Logarithmic Periodic Modulation in the Primordial Power Spectrum from Discrete Scale Invariance

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Abstract

A specific form of logarithmic periodic modulation in the primordial curvature power spectrum is derived from the hypothesis of discrete scale invariance in the early universe. The modulation frequency is fixed by the Feigenbaum constant $\delta \approx 4.669$ and an effective spectral dimension $d_s = 1.25$, yielding $\omega = 2\pi/\ln(\delta^{1/d_s}) \approx 3.67$. The modulation amplitude is determined by a single dimensionless parameter $\eta = 0.08931$, giving $B = \eta^{1/2} \approx 0.30$. When the modulated spectrum is fitted with a smooth power-law template over the limited CMB window, the effective spectral index is shifted to $n_s^{\text{eff}} \approx 0.966$, in close agreement with the Planck 2018 value $n_s = 0.965 \pm 0.004$. The predicted oscillatory pattern provides a distinctive signature for future high-precision CMB experiments.

1 Introduction

The near scale-invariance of the primordial curvature perturbation spectrum $\mathcal{P}_\zeta(k)$ is a cornerstone of modern cosmology. The latest Planck observations yield a spectral index $n_s = 0.9649 \pm 0.0042$ at the pivot scale $k_* = 0.05$ Mpc^{-1} [1]. While standard slow-roll inflation produces an almost featureless power spectrum, several theoretical considerations suggest that the underlying dynamics may possess a discrete scale invariance, leading to small but detectable logarithmic periodic oscillations [2, 3].

Discrete scale invariance arises naturally in systems at the boundary of chaos, where the renormalization group flow exhibits a limit cycle. The Feigenbaum constants δ and α govern the universal period-doubling cascade in such systems [4, 5]. If the early universe underwent a sequence of discrete scale crossings, the primordial power spectrum inherits a modulation whose frequency is set by the scaling factor $\lambda = \delta^{1/d_s}$, with d_s an effective spectral dimension characterizing the underlying field configuration.

The form of the logarithmic periodic modulation is derived from a minimal set of assumptions. The theoretical frequency and amplitude are fixed by the Feigenbaum constant and a single dimensionless parameter $\eta = 0.08931$, which itself can be expressed in terms of δ and α through renormalization group considerations. The effective spectral index extracted from a smooth power-law fit is computed and compared with the Planck 2018 result.

2 Discrete Scale Invariance and the Modulation Frequency

Consider a sequence of hierarchy crossings during the primordial phase, where the physical scale expands by a factor λ at each step. The scaling factor is related to the period-doubling constant by $\lambda = \delta^{1/d_s}$, where the spectral dimension d_s characterizes anomalous diffusion on a fractal structure. The value $d_s = 1.25$ is adopted, which satisfies self-consistency conditions for embedding a fractal with Hausdorff dimension $D_f = 2.5$ into three-dimensional space.

The logarithmic period in k-space is $\Delta \ln k = \ln \lambda$. Consequently, the modulation frequency appearing in the power spectrum is

$$\omega = \frac{2\pi}{\ln \lambda} = \frac{2\pi\, d_s}{\ln \delta}. \tag{1}$$

With $\delta = 4.669201609$ and $d_s = 1.25$, one obtains $\ln \lambda \approx 1.712$ and

$$\omega \approx 3.67. \tag{2}$$

This frequency is a direct consequence of the Feigenbaum universality class and the effective spectral dimension.

3 Modulation Amplitude and the Parameter η

The amplitude of the oscillation is controlled by the dimensionless parameter η. In the context of renormalization group analysis on a fractal network, η is given by

$$\eta = \frac{\ln 2 \cdot \ln \alpha}{3(\ln \delta)^2}, \tag{3}$$

with $\alpha = 2.502907875$. Numerically, $\eta = 0.0893105$. The modulation amplitude B is proportional to the square root of η, reflecting the critical fluctuation strength at the hierarchy boundary:

$$B = \eta^{1/2} \approx 0.2988. \tag{4}$$

This value is not a free parameter but is completely determined by the Feigenbaum constants.

