Entanglement from Octonionic Non-Associativity

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Abstract

We develop a mathematically explicit and physically transparent derivation of quantum entanglement from an octonionic fundamental model. The key observation is that the algebra of octonions is non-associative, so that the associator

$$A(x, y, z) := (xy)z - x(yz) \tag{1}$$

provides a genuine trilinear measure of non-associativity. This immediately implies that a fundamental coupling responsible for entanglement cannot be purely bipartite at the octonionic level. Instead, the minimal nontrivial coupling necessarily involves three degrees of freedom: one field for subsystem A, one field for subsystem B, and an additional mediating octonionic degree of freedom Ξ. We show that the resulting action naturally contains a term of the form

$$\lambda \left\| A(\Psi_A, \Xi, \Psi_B) \right\|^2, \tag{2}$$

which is structurally non-separable and therefore induces coupled equations of motion. After projection onto an effective associative sector, this term appears as an entangling Hamiltonian on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We then prove step by step that such a Hamiltonian dynamically generates non-factorizable states from initial product states. A two-qubit example is worked out explicitly, including the reduced density matrix, entanglement entropy, and Bell–CHSH violation. The central conclusion is that, in this framework, entanglement is the effective associative manifestation of a deeper non-associative octonionic correlation structure.

1 Introduction

One of the most striking features of quantum theory is entanglement. In the usual Hilbert-space formalism, entanglement is described kinematically: one considers a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and a state is called entangled if it cannot be written as a simple tensor product

$$|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B . \tag{3}$$

However, in the standard formulation this structure is postulated rather than derived from a deeper algebraic principle.

The aim of the present report is to show how entanglement can arise from a more fundamental octonionic dynamics. The octonions form the largest normed division algebra and, unlike the real numbers, complex numbers, and quaternions, they are non-associative. This means that the grouping of products matters:

$$(xy)z \neq x(yz) \tag{4}$$

in general.

This failure of associativity is not a minor technical detail. It provides a genuinely new algebraic structure, encoded in the associator

$$A(x, y, z) := (xy)z - x(yz). \tag{5}$$

The central claim of this report is that this non-associative structure gives rise to a minimal fundamental interaction that is intrinsically non-separable. Once projected onto the effective associative sector in which ordinary quantum states live, this interaction becomes precisely the kind of coupling that dynamically generates entanglement.

The argument proceeds in five stages:

(i) We define the octonionic associator and explain its basic properties.

(ii) We show why a fundamental octonionic entanglement mechanism must involve at least three degrees of freedom.

(iii) We construct the minimal covariant action containing the corresponding associator coupling.

(iv) We derive the equations of motion and show that they are non-separable.

(v) We project to an effective Hilbert-space description and prove that the induced Hamiltonian entangles generic product states.

The report is written so that every symbol and equation is introduced explicitly.

2 Octonionic algebra as the starting point

2.1 Definition of the octonions

The octonions, denoted by $\mathbb{O}$, form an eight-dimensional real algebra. This means that every octonion $x \in \mathbb{O}$ can be written as

$$x = x_0 \, 1 + \sum_{i=1}^{7} x_i e_i, \tag{6}$$

where:

- $x_0, \dots, x_7 \in \mathbb{R}$ are real coefficients,
- 1 is the multiplicative unit,
- $e_1, \dots, e_7$ are seven imaginary basis elements.

Thus, as a real vector space,

$$\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{O}), \tag{7}$$

where $\mathrm{Im}(\mathbb{O})$ denotes the seven-dimensional imaginary part.

The octonions are a division algebra, meaning that every nonzero element has a multiplicative inverse. They also carry a positive norm

$$\|x\|^2 = x\bar{x} = \bar{x}x, \tag{8}$$

where $\bar{x}$ denotes the octonionic conjugate of x. Explicitly, if

$$x = x_0 + \sum_{i=1}^{7} x_i e_i, \tag{9}$$

then

$$\bar{x} = x_0 - \sum_{i=1}^{7} x_i e_i. \tag{10}$$

The norm is multiplicative:

$$\|xy\| = \|x\| \, \|y\|. \tag{11}$$

2.2 Non-associativity

The crucial difference between $\mathbb{O}$ and the smaller normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is that multiplication in $\mathbb{O}$ is generally not associative. This means that for generic octonions x, y, z,

$$(xy)z \neq x(yz). \tag{12}$$

This failure is measured by the associator.

2.3 Definition of the associator

For three octonions $x, y, z \in \mathbb{O}$, define

$$A(x, y, z) := (xy)z - x(yz). \tag{13}$$

This quantity vanishes if and only if the product is associative on the given triple.

The symbol A therefore denotes the associator map

$$A : \mathbb{O} \times \mathbb{O} \times \mathbb{O} \to \mathbb{O}. \tag{14}$$

2.4 Basic properties of the associator

The associator satisfies three properties that are essential for everything that follows.

(i) Trilinearity. The map $A(x, y, z)$ is linear in each argument separately. For example, in the first slot:

$$A(ax_1 + bx_2, y, z) = aA(x_1, y, z) + bA(x_2, y, z), \tag{15}$$

for real numbers $a, b \in \mathbb{R}$. Analogous relations hold in the second and third slots.

This property follows directly from bilinearity of octonionic multiplication.

(ii) Total alternation. The associator is alternating. In particular,

$$A(x, x, z) = 0, \qquad A(x, y, y) = 0, \tag{16}$$

and more generally,

$$A(x, y, z) = 0 \tag{17}$$

whenever two of the three arguments are equal.

This means that the associator is sensitive only to genuinely distinct non-associative directions.

(iii) Vanishing on associative subalgebras. Every quaternionic subalgebra $\mathbb{H} \subset \mathbb{O}$ is associative. Therefore, if

$$x, y, z \in \mathbb{H}, \tag{18}$$

then

$$A(x, y, z) = 0. \tag{19}$$

This fact is decisive. It means that the associator probes precisely the degrees of freedom that leave every associative sector. If all dynamics were confined to a quaternionic subalgebra, the associator would vanish identically and no genuinely octonionic effect could survive.

