac{\rho_0}{\rho_m}}.}$$

12.3 Inertiality of the medium and transition to energy density

If space has an intrinsic energy density ρ_E, then the effective mass density is

$$\boxed{\rho_m = \frac{\rho_E}{c^2}.}$$

Substitution gives

$$u = c\sqrt{\frac{\rho_0}{\rho_E}}.$$

If disturbances in space propagate with the speed of light, $u = c$, then

$$\boxed{\rho_0 = \rho_E.}$$

Consequently,

$$\boxed{\rho_m = \frac{\rho_0}{c^2}.}$$

12.4 Transition from energy-density flux to momentum flux

If an energy density ρ_E is transported with velocity $\mathbf{v}$, the energy flux density is

$$\mathbf{J}_E = \rho_E \mathbf{v}.$$

The corresponding momentum density is

$$\mathbf{g} = \rho_m \mathbf{v} = \frac{\rho_E}{c^2}\mathbf{v} = \frac{\mathbf{J}_E}{c^2}.$$

Hence

$$\boxed{\mathbf{J}_p = \frac{1}{c^2}\mathbf{J}_E.}$$

Thus the replacement

$$\boxed{\rho_0 \longrightarrow \frac{\rho_0}{c^2}}$$

is not formal: it represents the transition from energy flux to momentum flux.

12.5 Summary

The model implies the chain of identifications

$$\boxed{B = \rho_0, \qquad \rho_E = \rho_0, \qquad \rho_m = \frac{\rho_0}{c^2}, \qquad \mathbf{J}_p = \frac{1}{c^2}\mathbf{J}_E.}$$

This chain makes it possible to pass from electrostatic interaction to magnetic-type interaction within one geometric model of the medium of space.

13 Interaction of Two Energy-Density Flows Generated by the Motion of Two Surface Sources of Curvature

Consider two surface sources of spatial curvature with geometric charges Q_1 and Q_2, located on spheres $S_{R_{1e}}$ and $S_{R_{2e}}$. Their centers satisfy

$$\mathbf{D} = \mathbf{X}_2 - \mathbf{X}_1, \qquad D = |\mathbf{D}|, \qquad D > R_{1e} + R_{2e}.$$

Although time is not yet introduced as an independent coordinate of the space of states, velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ may be treated as external geometric kinematic parameters characterizing directed transfer of the curvature state. The flows of energy and momentum density generated by the motion of sources are also treated as geometric quantities, without specifying their origin as time derivatives at this stage.

The motion of each charge is assumed to induce, in the external region, not the transport of the charge as a material point, but the transport of the energy density of space localized in the external curvature generated by that charge. Since the static interaction is bilinear in the two charge states, the dynamic joint flow is naturally assumed to be bilinear in the velocities. The minimal such structure is the tensor product of velocities.

13.1 Radial flux of spatial curvature

For each source, the rationalized external radial flux is

$$\boxed{\mathbf{S}_1(\mathbf{x}) = \frac{Q_1}{4\pi}\frac{\mathbf{x} - \mathbf{X}_1}{|\mathbf{x} - \mathbf{X}_1|^3}, \qquad |\mathbf{x} - \mathbf{X}_1| > R_{1e},}$$

$$\boxed{\mathbf{S}_2(\mathbf{x}) = \frac{Q_2}{4\pi}\frac{\mathbf{x}-\mathbf{X}_2}{|\mathbf{x}-\mathbf{X}_2|^3}, \qquad |\mathbf{x}-\mathbf{X}_2| > R_{2e}.}$$

They satisfy

$$\iint_{S_r} \mathbf{S}_i \cdot d\mathbf{S} = Q_i, \qquad r > R_{ie}.$$

The surface sources may be represented by delta-localized densities

$$\boxed{\varrho_{S,1}(\mathbf{r}_1) = \sigma_1\delta(r_1 - R_{1e}), \qquad \varrho_{S,2}(\mathbf{r}_2) = \sigma_2\delta(r_2 - R_{2e}),}$$

where

$$\boxed{\sigma_1 = \frac{Q_1}{4\pi R_{1e}^2}, \qquad \sigma_2 = \frac{Q_2}{4\pi R_{2e}^2}.}$$

13.2 Bilinear potential and gradient in the parameter space of two charges

The rationalized bilinear potential is

$$\boxed{\mathcal{P}_{12} = \iiint_{\mathbb{R}^3}\iiint_{\mathbb{R}^3} \varrho_{S,1}(\mathbf{r}_1)\varrho_{S,2}(\mathbf{r}_2)\frac{1}{4\pi|\mathbf{D}+\mathbf{r}_2-\mathbf{r}_1|}\,dV_1\,dV_2.}$$

