A SMOOTHED FUNCTIONAL DEFINED BY THE ZEROS OF THE RIEMANN ZETA FUNCTION

YUFEI LIU

Abstract. We define a function $L(x) = \sum_\rho (1 - e^{-x/\rho})$ for $x > 0$, where the sum runs over all non-trivial zeros ρ of the Riemann zeta function $\zeta(s)$, taken in the symmetric pairing ρ and $1-\rho$ to ensure absolute convergence. We prove that $L(x)$ converges absolutely for every $x > 0$, that it is differentiable, and that its derivative is given by $L'(x) = \sum_\rho \frac{1}{\rho} e^{-x/\rho}$ (with the same pairing). We also show that $L(x)$ is real-valued. This functional serves as a continuous analogue of the Li coefficients. We state the conjecture that the Riemann Hypothesis is equivalent to the strict positivity $L'(x) > 0$ for all $x > 0$. All results are unconditional and rely only on standard zero-density estimates.

Notation

Let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. By the functional equation, if ρ is a zero then so are $1-\rho$, $\bar{\rho}$, and $1-\bar{\rho}$. The sum $\sum_\rho$ is always understood in the symmetric pairing ρ and $1-\rho$:

$$\sum_\rho f(\rho) \;:=\; \lim_{T\to\infty} \sum_{0<\gamma\le T} \big(f(\rho) + f(1-\rho)\big).$$

This pairing guarantees absolute convergence of the series we consider, as shown below.

1. Introduction

The Riemann zeta function $\zeta(s)$ has non-trivial zeros ρ in the critical strip $0 < \Re(\rho) < 1$. The Li coefficients

$$\lambda_n = \sum_\rho \big(1 - (1 - 1/\rho)^n\big)$$

(summation in the symmetric pairing) satisfy $\lambda_n \ge 0$ for all n if and only if the Riemann Hypothesis (RH) holds [1]. In this paper we introduce a continuous analogue

$$L(x) = \sum_\rho \big(1 - e^{-x/\rho}\big), \qquad x > 0,$$

where the sum is taken in the symmetric pairing. We prove its basic analytic properties: absolute convergence, differentiability, and reality. We then state a conjecture that connects the positivity of $L'(x)$ to RH.

2. Preliminaries

The Riemann–von Mangoldt formula gives

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + O(\log T),$$

where $N(T)$ counts zeros with $0 < \gamma \le T$. Consequently $N(T) = O(T \log T)$ and the number of zeros with $\gamma \in [k, k+1]$ is $O(\log k)$. These estimates are unconditional.

Date: April 14, 2026.

3. Absolute convergence

Definition 3.1. *For $x > 0$, define*

$$L(x) = \sum_{\rho} \big(1 - e^{-x/\rho}\big),$$

using the symmetric pairing.

Theorem 3.1. *For every $x > 0$, the series defining $L(x)$ converges absolutely.*

Proof. For a zero ρ with $\Im(\rho) > 0$, consider the paired term

$$P(\rho) := \big(1 - e^{-x/\rho}\big) + \big(1 - e^{-x/(1-\rho)}\big).$$

We estimate $|P(\rho)|$. For any $z \in \mathbb{C}$ with $|z| \le 1/2$, the inequality $|1 - e^{-z} - z| \le e|z|^2$ holds (by Taylor's theorem with remainder). Take $z = x/\rho$ and $z = x/(1-\rho)$. For all sufficiently large $|\rho|$ we have $|\rho| \ge 2x$ and $|1-\rho| \ge 2x$, hence $|x/\rho| \le 1/2$ and $|x/(1-\rho)| \le 1/2$. Then

$$1 - e^{-x/\rho} = \frac{x}{\rho} + R_1(\rho), \qquad |R_1(\rho)| \le e\frac{x^2}{|\rho|^2},$$

and similarly

$$1 - e^{-x/(1-\rho)} = \frac{x}{1-\rho} + R_2(\rho), \qquad |R_2(\rho)| \le e\frac{x^2}{|1-\rho|^2}.$$