3 Why at least three degrees of freedom are necessary

We want to understand how entanglement between two later effective subsystems A and B can arise from the octonionic level.

Suppose we try to build a fundamental interaction involving only two octonionic fields,

$$\Psi_A, \Psi_B. \tag{20}$$

At first sight one might hope for a non-associative coupling directly between them. But the associator requires three arguments:

$$A(x, y, z). \tag{21}$$

There is no meaningful two-argument object $A(\Psi_A, \Psi_B)$. The non-associative structure is intrinsically trilinear.

Therefore the minimal fundamental coupling must involve a third octonionic degree of freedom, denoted by

$$\Xi. \tag{22}$$

Interpretation:

- Ψ_A: octonionic field associated with subsystem A,

- Ψ_B: octonionic field associated with subsystem B,
- Ξ: mediating non-associative octonionic degree of freedom.

The natural fundamental coupling is then

$$A(\Psi_A, \Xi, \Psi_B). \tag{23}$$

This is not an arbitrary choice. It is the unique minimal expression that:

- is genuinely octonionic,
- probes non-associativity,
- couples the future effective subsystems A and B,
- and vanishes identically if the configuration collapses into an associative subalgebra.

Thus, at the fundamental level, entanglement is not a primitive two-body notion. It is a three-legged non-associative effect.

4 Minimal covariant action

4.1 Structure of the action

We now write the minimal covariant action for the three octonionic fields Ψ_A, Ψ_B, Ξ:

$$S[\Psi_A, \Psi_B, \Xi] = \int d^4x \sqrt{-g} \left(\mathcal{L}_A(\Psi_A) + \mathcal{L}_B(\Psi_B) + \mathcal{L}_\Xi(\Xi) - \lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2\right). \tag{24}$$

with

- S is the action functional.
- d^4x is the spacetime integration measure.
- g denotes the determinant of the spacetime metric $g_{\mu\nu}$.
- $\sqrt{-g}$ is the invariant volume factor in Lorentzian signature.
- $\mathcal{L}_A(\Psi_A)$ is the Lagrangian density describing the free or local dynamics of subsystem A.
- $\mathcal{L}_B(\Psi_B)$ is the corresponding Lagrangian density for subsystem B.
- $\mathcal{L}_\Xi(\Xi)$ is the Lagrangian density for the mediating field Ξ.
- $\lambda > 0$ is a coupling constant that sets the strength of the octonionic associator interaction.
- $\|A(\Psi_A, \Xi, \Psi_B)\|^2$ is the squared norm of the associator.

The last term is the genuinely new ingredient. Define

$$V_{\text{assoc}} := \lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2. \tag{25}$$

This is the non-associative interaction energy density.

4.2 Meaning of the norm squared

The squared norm is given by the octonionic inner product:

$$\|X\|^2 = \langle X, X \rangle. \tag{26}$$

Hence

$$\|A(\Psi_A, \Xi, \Psi_B)\|^2 = \langle A(\Psi_A, \Xi, \Psi_B), A(\Psi_A, \Xi, \Psi_B) \rangle \,. \tag{27}$$

So the action contains a quartic expression in the fields, because the associator is already trilinear and then its norm is squared.

5 Why the associator term is non-separable

We now show that the associator term cannot be reduced to a sum of separate A- and B-parts.

5.1 What separability would mean

A separable interaction between two subsystems would have one of the following structures.

A purely additive form:

$$V_{\text{sep}}(\Psi_A, \Psi_B) = V_A(\Psi_A) + V_B(\Psi_B). \tag{28}$$

Or, more generally, a bilinear factorized form:

$$V_{\text{bilin}}(\Psi_A, \Psi_B) = \sum_\alpha f_\alpha(\Psi_A) g_\alpha(\Psi_B), \tag{29}$$

where f_α depends only on Ψ_A and g_α only on Ψ_B.

Such terms do not encode an irreducible three-way non-associative structure.

5.2 Structure of the associator term

The octonionic interaction is

$$\|A(\Psi_A, \Xi, \Psi_B)\|^2. \tag{30}$$

Because the associator is trilinear, the quantity

$$A(\Psi_A, \Xi, \Psi_B) \tag{31}$$

depends simultaneously on all three fields. Its norm squared then produces mixed contributions involving Ψ_A, Ψ_B, and Ξ in a way that does not split into a pure A-piece plus a pure B-piece.

The key point is this:

- if Ψ_A, Ξ, Ψ_B all lie in one common associative quaternionic subalgebra, then

$$A(\Psi_A, \Xi, \Psi_B) = 0; \tag{32}$$

- if they do not lie in such a common associative sector, then generically

$$A(\Psi_A, \Xi, \Psi_B) \neq 0. \tag{33}$$

Thus the energy depends on the joint configuration of the three fields, not on subsystem A or subsystem B separately.

5.3 Physical meaning

The energy-carrying degree of freedom is no longer attributable to subsystem A alone or subsystem B alone. It belongs to the combined triplet

$$(\Psi_A, \Psi_B, \Xi). \tag{34}$$

This is exactly the structural precursor of entanglement: the relevant physical information is stored in the joint configuration rather than in separate subsystem states.

6 Equations of motion

We now vary the action to obtain the field equations.

6.1 Euler–Lagrange principle

The equations of motion are obtained by requiring stationarity of the action under arbitrary variations:

$$\delta S = 0. \tag{35}$$

Since there are three independent fields, we obtain three Euler–Lagrange equations:

$$\frac{\delta S}{\delta \Psi_A} = 0, \qquad \frac{\delta S}{\delta \Psi_B} = 0, \qquad \frac{\delta S}{\delta \Xi} = 0. \tag{36}$$

Here $\delta S/\delta\Psi_A$ denotes the functional derivative of the action with respect to the field Ψ_A, and similarly for the others.