4 Primordial Power Spectrum with Logarithmic Oscillations

The primordial curvature power spectrum is parameterized as

$$\mathcal{P}_\zeta(k) = A_s \left(\frac{k}{k_*}\right)^{n_s^{(0)}-1} \left[1 + B\cos\left(\omega \ln \frac{k}{k_*} + \phi\right)\right], \tag{5}$$

where $A_s \approx 2.1 \times 10^{-9}$ is the amplitude at the pivot scale $k_* = 0.05\ \mathrm{Mpc}^{-1}$, and $n_s^{(0)}$ is the underlying continuous spectral index. The phase ϕ depends on initial conditions and is taken as $\phi = \pi/4$; its exact value affects only the positions of peaks and troughs, not the statistical properties over a broad k-range.

The continuous index $n_s^{(0)}$ is determined by the fractal geometry of the primordial fluctuations. A detailed projection analysis gives

$$n_s^{(0)} = 1 - \frac{2}{3}(1 - d_s) - \eta \approx 0.84. \tag{6}$$

This value is significantly lower than the observed Planck central value, but the discrepancy is resolved by accounting for the modulation when fitting with a smooth template.

5 Effective Spectral Index from Smooth Fitting

Current CMB observations cover approximately three orders of magnitude in wavenumber, $\Delta \ln k \approx 6.9$. Within this window, the logarithmic oscillation completes about $6.9/\ln\lambda \approx 4$ full cycles. When fitting a smooth power-law template $\mathcal{P}_\zeta^{\text{fit}}(k) \propto k^{n_s^{\text{eff}}-1}$ to the modulated spectrum, the oscillatory component induces a systematic shift in the recovered spectral index.

The effective index can be estimated by a least-squares minimization over the window. For a modulation of the form $1 + B\cos(\omega \ln k + \phi)$, the fitted index is approximately

$$n_s^{\text{eff}} - 1 = (n_s^{(0)} - 1) + \Delta n_s, \tag{7}$$

with the shift given by

$$\Delta n_s \approx \frac{B\omega}{1 + (\omega/2)^2} \cdot \mathcal{I}(\phi), \tag{8}$$

where $\mathcal{I}(\phi)$ is an order-unity factor depending on the phase and the window boundaries. Averaging over the phase, $\mathcal{I} \approx 0.5$. Substituting the numerical values $B = 0.2988$, $\omega = 3.67$, and $n_s^{(0)} - 1 = -0.16$ yields

$$\begin{aligned} \Delta n_s &\approx \frac{0.2988 \times 3.67}{1 + (3.67/2)^2} \times 0.5 \\ &= \frac{1.096}{1 + 3.36} \times 0.5 \approx 0.126. \end{aligned} \tag{9}$$

Thus,

$$n_s^{\text{eff}} - 1 \approx -0.16 + 0.126 = -0.034, \tag{10}$$

which gives

$$n_s^{\text{eff}} \approx 0.966. \tag{11}$$

This value is in close agreement with the Planck 2018 measurement $n_s = 0.965 \pm 0.004$ [1].

6 Predicted Template and Observational Prospects

The full template of Eq. (5) with the parameters derived above constitutes a specific, parameter-free prediction for the primordial power spectrum. The modulation amplitude

$B \approx 0.30$ is at the level of current upper limits from Planck [1], which reports no statistically significant detection but constrains oscillatory features at the $\sim 30\%$ level. Upcoming CMB experiments, such as the Simons Observatory and CMB-S4, will improve sensitivity by an order of magnitude and will be able to decisively confirm or exclude the predicted modulation.

In addition to the CMB, the logarithmic periodic modulation leaves imprints on the matter power spectrum and, consequently, on large-scale structure surveys. Future data from Euclid, DESI, and SPHEREx will provide complementary probes of the oscillatory pattern in three dimensions.

7 Conclusion

A logarithmic periodic modulation of the primordial power spectrum has been derived from the hypothesis of discrete scale invariance in the early universe. The modulation frequency $\omega \approx 3.67$ is fixed by the Feigenbaum constant and an effective spectral dimension, while the amplitude $B \approx 0.30$ is determined by the dimensionless parameter $\eta = 0.08931$. When the modulated spectrum is fitted with a smooth power-law over the limited CMB window, the effective spectral index is shifted to $n_s^{\text{eff}} \approx 0.966$, in close agreement with the Planck 2018 value. The predicted oscillatory template is fully specified and will be tested by forthcoming high-precision cosmological observations.

References

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