Thus

$$P(\rho) = x\Big(\frac{1}{\rho} + \frac{1}{1-\rho}\Big) + R(\rho), \qquad |R(\rho)| \le ex^2\Big(\frac{1}{|\rho|^2} + \frac{1}{|1-\rho|^2}\Big).$$

Now $\frac{1}{\rho} + \frac{1}{1-\rho} = \frac{1}{\rho(1-\rho)}$. Since $|\rho(1-\rho)| \ge \gamma^2$ (because $|\rho| \ge \gamma$ and $|1-\rho| \ge \gamma$), we have

$$\Big|\frac{1}{\rho} + \frac{1}{1-\rho}\Big| \le \frac{1}{\gamma^2}.$$

Also $1/|\rho|^2 \le 1/\gamma^2$ and $1/|1-\rho|^2 \le 1/\gamma^2$. Therefore

$$|P(\rho)| \le \frac{x}{\gamma^2} + 2ex^2\frac{1}{\gamma^2} \le \frac{K}{\gamma^2},$$

where $K = x + 2ex^2$ is a constant depending only on x.

The finitely many zeros with $\gamma \le 2x$ contribute a finite sum. For the remaining zeros, the series $\sum_{\gamma > 2x} |P(\rho)|$ is bounded by $K\sum_{\gamma>2x} 1/\gamma^2$. Since the number of zeros with $\gamma \in [k, k+1]$ is $O(\log k)$,

$$\sum_{\gamma>2x} \frac{1}{\gamma^2} \ll \sum_{k=1}^{\infty} \frac{\log k}{k^2} < \infty.$$

Thus the series converges absolutely. □

4. Differentiability

Theorem 4.1. *$L(x)$ is differentiable on $(0,\infty)$, and for each $x > 0$,*

$$L'(x) = \sum_{\rho} \frac{1}{\rho} e^{-x/\rho},$$

with the same symmetric pairing.

Proof. For each zero ρ with $\Im(\rho) > 0$, define

$$g_\rho(x) = \bigl(1 - e^{-x/\rho}\bigr) + \bigl(1 - e^{-x/(1-\rho)}\bigr), \qquad h_\rho(x) = g_\rho'(x) = \frac{1}{\rho}e^{-x/\rho} + \frac{1}{1-\rho}e^{-x/(1-\rho)}.$$

We first show that the series $\sum_{\Im(\rho)>0} h_\rho(x)$ converges uniformly on any compact interval $[a, b] \subset (0, \infty)$. For $\gamma \geq 2b$, we have $|x/\rho| \leq 1/2$ and $|x/(1-\rho)| \leq 1/2$ for all $x \in [a, b]$. By the Taylor expansion of e^{-z} with remainder,

$$\frac{1}{\rho}e^{-x/\rho} = \frac{1}{\rho} - \frac{x}{\rho^2} + R_\rho(x), \qquad |R_\rho(x)| \leq \sum_{k=2}^{\infty} \frac{|x|^k}{k!|\rho|^{k+1}} \leq \frac{|x|^2}{|\rho|^3} \sum_{m=0}^{\infty} \frac{1}{2^m} = \frac{2x^2}{|\rho|^3} \leq \frac{2b^2}{\gamma^3}.$$

Similarly,

$$\frac{1}{1-\rho}e^{-x/(1-\rho)} = \frac{1}{1-\rho} - \frac{x}{(1-\rho)^2} + S_\rho(x), \qquad |S_\rho(x)| \leq \frac{2b^2}{\gamma^3}.$$

Thus

$$h_\rho(x) = \Bigl(\frac{1}{\rho} + \frac{1}{1-\rho}\Bigr) - x\Bigl(\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2}\Bigr) + T_\rho(x),$$

with $|T_\rho(x)| \leq \frac{4b^2}{\gamma^3}$. Using the bounds $|\frac{1}{\rho} + \frac{1}{1-\rho}| \leq 1/\gamma^2$ and $|\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2}| \leq 2/\gamma^2$, we obtain for all large γ,

$$|h_\rho(x)| \leq \frac{1}{\gamma^2} + x \cdot \frac{2}{\gamma^2} + \frac{4b^2}{\gamma^3} \leq \frac{1 + 2b + 4b^2}{\gamma^2} \quad \text{for } x \in [a, b].$$

Let $K = 1 + 2b + 4b^2$ (depending only on $[a, b]$). Then $|h_\rho(x)| \leq K/\gamma^2$ for all $x \in [a, b]$ and all sufficiently large γ.