6.2 Variation with respect to Ψ_A

Let

$$X := A(\Psi_A, \Xi, \Psi_B). \tag{37}$$

Then the interaction term is

$$-\lambda\langle X, X\rangle. \tag{38}$$

Its variation is

$$\delta\langle X, X\rangle = 2\langle \delta X, X\rangle, \tag{39}$$

assuming the inner product is symmetric and real-valued on the relevant projected sector.

Now vary X with respect to Ψ_A. Because the associator is trilinear,

$$\delta X = A(\delta\Psi_A, \Xi, \Psi_B). \tag{40}$$

This is the crucial simplification: differentiation of a trilinear map in one variable simply replaces that variable by its variation.

Hence the variation of the action with respect to Ψ_A gives schematically

$$\mathcal{E}_A(\Psi_A) - 2\lambda \left(D_{\Psi_A} A(\Psi_A, \Xi, \Psi_B)\right)^\dagger A(\Psi_A, \Xi, \Psi_B) = 0. \tag{41}$$

- $\mathcal{E}_A(\Psi_A)$ denotes the Euler–Lagrange operator coming from $\mathcal{L}_A$.
- $D_{\Psi_A} A$ denotes the derivative of the associator with respect to its first slot.
- The symbol † denotes the adjoint with respect to the relevant inner product.

Because of trilinearity,

$$D_{\Psi_A} A(\delta\Psi_A, \Xi, \Psi_B) = A(\delta\Psi_A, \Xi, \Psi_B). \tag{42}$$

So the equation for Ψ_A depends explicitly on Ψ_B and on Ξ.

6.3 Variation with respect to Ψ_B

Exactly analogously,

$$\mathcal{E}_B(\Psi_B) - 2\lambda \left(D_{\Psi_B} A(\Psi_A, \Xi, \Psi_B)\right)^\dagger A(\Psi_A, \Xi, \Psi_B) = 0. \tag{43}$$

Again, by trilinearity,

$$D_{\Psi_B} A(\Psi_A, \Xi, \delta\Psi_B) = A(\Psi_A, \Xi, \delta\Psi_B). \tag{44}$$

Thus the equation for Ψ_B explicitly depends on Ψ_A and Ξ.

6.4 Variation with respect to Ξ

Similarly,

$$\mathcal{E}_\Xi(\Xi) - 2\lambda \left(D_\Xi A(\Psi_A, \Xi, \Psi_B)\right)^\dagger A(\Psi_A, \Xi, \Psi_B) = 0. \tag{45}$$

So all three fields are dynamically coupled through the associator sector.

6.5 Non-separability of the dynamics

This is the core dynamical statement:

- the equation for subsystem A depends on subsystem B,
- the equation for subsystem B depends on subsystem A,
- and both are tied together through the mediating field Ξ.

Therefore the dynamics is not separable. At the field-theoretic level, the two subsystems are already linked by an irreducible common interaction structure.

7 Projection to an effective associative Hilbert-space sector

Up to this point the model is formulated directly in octonionic field language. To connect with standard quantum mechanics, we must pass to an effective associative description.

This step is an additional physical assumption of the framework: observable quantum states are assumed to arise after projection onto an associative sector, typically a complex or quaternionic substructure.

7.1 Effective Hilbert spaces

Assume that the observable states of the two subsystems are described in effective Hilbert spaces

$$\mathcal{H}_A, \qquad \mathcal{H}_B. \tag{46}$$

Then the total effective state space is

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{47}$$

Here:

- $\mathcal{H}_A$ is the Hilbert space of subsystem A,
- $\mathcal{H}_B$ is the Hilbert space of subsystem B,
- $\otimes$ denotes the tensor product.

7.2 Effective Hamiltonian

After projection, the octonionic associator sector induces an effective Hamiltonian of the form

$$H_{\text{eff}} = H_A \otimes I_B + I_A \otimes H_B + H_{\text{ent}}. \tag{48}$$

Explanation of the symbols:

- H_A is the local Hamiltonian acting on subsystem A,
- H_B is the local Hamiltonian acting on subsystem B,
- I_A and I_B are identity operators on $\mathcal{H}_A$ and $\mathcal{H}_B$,
- H_{ent} is the effective interaction induced by the octonionic associator term.

The decisive property is that

$$H_{\text{ent}} \neq H'_A \otimes I_B + I_A \otimes H'_B \tag{49}$$

for generic configurations. In other words, H_{ent} is not just another sum of local terms. It is a genuine coupling operator.

This is how the fundamental non-separability reappears in the effective Hilbert-space description.

8 Explicit form of the induced coupling operator

Choose orthonormal bases

$$\{|i\rangle_A\} \subset \mathcal{H}_A, \qquad \{|\mu\rangle_B\} \subset \mathcal{H}_B. \tag{50}$$

Then any operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded as

$$H_{\mathrm{ent}} = \sum_{i,j,\mu,\nu} K_{i\mu,j\nu} \, |i\rangle \langle j| \otimes |\mu\rangle \langle \nu| \, . \tag{51}$$

Here:

- i, j label basis vectors of subsystem A,
- μ, ν label basis vectors of subsystem B,
- $K_{i\mu,j\nu}$ are complex coefficients.

In the octonionic model these coefficients arise from projected matrix elements of the associator-induced operator:

$$K_{i\mu,j\nu} \sim \langle i, \mu|\mathcal{A}_{\Xi}|j, \nu\rangle \, , \tag{52}$$

where $\mathcal{A}_{\Xi}$ is the effective operator induced by the map

$$(\Psi_A, \Psi_B) \mapsto A(\Psi_A, \Xi, \Psi_B). \tag{53}$$

The sign $\sim$ means "arises proportionally from" or "is induced by" after projection and normalization.

If the coefficient tensor factorizes as

$$K_{i\mu,j\nu} = a_{ij} b_{\mu\nu}, \tag{54}$$

then the operator has product structure and is comparatively trivial. But if this factorization fails, then H_{ent} is a genuinely bipartite coupling operator capable of creating entanglement.

Thus the mathematical criterion is:

- factorizable coefficients $\Rightarrow$ no genuine entangling structure,
- non-factorizable coefficients $\Rightarrow$ genuine entangling interaction.

