

Proof of the Collatz Function through Identity Reduction, Catalans Conjecture, and Implementation of Infinite Expansion

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Abstract

The Collatz function is one of the simplest difficult problems in modern mathematics. For any positive integer, multiply any odd integer by 3 and add 1, while any even integer is divided by 2. Take the result and re-insert it into the function. Every integer will eventually fall to 1, and begin a loop of the sequence $1 - 4 - 2 - 1$. Will this function produce another loop at some point? Will the jumps $(3n + 1)$ overtake the drops $(/2)$ and climb to infinity? Through a nested fractal implementation as well as the reduction principle set in Catalan's Conjecture, it is shown that both of these questions are topologically impossible within the system.

1 Base Function(Single Jump)

In its simplest form, we can see the inherent mechanism of the function. Here, "k" is the total drops.

$$f(x) = \frac{3(f(x)) + 1}{2^k} \quad (1)$$

$$2^k(f(x)) = 3(f(x)) + 1 \quad (2)$$

$$2^k(f(x)) - 3(f(x)) = 1 \quad (3)$$

$$(f(x))(2^k - 3) = 1 \quad (4)$$

$$f(x) = \frac{1}{2^k - 3} \quad (5)$$

1.1 Base Function Shortcut(Single Jump)

From here, "k" is only the extra drops, with m representing inherent drops (equal to the number of jumps) while at the same time functionally facilitating normalization.

$$f(x) = \frac{1.5(f(x)) + .5}{2^k} \quad (6)$$

$$2^k(f(x)) = 1.5(f(x)) + .5 \quad (7)$$

$$2^k(f(x)) - 1.5(f(x)) = .5 \quad (8)$$

$$(f(x))(2^k - 1.5) = .5 \quad (9)$$

$$f(x) = \frac{.5}{2^k - 1.5} \quad (10)$$

Normalize by removing decimals(multiply by 2(as 2^m))

$$f(x) = \frac{2(.5)}{2^m(2^k - 1.5)} \quad (11)$$

$$f(x) = \frac{1}{2^{k+1} - 3} \quad (12)$$

$$f(x) = \frac{1}{2^{k+m} - 3^m} \quad (13)$$

2 Nesting Function

2.1 Fractal Nesting(Double Jump)

Now, we substitute the function into its own equation, showing a self-referential pattern.

$$f(x) = \frac{1.5\left(\frac{1.5(f(x))+.5}{2^{k_1}}\right) + .5}{2^{k_2}} \quad (14)$$

$$2^{k_2} \cdot f(x) = 1.5 \left(\frac{1.5f(x) + 0.5}{2^{k_1}} \right) + 0.5 \quad (15)$$

$$2^{k_1+k_2} \cdot f(x) = 1.5(1.5f(x) + 0.5) + 0.5 \cdot 2^{k_1} \quad (16)$$

$$2^{k_1+k_2} \cdot f(x) = 2.25f(x) + 0.75 + 0.5 \cdot 2^{k_1} \quad (17)$$

$$f(x)(2^{k_1+k_2} - 2.25) = 0.5 \cdot 2^{k_1} + 0.75 \quad (18)$$

Normalize by removing decimals(multiply by 4(as 2^m))

$$f(x)(4 \cdot 2^{k_1+k_2} - 9) = 2 \cdot 2^{k_1} + 3 \quad (19)$$

$$f(x)(2^{k_1+k_2+2} - 9) = 2^{k_1+1} + 3 \quad (20)$$

$$f(x) = \frac{2^{k_1+1} + 3}{2^{k_1+k_2+2} - 9} \quad (21)$$

$$f(x) = \frac{2^{k_1+1} + 3}{2^{k_1+k_2+2} - 3^2} \quad (22)$$

At this point, K becomes the sum of the total extra drops from every layer(k_n).

$$f(x) = \frac{2^{k_1+1} + 3}{2^{K+m} - 3^m} \quad (23)$$

2.2 Fractal Nesting(Triple Jump)

$$f(x) = \frac{1.5\left(\frac{1.5\left(\frac{1.5(f(x))+.5}{2^{k_1}}\right)+.5}{2^{k_2}}\right) + .5}{2^{k_3}} \quad (24)$$

$$2^{k_1+k_2+k_3} \cdot f(x) = 1.5(1.5(1.5f(x) + 0.5) + 0.5 \cdot 2^{k_1}) + 0.5 \cdot 2^{k_1+k_2} \quad (25)$$

$$2^{k_1+k_2+k_3} \cdot f(x) = 3.375f(x) + 1.125 + 0.75 \cdot 2^{k_1} + 0.5 \cdot 2^{k_1+k_2} \quad (26)$$

$$f(x)(2^{k_1+k_2+k_3} - 3.375) = 0.5 \cdot 2^{k_1+k_2} + 0.75 \cdot 2^{k_1} + 1.125 \quad (27)$$

Normalize by removing decimals(multiply by 8(as 2^m))

$$f(x)(8 \cdot 2^{k_1+k_2+k_3} - 27) = 4 \cdot 2^{k_1+k_2} + 6 \cdot 2^{k_1} + 9 \quad (28)$$

$$f(x)(2^3 \cdot 2^{k_1+k_2+k_3} - 27) = 2^2 \cdot 2^{k_1+k_2} + 6 \cdot 2^{k_1} + 9 \quad (29)$$

$$f(x)(2^{k_1+k_2+k_3+3} - 27) = 2^{k_1+k_2+2} + 3 \cdot 2^{k_1+1} + 9 \quad (30)$$

$$f(x) = \frac{2^{k_1+k_2+2} + 3 \cdot 2^{k_1+1} + 9}{2^{k_1+k_2+k_3+3} - 27} \quad (31)$$

$$f(x) = \frac{2^{k_1+k_2+2} + 3 \cdot 2^{k_1+1} + 3^2}{2^{k_1+k_2+k_3+3} - 3^3} \quad (32)$$