Let

$$\mathbf{R}_{12} = \mathbf{D} + \mathbf{r}_2 - \mathbf{r}_1.$$

Introduce the gradient in the direct sum of the two parameter spaces:

$$\boxed{\nabla_{12} = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}).}$$

The unified two-component force is

$$\boxed{\mathbb{F}_{12} := -\rho_0 \iiiint\!\!\iint \varrho_{S,1}(\mathbf{r}_1)\varrho_{S,2}(\mathbf{r}_2)\nabla_{12}\frac{1}{4\pi|\mathbf{R}_{12}|}\,dV_1\,dV_2.}$$

Since

$$\nabla_{\mathbf{r}_1}\frac{1}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|^3}, \qquad \nabla_{\mathbf{r}_2}\frac{1}{|\mathbf{R}_{12}|} = -\frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|^3},$$

we obtain

$$\boxed{\mathbb{F}_{12} = \rho_0 \iiiint\!\!\iint \varrho_{S,1}\varrho_{S,2}\left(-\frac{\mathbf{R}_{12}}{4\pi|\mathbf{R}_{12}|^3}, \frac{\mathbf{R}_{12}}{4\pi|\mathbf{R}_{12}|^3}\right)\,dV_1\,dV_2.}$$

Thus $\mathbb{F}_{12} = (\mathbf{F}_1, \mathbf{F}_2)$ with $\mathbf{F}_1 = -\mathbf{F}_2$.

13.3 Surface form and evaluation

After delta localization, define

$$\boxed{\mathbf{K}_{12} := \iint_{S_{R_{1e}}} \iint_{S_{R_{2e}}} \sigma_1\sigma_2 \frac{\mathbf{D}+\mathbf{r}_2-\mathbf{r}_1}{4\pi|\mathbf{D}+\mathbf{r}_2-\mathbf{r}_1|^3}\, dS_1\, dS_2.}$$

Then

$$\boxed{\mathbb{F}_{12} = \rho_0(-\mathbf{K}_{12}, \mathbf{K}_{12}).}$$

Sequential evaluation gives

$$\boxed{\mathbf{K}_{12} = \frac{Q_1Q_2}{4\pi D^2}\hat{\mathbf{D}}.}$$

Therefore the static two-component force pair is

$$\boxed{\mathbb{F}_{12}^{(0)} = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}(\hat{\mathbf{D}}_1, \hat{\mathbf{D}}_2), \qquad \hat{\mathbf{D}}_1 = -\hat{\mathbf{D}}, \quad \hat{\mathbf{D}}_2 = +\hat{\mathbf{D}}.}$$

13.4 Tensor description of the joint energy-density flow

For moving sources the joint flow of energy density is assumed to be bilinear in the velocities:

$$\boxed{\mathbb{J}_{E,12} = \frac{\rho_0 Q_1 Q_2}{4\pi D^2}\Big((\mathbf{V}_2 \otimes \mathbf{V}_1)\hat{\mathbf{D}}_1, (\mathbf{V}_1 \otimes \mathbf{V}_2)\hat{\mathbf{D}}_2\Big).}$$

Using the relation between energy-flux density and momentum-flux density,

$$\boxed{\mathbb{J}_{p,12} = \frac{1}{c^2}\mathbb{J}_{E,12},}$$

the transition $\rho_0 \to \rho_0/c^2$ is justified.

13.5 From tensor products to double vector products

For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the tensor product acts according to

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}).$$

Therefore,

$$(\mathbf{V}_2 \otimes \mathbf{V}_1)\hat{\mathbf{D}}_1 = \mathbf{V}_2(\mathbf{V}_1 \cdot \hat{\mathbf{D}}_1),$$

and

$$(\mathbf{V}_1 \otimes \mathbf{V}_2)\hat{\mathbf{D}}_2 = \mathbf{V}_1(\mathbf{V}_2 \cdot \hat{\mathbf{D}}_2).$$

Now use the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

For the first component, set

$$\mathbf{a} = \mathbf{V}_1, \qquad \mathbf{b} = \mathbf{V}_2, \qquad \mathbf{c} = \hat{\mathbf{D}}_1.$$

Then

$$\mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}_1) = \mathbf{V}_2(\mathbf{V}_1 \cdot \hat{\mathbf{D}}_1) - \hat{\mathbf{D}}_1(\mathbf{V}_1 \cdot \mathbf{V}_2).$$