9 Dynamical generation of entanglement

We now show how an initial product state evolves into an entangled state.

9.1 Initial product state

Assume the initial state is unentangled:

$$|\psi(0)\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B \,. \tag{55}$$

This means the state is exactly separable at time $t = 0$.

9.2 Time evolution

The Schrödinger evolution generated by the effective Hamiltonian is

$$|\psi(t)\rangle = e^{-itH_{\text{eff}}} |\psi(0)\rangle \,. \tag{56}$$

Here:

- t denotes time,
- $e^{-itH_{\text{eff}}}$ is the unitary time-evolution operator,
- we use units in which $\hbar = 1$.

For small t, expand the exponential:

$$e^{-itH_{\text{eff}}} = I - itH_{\text{eff}} + \mathcal{O}(t^2), \tag{57}$$

where:

- I is the identity operator,
- $\mathcal{O}(t^2)$ denotes terms of order t^2 and higher.

Hence

$$|\psi(t)\rangle = \Big(I - itH_{\text{eff}} + \mathcal{O}(t^2)\Big) |\alpha\rangle |\beta\rangle \,. \tag{58}$$

Insert

$$H_{\text{eff}} = H_{\text{loc}} + H_{\text{ent}}, \tag{59}$$

with

$$H_{\text{loc}} := H_A \otimes I_B + I_A \otimes H_B. \tag{60}$$

Then

$$|\psi(t)\rangle = |\alpha\rangle |\beta\rangle - itH_{\text{loc}} |\alpha\rangle |\beta\rangle - itH_{\text{ent}} |\alpha\rangle |\beta\rangle + \mathcal{O}(t^2). \tag{61}$$

9.3 Why the local part does not entangle

The operator H_{loc} acts separately on the two subsystems:

$$(H_A \otimes I_B) |\alpha\rangle |\beta\rangle = (H_A |\alpha\rangle) \otimes |\beta\rangle \,, \tag{62}$$

$$(I_A \otimes H_B) |\alpha\rangle |\beta\rangle = |\alpha\rangle \otimes (H_B |\beta\rangle). \tag{63}$$

So it only generates local changes of the two factors. It does not introduce irreducible non-factorizable correlations.

9.4 Why the interaction part entangles

The term

$$H_{\text{ent}} |\alpha\rangle |\beta\rangle \tag{64}$$

is the decisive one.

If this vector is not proportional to a product vector, then already at arbitrarily small nonzero time the state cannot remain separable. Therefore entanglement is generated immediately.

So the criterion is:

$$H_{\text{ent}} |\alpha\rangle |\beta\rangle \text{ non-factorizable} \quad \Longrightarrow \quad |\psi(t)\rangle \text{ entangled for generic } t \neq 0. \tag{65}$$

10 Explicit two-qubit example

We now work out the simplest concrete example.

10.1 Effective Hilbert spaces

Take

$$\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2. \tag{66}$$

Each subsystem is therefore a qubit.

Let $|0\rangle, |1\rangle$ be the standard computational basis.

10.2 Effective entangling Hamiltonian

Assume the projected associator sector induces

$$H_{\text{ent}} = g\, \sigma_x \otimes \sigma_x, \tag{67}$$

where:

- g is a real effective coupling constant,
- σ_x is the Pauli x-matrix,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{68}$$

The relation to the octonionic theory is that g arises effectively from the fundamental coupling strength and the mediator amplitude, schematically

$$g \propto \lambda \|\Xi\|. \tag{69}$$

10.3 Initial state

Take the initial product state

$$|\psi(0)\rangle = |0\rangle_A \otimes |0\rangle_B = |00\rangle . \tag{70}$$

10.4 Action of the Hamiltonian

Since

$$\sigma_x |0\rangle = |1\rangle , \tag{71}$$

it follows that

$$(\sigma_x \otimes \sigma_x) |00\rangle = |11\rangle . \tag{72}$$

Therefore, for short times,

$$|\psi(t)\rangle = |00\rangle - igt\,|11\rangle + \mathcal{O}(t^2). \tag{73}$$

This state is already non-factorizable for $gt \neq 0$, as we prove below.

10.5 Exact time evolution

Because

$$(\sigma_x \otimes \sigma_x)^2 = I, \tag{74}$$

the exponential can be evaluated exactly:

$$e^{-igt\,\sigma_x\otimes\sigma_x} = \cos(gt)\,I - i\sin(gt)\,\sigma_x \otimes \sigma_x. \tag{75}$$

Applying this to $|00\rangle$ gives

$$|\psi(t)\rangle = \cos(gt)\,|00\rangle - i\sin(gt)\,|11\rangle . \tag{76}$$

This is a Bell-type state.

For

$$gt = \frac{\pi}{4}, \tag{77}$$

we obtain

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle - i\,|11\rangle)\,, \tag{78}$$

which is maximally entangled.

11 Why the state is entangled

We now prove rigorously that a state of the form

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle\,, \qquad \alpha\beta \neq 0, \tag{79}$$

is not separable.

11.1 General product state

A general product state in $\mathbb{C}^2 \otimes \mathbb{C}^2$ has the form

$$(a\left|0\right\rangle + b\left|1\right\rangle) \otimes (c\left|0\right\rangle + d\left|1\right\rangle), \tag{80}$$

where $a, b, c, d \in \mathbb{C}$.

Expanding gives

$$ac\left|00\right\rangle + ad\left|01\right\rangle + bc\left|10\right\rangle + bd\left|11\right\rangle. \tag{81}$$

11.2 Matching to $\alpha\left|00\right\rangle + \beta\left|11\right\rangle$

To reproduce only the terms $\left|00\right\rangle$ and $\left|11\right\rangle$, one must require

$$ad = 0, \qquad bc = 0. \tag{82}$$

Now:

- from $ad = 0$, either $a = 0$ or $d = 0$,
- from $bc = 0$, either $b = 0$ or $c = 0$.