2.3 Emergent Pattern

By continuing to add nested layers, a pattern emerges as the following formula, with j defined as the jump stage, and S defined as the partial sum of extra drops at that stage.

$$f(x) = \frac{\sum_{j=0}^{m-1} 3^j \cdot 2^{S_{m-1-j}+(m-1-j)}}{2^{K+m} - 3^m} \quad (33)$$

3 Reduction Principle

For any looping number, $f(x)$ is an odd integer that satisfies the above equation. In addition to this, there is another equation that looks quite similar that also needs to be satisfied for any integer, which is when a whole integer is converted to a rational number as an identity.

$$f(x) = \frac{f(x)y}{1y} = \frac{f(x)}{1} = \frac{\sum_{j=0}^{m-1} 3^j \cdot 2^{S_{m-1-j}+(m-1-j)}}{2^{K+m} - 3^m} \quad (34)$$

The specific numbers in the fraction do not matter, only that they divide equally, but they will always reduce to $\frac{f(x)}{1}$ for ANY integer.

$$5 = \frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4} \quad (35)$$

So, for any looping integer,

$$2^{K+m} - 3^m = 1 \quad (36)$$

There is only ONE set of exponents that satisfy this equation (Mihalescu, 2004), resulting in the 4-2-1 loop, and showing there are no other possible loops.

$$2^{1+1} - 3^1 = 1 \quad (37)$$

This has 1 jump, 1 inherent drop, and 1 extra drop. Since there is only one jump, we can use the simplified equation (Line 13) to find the only looping positive integer.

$$f(x) = \frac{1}{2^{k+m} - 3^m} = \frac{1}{2^{1+1} - 3^1} = \frac{1}{2^2 - 3} = \frac{1}{4 - 3} = \frac{1}{1} = 1 \quad (38)$$

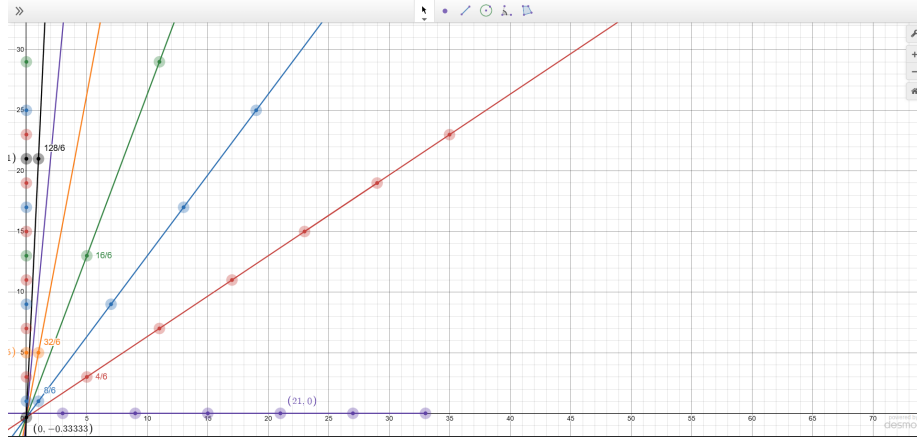


Figure 1: Here, x is the resulting number and y is the origin, with similar drop points connected linearly. The drop lines converge at $(0, -1/3)$

4 Klein Inversion Point

4.1 Limit Operator

$$f(x) = \lim_{m \rightarrow \infty} \frac{\sum_{j=0}^{m-1} 3^j \cdot 2^{S_{m-1-j} + (m-1-j)}}{2^{K+m} - 3^m} \quad (39)$$

4.2 Dominant Terms

- Numerator: The final term (3^j , $j = m - 1$) holds the most influence.
- Denominator: As $m \rightarrow \infty$, 2^{K+m} loses all strength to 3^m

4.3 Normalization

- Numerator: $\frac{3^{m-1}}{3^m} \rightarrow 1/3$
- Denominator: $\frac{2^{K+m}}{3^m} - \frac{3^m}{3^m} \rightarrow 0 - 1 = -1$

4.4 Implementation and Interpretation

$$f(x) = \frac{1/3}{-1} = -\frac{1}{3} \quad (40)$$

Essentially this shows the system is self limiting. Even when attempting to jump infinite times the system rebounds to $-1/3$. This $-1/3$ showed up in another place, though, as the convergence point for the drop lines while graphing drop mechanics (Figure 1).

5 Conclusion

Applying the Reduction Principle as well as Catalan's Conjecture, it is shown that the only possible loop in the Collatz function is $1 - 4 - 2 - 1$. In addition, when pushing the limits of the function towards infinity, the system itself topologically rebounds to a stopping point as a negative fraction of $-\frac{1}{3}$, showing the system as self-bound.

6 Sources

Mihăilescu, Preda (2004). "Primary Cyclotomic Units and a Proof of Catalan's Conjecture". *Journal für die reine und angewandte Mathematik (Crelle's Journal)*. 2004 (572): 167–195. doi:10.1515/crll.2004.048.

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- v2:Modified title to reflect specifics. Expanded the equations in Section 3(Reduction Principle) for clarity. Removed Section 6(Heuristics) for brevity.