Let n_k be the number of zeros with $\gamma \in [k-1, k]$. Then $n_k = O(\log k)$. The total contribution from zeros with $\gamma \geq M$ is bounded by

$$\sum_{k=M}^{\infty} \frac{K n_k}{(k-1)^2} \ll \sum_{k=M}^{\infty} \frac{\log k}{k^2} \xrightarrow[M\to\infty]{} 0.$$

Hence the series $\sum_{\Im(\rho)>0} h_\rho(x)$ converges uniformly on $[a, b]$.

Now consider the partial sums $L_N(x) = \sum_{\Im(\rho)>0,\ \gamma\leq N} g_\rho(x)$. Each g_ρ is smooth, and $g_\rho' = h_\rho$. The series of derivatives converges uniformly on $[a, b]$, and the series of functions converges at least at one point (e.g., at any fixed $x_0 \in [a, b]$, by the absolute convergence proved in Section 4). By Theorem 7.17 of Rudin [3], $L(x) = \lim_{N\to\infty} L_N(x)$ is differentiable on $[a, b]$ and $L'(x) = \sum h_\rho(x)$. Since $[a, b]$ is arbitrary, the result holds on $(0, \infty)$. Unwrapping the pairing gives the stated formula for $L'(x)$. □

5. Reality

Proposition 5.1. *$L(x)$ is real-valued for all $x > 0$.*

Proof. We use the definition of the sum in the symmetric pairing. For each zero ρ with $\Im(\rho) > 0$, consider the quadruple $\{\rho, 1-\rho, \bar\rho, 1-\bar\rho\}$. The contribution of this quadruple to $L(x)$ is

$$(1 - e^{-x/\rho}) + (1 - e^{-x/(1-\rho)}) + (1 - e^{-x/\bar\rho}) + (1 - e^{-x/(1-\bar\rho)}).$$

Since $\bar\rho$ and $1-\bar\rho$ are the complex conjugates of ρ and $1-\rho$ respectively, this sum is equal to $2\Re\bigl((1 - e^{-x/\rho}) + (1 - e^{-x/(1-\rho)})\bigr)$, which is real. Summing over all ρ with $\Im(\rho) > 0$ yields a real number. The same reasoning applied to the derivative series shows that $L'(x)$ is also real. □

6. Relation to Li coefficients

Remark 6.1. *Expanding* $1-e^{-x/\rho}$ *as a power series in* x *and interchanging sum and series (justified by absolute convergence) gives*

$$L(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\sigma_n x^n, \qquad \sigma_n=\sum_{\rho}\frac{1}{\rho^n}$$

(the sum is taken in the symmetric pairing). The classical Li coefficients are

$$\lambda_n=\sum_{\rho}\big(1-(1-1/\rho)^n\big)=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\sigma_k.$$

Thus $L(x)$ *is a continuous generating function for the Li coefficients.*

7. Conjecture

Motivated by the Li criterion and the formal analogy, we propose the following:

Conjecture 7.1. *The Riemann Hypothesis is equivalent to the strict positivity* $L'(x)>0$ *for all* $x>0$.

A proof of this equivalence would require an arithmetic representation of $L'(x)$ via the explicit formula, which is beyond the scope of this paper.

8. Conclusion

We have introduced a smoothed functional $L(x)$ defined by the zeros of the Riemann zeta function and established its absolute convergence, differentiability, and reality. This functional is a continuous analogue of the Li coefficients. We have stated a conjecture linking its derivative to the Riemann Hypothesis. The analytic properties proved here provide a foundation for further investigation.

References

[1] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Theory* **65** (1997), 325–333.

[2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford Univ. Press, 1986.

[3] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, 1976.

Independent Researcher, Beijing, China
Email address: `13718705197@139.com`