Check all possibilities:

- if $a = 0$, then $ac = 0$, so the $\left|00\right\rangle$-coefficient vanishes;
- if $d = 0$, then $bd = 0$, so the $\left|11\right\rangle$-coefficient vanishes;
- similarly, $b = 0$ kills $\left|11\right\rangle$, and $c = 0$ kills $\left|00\right\rangle$.

Therefore one cannot have both coefficients nonzero simultaneously.

Hence no product state can equal

$$\alpha\left|00\right\rangle + \beta\left|11\right\rangle \tag{83}$$

when $\alpha\beta \neq 0$.

So such a state is entangled.

12 Density matrix and reduced states

Entanglement is also visible in the reduced density matrix.

12.1 Full density matrix

For the pure state

$$\left|\psi(t)\right\rangle = \cos(gt)\left|00\right\rangle - i\sin(gt)\left|11\right\rangle, \tag{84}$$

the density matrix is

$$\rho = \left|\psi(t)\right\rangle\left\langle\psi(t)\right|. \tag{85}$$

12.2 Reduced density matrix of subsystem A

The reduced density matrix is obtained by tracing out subsystem B:

$$\rho_A = \mathrm{Tr}_B(\rho). \tag{86}$$

Using orthonormality of $|0\rangle$ and $|1\rangle$, one finds

$$\rho_A = \cos^2(gt)\,|0\rangle\,\langle 0| + \sin^2(gt)\,|1\rangle\,\langle 1|\,. \tag{87}$$

12.3 Why this signals entanglement

A density matrix is pure if and only if

$$\rho_A^2 = \rho_A. \tag{88}$$

Here,

$$\rho_A^2 = \cos^4(gt)\,|0\rangle\,\langle 0| + \sin^4(gt)\,|1\rangle\,\langle 1|\,, \tag{89}$$

which equals ρ_A only if one of the probabilities is 0 or 1, that is, only if

$$gt = 0,\ \frac{\pi}{2},\ \pi, \ldots \tag{90}$$

For generic t, the reduced density matrix is mixed.

But the total state ρ is pure. A pure bipartite state with mixed reduced subsystem states is exactly the hallmark of entanglement.

12.4 Entanglement entropy

The entanglement entropy of subsystem A is

$$S_A = -\,\mathrm{Tr}(\rho_A \ln \rho_A). \tag{91}$$

Since the values of ρ_A are

$$\lambda_1 = \cos^2(gt), \qquad \lambda_2 = \sin^2(gt), \tag{92}$$

we obtain

$$S_A = -\cos^2(gt)\ln\Big(\cos^2(gt)\Big) - \sin^2(gt)\ln\Big(\sin^2(gt)\Big). \tag{93}$$

For

$$gt = \frac{\pi}{4}, \tag{94}$$

both values are 1/2, so

$$\rho_A = \frac{1}{2}I, \qquad S_A = \ln 2. \tag{95}$$

This is the maximal entanglement entropy for a two-qubit pure state.

13 Why the octonions are indispensable

At this point one might object that the effective Hamiltonian

$$H_{\text{ent}} = g\,\sigma_x \otimes \sigma_x \tag{96}$$

looks like an ordinary entangling interaction from standard quantum mechanics.

That is true at the effective level. But the novelty lies deeper: in the octonionic framework this term is not simply postulated. It is derived from the fundamental non-associative structure.

The chain is:

1. Octonions are non-associative.
2. Therefore the associator
$$A(\Psi_A, \Xi, \Psi_B) \tag{97}$$
can be nonzero.
3. Hence the action admits the fundamental term
$$\lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2. \tag{98}$$
4. This term is non-separable.
5. After projection it becomes an entangling effective Hamiltonian.

If the algebra were associative, then

$$A(\Psi_A, \Xi, \Psi_B) = 0 \tag{99}$$

identically. In that case the entire non-associative interaction sector would disappear. Thus, in this model, the origin of entanglement is directly tied to non-associativity.

In compact form:

$$\text{non-associativity} \Longrightarrow \text{associator term} \Longrightarrow \text{non-separable dynamics} \Longrightarrow \text{entanglement.} \tag{100}$$

14 Formal theorem

Theorem 14.1. *Let Ψ_A, Ψ_B, Ξ be octonionic fields with action*

$$S = \int d^4x\,\sqrt{-g}\,\Big(\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_\Xi - \lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2\Big). \tag{101}$$

Assume that projection onto the effective associative sector yields a Hamiltonian

$$H_{\text{eff}} = H_A \otimes I + I \otimes H_B + H_{\text{ent}}, \tag{102}$$

where H_{ent} is induced by the associator sector and is not reducible to a sum of local operators. Then, for generic product initial states

$$|\alpha\rangle \otimes |\beta\rangle \,, \tag{103}$$

the time evolution generated by H_{eff} produces entangled states for generic $t \neq 0$.

Proof. Start from the initial product state

$$|\psi(0)\rangle = |\alpha\rangle \otimes |\beta\rangle \,. \tag{104}$$

For small time,

$$|\psi(t)\rangle = \Big(I - itH_{\text{eff}} + \mathcal{O}(t^2)\Big) |\psi(0)\rangle \,. \tag{105}$$

Split the Hamiltonian as

$$H_{\text{eff}} = H_{\text{loc}} + H_{\text{ent}}, \qquad H_{\text{loc}} := H_A \otimes I + I \otimes H_B. \tag{106}$$

Then

$$|\psi(t)\rangle = |\alpha\rangle |\beta\rangle - itH_{\text{loc}} |\alpha\rangle |\beta\rangle - itH_{\text{ent}} |\alpha\rangle |\beta\rangle + \mathcal{O}(t^2). \tag{107}$$

The local part H_{loc} preserves product structure up to local rotations, because it acts independently on the two tensor factors. By assumption, however, H_{ent} is not a sum of local operators. Therefore, for generic product vectors $|\alpha\rangle |\beta\rangle$, the vector

$$H_{\text{ent}} |\alpha\rangle |\beta\rangle \tag{108}$$

is not proportional to a product vector.