Hence

$$\boxed{(\mathbf{V}_2 \otimes \mathbf{V}_1)\hat{\mathbf{D}}_1 = \mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}_1) + \hat{\mathbf{D}}_1(\mathbf{V}_1 \cdot \mathbf{V}_2).}$$

For the second component, set

$$\mathbf{a} = \mathbf{V}_2, \qquad \mathbf{b} = \mathbf{V}_1, \qquad \mathbf{c} = \hat{\mathbf{D}}_2.$$

Then

$$\mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}_2) = \mathbf{V}_1(\mathbf{V}_2 \cdot \hat{\mathbf{D}}_2) - \hat{\mathbf{D}}_2(\mathbf{V}_1 \cdot \mathbf{V}_2).$$

Therefore

$$\boxed{(\mathbf{V}_1 \otimes \mathbf{V}_2)\hat{\mathbf{D}}_2 = \mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}_2) + \hat{\mathbf{D}}_2(\mathbf{V}_1 \cdot \mathbf{V}_2).}$$

Now add the two expressions:

$$(\mathbf{V}_2 \otimes \mathbf{V}_1)\hat{\mathbf{D}}_1 + (\mathbf{V}_1 \otimes \mathbf{V}_2)\hat{\mathbf{D}}_2$$

$$= \mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}_1) + \mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}_2) + (\hat{\mathbf{D}}_1 + \hat{\mathbf{D}}_2)(\mathbf{V}_1 \cdot \mathbf{V}_2).$$

Since

$$\hat{\mathbf{D}}_1 + \hat{\mathbf{D}}_2 = 0,$$

the longitudinal scalar-product term cancels. Thus

$$\boxed{(\mathbf{V}_2 \otimes \mathbf{V}_1)\hat{\mathbf{D}}_1 + (\mathbf{V}_1 \otimes \mathbf{V}_2)\hat{\mathbf{D}}_2 = \mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}_1) + \mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}_2).}$$

Since

$$\hat{\mathbf{D}}_1 = -\hat{\mathbf{D}}, \qquad \hat{\mathbf{D}}_2 = +\hat{\mathbf{D}},$$

we have

$$\boxed{\mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}_1) = -\mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}),}$$

and

$$\boxed{\mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}_2) = \mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}}).}$$

Therefore the dynamic force pair takes the final form

$$\boxed{\mathbb{F}_{12}^{(\mathrm{dyn})} = \frac{\rho_0 Q_1 Q_2}{4\pi c^2 D^2}\Big(-\mathbf{V}_1 \times (\mathbf{V}_2 \times \hat{\mathbf{D}}), \mathbf{V}_2 \times (\mathbf{V}_1 \times \hat{\mathbf{D}})\Big).}$$

This expression is the desired vector form of the tensor bilinear interaction. It shows explicitly that the magnetic-type force is obtained from the transverse part of the velocity-tensor flux after the cancellation of the longitudinal term proportional to $\mathbf{V}_1 \cdot \mathbf{V}_2$.

13.6 Single fundamental characteristic of the medium of space and the origin of the relation between ε_0 and μ_0

A significant consequence of the model is that it naturally introduces a single fundamental characteristic of the physical medium of space: the volumetric stiffness ρ_0, which also admits the interpretation of volumetric energy density. In the electrostatic sector,

$$\boxed{\rho_0 = \frac{1}{\varepsilon_0 R_e^4}.}$$

Thus ε_0 is not an independent phenomenological parameter, but characterizes how the volumetric stiffness of space manifests itself in the rationalized Coulomb law after the transition from geometric charge to electric normalization.

In the dynamic sector the same ρ_0 determines magnetic-type interaction. Since

$$\boxed{\mathbf{J}_p = \frac{1}{c^2}\mathbf{J}_E,}$$

the dynamic coefficient becomes

$$\frac{1}{4\pi\varepsilon_0 c^2}.$$

Comparison with the standard rationalized magnetic coefficient $\mu_0/(4\pi)$ gives

$$\boxed{\mu_0 = \frac{1}{\varepsilon_0 c^2},}$$

and hence

$$\boxed{\varepsilon_0 \mu_0 = \frac{1}{c^2}.}$$

In this model this relation is not introduced as an external empirical rule; it follows from the transition from energy density to momentum density in the same medium of space.

The common ontological source of electric, magnetic, and possibly gravitational interactions may therefore be expressed as

volumetric stiffness of space $\Longleftrightarrow$ energy density of space
$\Longrightarrow$ observable interactions.