Hence the first-order correction already leaves the separable manifold. Therefore $|\psi(t)\rangle$ is non-separable for generic $t \neq 0$. A non-separable pure state is entangled. This proves the claim. □

15 Physical interpretation

The interpretation of the result is the following.

At the observable level, subsystems A and B appear as ordinary quantum subsystems living in a tensor-product Hilbert space. Their entanglement is then visible as usual through non-factorizable states, mixed reduced density matrices, and nonzero entanglement entropy.

But at the deeper level, this entanglement is not fundamental in the usual Hilbert-space sense. It is the projected trace of a more primitive octonionic structure.

More precisely:

- the fields Ψ_A and Ψ_B do not interact merely pairwise;
- instead, both are tied to a deeper mediating non-associative degree of freedom Ξ;
- the irreducible triplet structure

$$A(\Psi_A, \Xi, \Psi_B) \tag{109}$$

generates a common interaction energy;

- after projection onto the effective associative sector, this becomes an entangling Hamiltonian.

Thus entanglement is the effective associative shadow of a deeper non-associative correlation.

16 What has been shown rigorously and what remains an assumption

It is important to separate strict mathematical results from additional physical assumptions.

16.1 Rigorous within the model structure

The following statements are mathematically established within the framework:

- The octonions possess a nontrivial associator.
- A term
$$\|A(\Psi_A, \Xi, \Psi_B)\|^2 \tag{110}$$
is a well-defined fundamental interaction.
- This term is structurally non-separable.
- It produces coupled Euler–Lagrange equations.
- After projection, it yields an effective nonlocal coupling operator.
- Such an operator generically entangles product states.

16.2 Additional physical assumption

The main extra assumption is the existence of a precise map from the fundamental octonionic field theory to the effective Hilbert-space description. This is the same conceptual step at which, in the broader program, one must also derive:

- the Schrödinger equation,
- the complex Hilbert-space structure,
- the Born rule,
- and the measurement projection structure.

So the present report establishes the entanglement mechanism conditionally on that projection step.

17 One-line logical chain

The entire derivation can be summarized in the following single chain:

$$\mathbb{O} \text{ non-associative} \Longrightarrow A(\Psi_A, \Xi, \Psi_B) \neq 0 \Longrightarrow V_{\text{assoc}} = \lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2 \Longrightarrow \text{non-separable field equations} \Longrightarrow \tag{111}$$

18 CHSH inequality violation from the octonionic entanglement sector

A central question is whether the entanglement generated by the octonionic associator sector is strong enough to violate a Bell inequality. In the effective two-qubit reduction discussed above, the answer is yes. We now derive this explicitly.

18.1 CHSH setup

Consider two spatially separated effective subsystems A and B, each described by a two-dimensional Hilbert space

$$\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2. \tag{112}$$

Each side admits two possible dichotomic observables with outcomes ± 1.

We denote the two observables on subsystem A by

$$\hat{A}, \qquad \hat{A}', \tag{113}$$

and the two observables on subsystem B by

$$\hat{B}, \qquad \hat{B}'. \tag{114}$$

Each of these operators is Hermitian and has values ± 1. In the qubit case, such observables can be written as spin projections along unit vectors in $\mathbb{R}^3$:

$$\hat{A} = \boldsymbol{a} \cdot \boldsymbol{\sigma}, \qquad \hat{A}' = \boldsymbol{a}' \cdot \boldsymbol{\sigma}, \qquad \hat{B} = \boldsymbol{b} \cdot \boldsymbol{\sigma}, \qquad \hat{B}' = \boldsymbol{b}' \cdot \boldsymbol{\sigma}, \tag{115}$$

where

- $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^3$ are unit vectors,
- $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector.

The corresponding correlation function in a bipartite pure state $|\psi\rangle$ is

$$E(\boldsymbol{a}, \boldsymbol{b}) = \langle \psi | (\boldsymbol{a} \cdot \boldsymbol{\sigma}) \otimes (\boldsymbol{b} \cdot \boldsymbol{\sigma}) | \psi \rangle \,. \tag{116}$$

The CHSH combination is

$$S_{\text{CHSH}} = E(\boldsymbol{a}, \boldsymbol{b}) + E(\boldsymbol{a}, \boldsymbol{b}') + E(\boldsymbol{a}', \boldsymbol{b}) - E(\boldsymbol{a}', \boldsymbol{b}'). \tag{117}$$

For every local hidden-variable theory one has the Bell–CHSH bound

$$|S_{\mathrm{CHSH}}| \leq 2. \tag{118}$$

Quantum mechanics allows stronger correlations, up to the Tsirelson bound

$$|S_{\mathrm{CHSH}}| \leq 2\sqrt{2}. \tag{119}$$

Our task is to show that the octonionically induced effective state indeed reaches values larger than 2.

18.2 Effective Bell-type state generated by the octonionic sector

From the previous section, the octonionic associator term induces in the two-qubit sector an effective entangling Hamiltonian of the form

$$H_{\mathrm{ent}} = g\,\sigma_x \otimes \sigma_x. \tag{120}$$

Starting from the product state $|00\rangle$, the exact evolution gives

$$|\psi(t)\rangle = \cos(gt)\,|00\rangle - i\sin(gt)\,|11\rangle\,. \tag{121}$$

At the special interaction time

$$gt = \frac{\pi}{4}, \tag{122}$$

this becomes

$$|\psi_{\mathrm{Bell}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - i\,|11\rangle). \tag{123}$$

This state differs from the standard Bell state only by a relative phase. Since local unitary transformations do not change entanglement or CHSH-violation strength, this phase does not obstruct Bell inequality violation. Still, it is useful to show this explicitly.

Define a local phase rotation on subsystem B,

$$U_B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{124}$$

Then

$$(I \otimes U_B)\frac{1}{\sqrt{2}}(|00\rangle - i\,|11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{125}$$

Thus the octonionically generated maximally entangled state is locally equivalent to the standard Bell state

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{126}$$

Therefore it suffices to evaluate CHSH for $\left|\Phi^+\right\rangle$, because the same maximal violation is achievable for the original state after a corresponding redefinition of local measurement axes.

18.3 Correlation tensor for the Bell state

We now compute the correlation functions for

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{127}$$

For a general two-qubit state, define the correlation tensor T by

$$T_{ij} = \left\langle \Phi^+ \middle| \sigma_i \otimes \sigma_j \middle| \Phi^+ \right\rangle, \qquad i, j \in \{x, y, z\}. \tag{128}$$

We compute the diagonal entries explicitly.

xx-correlator. Using

$$\sigma_x |0\rangle = |1\rangle, \qquad \sigma_x |1\rangle = |0\rangle, \tag{129}$$

we obtain

$$(\sigma_x \otimes \sigma_x) |00\rangle = |11\rangle, \qquad (\sigma_x \otimes \sigma_x) |11\rangle = |00\rangle. \tag{130}$$

Therefore

$$(\sigma_x \otimes \sigma_x) \left|\Phi^+\right\rangle = \left|\Phi^+\right\rangle, \tag{131}$$

so

$$\left\langle \Phi^+ \middle| \sigma_x \otimes \sigma_x \middle| \Phi^+ \right\rangle = 1. \tag{132}$$

yy-correlator. Using

$$\sigma_y |0\rangle = i |1\rangle, \qquad \sigma_y |1\rangle = -i |0\rangle, \tag{133}$$

we find

$$(\sigma_y \otimes \sigma_y) |00\rangle = - |11\rangle, \qquad (\sigma_y \otimes \sigma_y) |11\rangle = - |00\rangle. \tag{134}$$

Hence

$$(\sigma_y \otimes \sigma_y) \left|\Phi^+\right\rangle = - \left|\Phi^+\right\rangle, \tag{135}$$

and thus

$$\left\langle \Phi^+ \middle| \sigma_y \otimes \sigma_y \middle| \Phi^+ \right\rangle = -1. \tag{136}$$

zz-correlator. Since

$$\sigma_z |0\rangle = |0\rangle, \qquad \sigma_z |1\rangle = - |1\rangle, \tag{137}$$

we get

$$(\sigma_z \otimes \sigma_z) |00\rangle = |00\rangle, \qquad (\sigma_z \otimes \sigma_z) |11\rangle = |11\rangle. \tag{138}$$

So

$$(\sigma_z \otimes \sigma_z) \left|\Phi^+\right\rangle = \left|\Phi^+\right\rangle, \tag{139}$$

hence

$$\left\langle \Phi^+ \middle| \sigma_z \otimes \sigma_z \middle| \Phi^+ \right\rangle = 1. \tag{140}$$

All mixed correlators such as

$$\left\langle \Phi^+ \middle| \sigma_x \otimes \sigma_z \middle| \Phi^+ \right\rangle \tag{141}$$

vanish. Therefore the correlation tensor is

$$T = \mathrm{diag}(1, -1, 1). \tag{142}$$

This means that for arbitrary measurement directions $\boldsymbol{a}, \boldsymbol{b}$,

$$E(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a}^T T \boldsymbol{b}. \tag{143}$$

18.4 Choice of measurement directions

To obtain a CHSH violation, choose all measurement directions in the x-z plane. Then the y-component plays no role, and because T acts as $+1$ on both the x- and z-directions, the correlation function reduces to the Euclidean scalar product in that plane.

Choose

$$\boldsymbol{a} = (0, 0, 1), \qquad \boldsymbol{a}' = (1, 0, 0), \tag{144}$$

that is,

$$\hat{A} = \sigma_z, \qquad \hat{A}' = \sigma_x. \tag{145}$$

For subsystem B, choose

$$\boldsymbol{b} = \frac{1}{\sqrt{2}}(1, 0, 1), \qquad \boldsymbol{b}' = \frac{1}{\sqrt{2}}(-1, 0, 1). \tag{146}$$

Thus

$$\hat{B} = \frac{\sigma_x + \sigma_z}{\sqrt{2}}, \qquad \hat{B}' = \frac{-\sigma_x + \sigma_z}{\sqrt{2}}. \tag{147}$$

Now compute the four correlations.

First,

$$E(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a}^T T \boldsymbol{b} = (0, 0, 1) \cdot \frac{1}{\sqrt{2}}(1, 0, 1) = \frac{1}{\sqrt{2}}. \tag{148}$$

Second,

$$E(\boldsymbol{a}, \boldsymbol{b}') = (0, 0, 1) \cdot \frac{1}{\sqrt{2}}(-1, 0, 1) = \frac{1}{\sqrt{2}}. \tag{149}$$

Third,

$$E(\boldsymbol{a}', \boldsymbol{b}) = (1, 0, 0) \cdot \frac{1}{\sqrt{2}}(1, 0, 1) = \frac{1}{\sqrt{2}}. \tag{150}$$

Fourth,

$$E(\boldsymbol{a}', \boldsymbol{b}') = (1, 0, 0) \cdot \frac{1}{\sqrt{2}}(-1, 0, 1) = -\frac{1}{\sqrt{2}}. \tag{151}$$

Insert these into the CHSH expression:

$$S_{\mathrm{CHSH}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right). \tag{152}$$

Therefore

$$S_{\text{CHSH}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}. \tag{153}$$

This exceeds the classical Bell bound 2. Hence the effective state generated by the octonionic entangling sector violates the CHSH inequality maximally.

18.5 Direct evaluation for the time-dependent state

We can also keep the full time dependence of

$$|\psi(t)\rangle = \cos(gt)\,|00\rangle - i\sin(gt)\,|11\rangle \tag{154}$$

and ask when CHSH violation begins.

Since this state is locally equivalent to the Schmidt-form state

$$|\psi_\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle\,, \qquad \theta = gt, \tag{155}$$

its maximal CHSH value is the standard two-qubit result

$$S_{\max}(t) = 2\sqrt{1+\sin^2(2gt)}. \tag{156}$$

Let us explain this formula.