14 Conclusion and Prospects for Further Development of the Model

The geometric scheme constructed in this paper shows that electric charge, interaction energy, and elementary-particle mass may be described as consequences of local curvature of space. The initial quantity is not charge as an independent physical entity, but the intensity of volumetric curvature distributed inside a fundamental region and compensated by an external distribution obtained from a variational principle and a global conservation law.

On this basis the following chain was obtained:

geometric charge
$\Longrightarrow$ Coulomb interaction
$\Longrightarrow$ magnetic-type interaction,

and also

creation energy
$\Longrightarrow$ rest mass
$\Longrightarrow$ characteristic particle radius.

The leading term of the force gives the rationalized Coulomb law, while the volume integral over the parameter space gives a more complete formula containing finite-size corrections interpreted as screening and prenormalization at small distances.

14.1 Possible transition to a six-dimensional space with signature $(1, 1, 1, -1, -1, -1)$

A natural generalization of the three-dimensional geometric scheme is a six-dimensional space with metric

$$\boxed{ds_6^2 = dx_1^2 + dx_2^2 + dx_3^2 - dy_1^2 - dy_2^2 - dy_3^2.}$$

The coordinates (x_1, x_2, x_3) form the ordinary spacelike sector, while (y_1, y_2, y_3) form a hidden timelike sector. The usual four-dimensional spacetime may then be interpreted as an effective projection of a deeper six-dimensional geometry. If a fixed timelike direction is selected and $|\mathbf{Y}| = ct$, then

$$ds_6^2 = d\mathbf{x}^2 - d\mathbf{y}^2$$

reduces to the Minkowski interval

$$\boxed{ds_4^2 = dx^2 + dy^2 + dz^2 - c^2dt^2.}$$

This connects the present approach with higher-dimensional geometric programs such as Kaluza–Klein theory [1, 2, 14], conformal higher-dimensional constructions [16, 22], and two-time physics [17, 18].

14.2 Infinite family of three-dimensional fibers and the geometric origin of time

In the six-dimensional setting one may consider an infinite family of three-dimensional fibers,

$$\boxed{\mathbb{R}^{3,3} \supset \{\Sigma_{\mathbf{Y}}\}_{\mathbf{Y}\in\mathbb{R}^3}.}$$

The observed flow of time may then be related not to an external independent entity, but to transitions between different three-dimensional fibers. Time becomes a geometric parameter characterizing a sequence of projections of the same higher-dimensional object onto different three-dimensional slices.

14.3 Possible relation to the quantum nature of the Universe

In this higher-dimensional interpretation, quantum description may be projective rather than fundamental. What appears in standard quantum theory as probability, superposition, discreteness of spectra, or dependence on measurement configuration may reflect the fact that the observed three-dimensional picture is only a partial projection of a richer six-dimensional geometric structure.

14.4 Electron and proton as effective three-dimensional projections

In this paper the electron and the proton were modeled as spherical curvature regions with constant internal volumetric curvature intensity. This spherical model is likely only a first effective approximation. If particles are fundamentally six-dimensional structures, their observed spherical form may arise as a projection of a more complex multidimensional configuration. A possible candidate is a six-dimensional toroidal structure whose three-dimensional projection appears spherical or quasi-spherical.

14.5 Possible explanation of the difference between theoretical and experimental particle sizes

If experiments observe not the full multidimensional object but only its effective three-dimensional projection, the measured radius need not coincide with the true geometric scale of the fundamental structure. It may represent an effective projection radius, an interaction scale, or an integral characteristic of the multidimensional structure.

14.6 Possible connection between electricity and gravitation

The present work associates electric interaction with the trace, volumetric part of spatial deformation. A natural extension is to consider the traceless part of the curvature tensor. If this traceless part satisfies a divergence equilibrium condition,

$$\partial_i \mathcal{K}_{ij} = 0,$$

it may provide a geometric candidate for gravitation. In this case,

$$\boxed{\begin{aligned}&\text{electricity} \sim \text{volumetric part of curvature,}\\ &\text{gravitation} \sim \text{traceless compensating part.}\end{aligned}}$$

This differs from standard unification schemes because gravity is not added externally; it may arise as the geometric self-consistency condition of the electric curvature already introduced.