For a pure two-qubit state in Schmidt form

$$|\psi_\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle\,, \tag{157}$$

the concurrence is

$$C = |\sin(2\theta)|. \tag{158}$$

The Horodecki criterion gives the maximal CHSH value as

$$S_{\max} = 2\sqrt{1+C^2}. \tag{159}$$

Substituting $C = |\sin(2gt)|$ yields

$$S_{\max}(t) = 2\sqrt{1+\sin^2(2gt)}. \tag{160}$$

Now:

- at $t = 0$, one has $\sin(2gt) = 0$, so

  $$S_{\max}(0) = 2, \tag{161}$$

  which is exactly the classical bound;

- for any generic time with

  $$\sin(2gt) \neq 0, \tag{162}$$

  one gets

  $$S_{\max}(t) > 2, \tag{163}$$

so the CHSH inequality is violated;

- at the maximally entangled time

$$gt = \frac{\pi}{4}, \tag{164}$$

one has $\sin(2gt) = 1$, hence

$$S_{\max} = 2\sqrt{2}. \tag{165}$$

So CHSH violation appears dynamically as soon as the associator-induced interaction has generated a nonzero entanglement amplitude.

18.6 Interpretation in the octonionic model

The effective CHSH violation is not inserted by hand. It arises because the octonionic non-associative term generates an effective interaction Hamiltonian that produces Bell-type states.

The logical chain is therefore now strengthened to

$$\mathbb{O}\ \text{non-associative} \Longrightarrow A(\Psi_A, \Xi, \Psi_B) \neq 0 \Longrightarrow \lambda \|A(\Psi_A, \Xi, \Psi_B)\|^2 \Longrightarrow H_{\text{ent}} \neq H'_A \otimes I + I \otimes H'_B \Longrightarrow |\psi(t)\rangle\ \text{entangle} \tag{166}$$

for generic nontrivial interaction times.

This is important conceptually. In the standard formalism, Bell-violating states are usually introduced as abstract entangled states in Hilbert space. Here they are traced back to a specific underlying algebraic source: the non-associativity of the octonions and the resulting trilinear associator coupling.

19 Formal proposition on CHSH violation

Proposition 19.1. *Assume that the projection of the octonionic associator sector onto the effective two-qubit subspace yields an entangling Hamiltonian*

$$H_{\text{ent}} = g\, \sigma_x \otimes \sigma_x \tag{167}$$

with $g \neq 0$*, and let the initial state be*

$$|\psi(0)\rangle = |00\rangle\,. \tag{168}$$

Then the evolved state

$$|\psi(t)\rangle = e^{-igt\, \sigma_x \otimes \sigma_x}\, |00\rangle = \cos(gt)\, |00\rangle - i \sin(gt)\, |11\rangle \tag{169}$$

violates the CHSH inequality for every t *such that* $\sin(2gt) \neq 0$*. Its maximal CHSH value is*

$$S_{\max}(t) = 2\sqrt{1 + \sin^2(2gt)}, \tag{170}$$

and reaches the Tsirelson bound $2\sqrt{2}$ at

$$gt = \frac{\pi}{4} \mod \frac{\pi}{2}. \tag{171}$$

Proof. From the exact time evolution,

$$|\psi(t)\rangle = \cos(gt)\,|00\rangle - i\sin(gt)\,|11\rangle\,, \tag{172}$$

the state is locally unitary equivalent to the Schmidt-form state

$$|\psi_\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle \tag{173}$$

with $\theta = gt$. For such a pure two-qubit state the concurrence is

$$C = |\sin(2\theta)| = |\sin(2gt)|. \tag{174}$$

By the Horodecki formula, the maximal CHSH value is

$$S_{\max} = 2\sqrt{1 + C^2}. \tag{175}$$

Therefore

$$S_{\max}(t) = 2\sqrt{1 + \sin^2(2gt)}. \tag{176}$$

If $\sin(2gt) \neq 0$, then $S_{\max}(t) > 2$, so the CHSH inequality is violated. If $\sin(2gt) = \pm 1$, then

$$S_{\max} = 2\sqrt{2}, \tag{177}$$

which is the maximal quantum value. □

20 Suggested concluding physical statement

A direct consequence of the octonionically induced entangling sector is Bell nonlocality at the effective level. In the two-qubit reduction, the associator-generated interaction produces states of the form

$$|\psi(t)\rangle = \cos(gt)\,|00\rangle - i\sin(gt)\,|11\rangle\,, \tag{178}$$

whose maximal CHSH value is

$$S_{\max}(t) = 2\sqrt{1 + \sin^2(2gt)}. \tag{179}$$

Hence, for every generic nontrivial interaction time, one has

$$S_{\max}(t) > 2, \tag{180}$$

and at maximal entanglement the Tsirelson bound $2\sqrt{2}$ is reached. Thus the violation of the CHSH inequality is not an additional postulate of the effective quantum description, but follows from the same non-associative octonionic mechanism that generates entanglement in the first place.

21 Conclusion

We have derived, in mathematical steps, how entanglement can arise in an octonionic framework. The essential point is that octonionic non-associativity introduces a trilinear associator, and this makes the minimal fundamental coupling intrinsically three-legged rather than bipartite. The corresponding interaction term

$$\lambda \| A(\Psi_A, \Xi, \Psi_B) \|^2 \tag{181}$$

is non-separable and leads to coupled equations of motion already at the field-theoretic level.

After projection onto an effective associative sector, this structure appears as an entangling Hamiltonian on $\mathcal{H}_A \otimes \mathcal{H}_B$. We showed explicitly that such a Hamiltonian drives product states into entangled states, and we demonstrated this in detail for a two-qubit example. The reduced density matrix, the entanglement entropy, and the CHSH analysis confirm the result.

The conceptual payoff is sharp: in this framework, entanglement is not merely postulated as a kinematical possibility of tensor-product Hilbert spaces. It is explained as the effective associative manifestation of a deeper octonionic non-associative interaction structure.

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