14.7 Possible six-dimensional Lagrangian of the medium of space and the dynamic origin of the speed of light

The next natural step is the construction of a dynamic theory in $\mathbb{R}^{3,3}$. Introduce coordinates

$$X^A = (x^1, x^2, x^3, y^1, y^2, y^3), \qquad A = 1, \ldots, 6,$$

with metric

$$\eta_{AB} = \text{diag}(+1, +1, +1, -1, -1, -1).$$

Let $U^A(X)$ be a displacement field of the six-dimensional medium, and define the deformation tensor

$$\boxed{\varepsilon_{AB} = \frac{1}{2}(\partial_A U_B + \partial_B U_A).}$$

A natural candidate for the Lagrangian density of a linear isotropic six-dimensional medium is

$$\boxed{\mathcal{L}_6 = \frac{1}{2}\lambda(\varepsilon_A^{\ A})^2 + \mu\varepsilon_{AB}\varepsilon^{AB} + \frac{1}{2}\rho_0\eta^{AB}\partial_A U_C \partial_B U^C.}$$

The action is

$$\boxed{S_6 = \int d^6X\, \mathcal{L}_6.}$$

Choosing $y^1 = ct$ and assuming independence from y^2 and y^3, one has

$$\frac{\partial}{\partial y^1} = \frac{1}{c}\frac{\partial}{\partial t}.$$

The six-dimensional wave equation

$$\boxed{\Box_6\Phi = 0, \qquad \Box_6 = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} - \sum_{a=1}^{3}\frac{\partial^2}{\partial y_a^2},}$$

then reduces, for dependence on $y^1 = ct$ only, to

$$\boxed{\frac{\partial^2\Phi}{\partial t^2} = c^2\sum_{i=1}^{3}\frac{\partial^2\Phi}{\partial x_i^2}.}$$

Thus in the dynamic version of the model the speed of light should arise not as an external postulate, but as the propagation speed of small disturbances of the six-dimensional medium. This will be the subject of the next study.

14.8 Possible origin of Lorentz transformations from six-dimensional geometry

The same six-dimensional geometry may naturally lead to the kinematic structure of special relativity. The metric

$$ds_6^2 = dX_1^2 + dX_2^2 + dX_3^2 - dY_1^2 - dY_2^2 - dY_3^2$$

is invariant under transformations preserving signature $(3,3)$. If one chooses a timelike direction Y_1 and sets

$$Y_1 = ct,$$

then transformations mixing X_1 and Y_1 take the form of hyperbolic rotations:

$$\boxed{\begin{aligned} X_1' &= X_1\cosh\psi + Y_1\sinh\psi, \\ Y_1' &= X_1\sinh\psi + Y_1\cosh\psi. \end{aligned}}$$

Substitution of $Y_1 = ct$ gives

$$\boxed{\begin{aligned} x' &= x\cosh\psi + ct\sinh\psi, \\ ct' &= x\sinh\psi + ct\cosh\psi. \end{aligned}}$$

With

$$\cosh\psi = \gamma, \qquad \sinh\psi = \gamma\frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

these become the Lorentz transformations

$$\boxed{\begin{aligned} x' &= \gamma(x + vt), \\ t' &= \gamma\left(t + \frac{v}{c^2}x\right). \end{aligned}}$$

The sign of v depends on the chosen direction of the hyperbolic rotation. The essential point is that the Lorentz structure appears as a projection of hyperbolic rotations in the plane (X_1, Y_1) of the six-dimensional space with signature $(3, 3)$. This connects the model with the broader literature on higher-dimensional and two-time formulations of relativistic physics [16, 17, 18, 19].

14.9 General conclusion

The model developed in this article shows that charge, mass, and electromagnetic interaction may be described geometrically through local curvature of space without introducing electric charge as a primary object. The results can be developed toward a broader multidimensional theory in which

$$\boxed{\begin{array}{l} \text{three-dimensional model} \\ \implies \text{effective projection of six-dimensional geometry,} \end{array}}$$

$$\boxed{\begin{array}{l} \text{time} \\ \implies \text{geometric transition between 3D fibers,} \end{array}}$$

$$\boxed{\begin{array}{l} \text{quantumness} \\ \implies \text{projection effect of a higher-dimensional structure,} \end{array}}$$

$$\boxed{\begin{array}{l} \text{electron and proton} \\ \implies \text{not fundamental spheres, but projections of 6D objects.} \end{array}}$$

If this program can be realized, the geometric scheme constructed here may serve not only as a model of electromagnetic interaction, but also as a basis for a more general theory in which spacetime, charge, mass, quantum properties, and possibly gravitation are different manifestations of one multidimensional geometric structure.

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