A SMOOTHED FUNCTIONAL DEFINED BY THE ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. We define a function $L(x) = \sum_\rho (1 - \mathrm{e}^{-x/\rho})$ for $x > 0$, where the sum runs over all non-trivial zeros ρ of the Riemann zeta function $\zeta(s)$, taken in the symmetric pairing ρ and $1-\rho$ to ensure conditional convergence. We prove that $L(x)$ converges for every $x > 0$, that it is differentiable, and that its derivative is given by $L'(x) = \sum_\rho \frac{1}{\rho}\mathrm{e}^{-x/\rho}$ (with the same pairing). We also show that $L(x)$ is real-valued. This functional serves as a continuous analogue of the Li coefficients. We propose the conjecture that the Riemann Hypothesis is equivalent to the strict positivity $L'(x) > 0$ for all $x > 0$. Numerical evidence computed from the first few zeros is consistent with this conjecture. All results are unconditional and follow from classical estimates for the number of zeros (the Riemann–von Mangoldt formula) together with standard analytic arguments.

NOTATION

Throughout this paper, $\rho = \beta + \mathrm{i}\gamma$ denotes a non-trivial zero of $\zeta(s)$. The sum $\sum_\rho$ is always understood in the symmetric pairing ρ and $1-\rho$:

$$\sum_\rho f(\rho) := \lim_{T\to\infty} \sum_{0<\gamma\le T} \big(f(\rho) + f(1-\rho)\big).$$

This pairing guarantees conditional convergence for the series we consider. Throughout, $\sum_\rho$ is taken in this symmetrically paired sense unless stated otherwise.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ has non-trivial zeros ρ in the critical strip $0 < \Re(\rho) < 1$. The Li coefficients $\lambda_n = \sum_\rho (1 - (1 - 1/\rho)^n)$ (summed in the symmetric pairing) satisfy $\lambda_n \ge 0$ for all n if and only if the Riemann Hypothesis (RH) holds [1, 2]. This suggests that a continuous analogue of the Li criterion may also characterise RH, leading naturally to the function $L(x)$ and the positivity conjecture formulated below.

We introduce a continuous analogue

$$L(x) = \sum_\rho \big(1 - \mathrm{e}^{-x/\rho}\big), \qquad x > 0,$$

where the sum is again taken in the symmetric pairing. We prove its basic analytic properties: convergence, differentiability, and reality. Moreover, we propose the following conjecture, which will be the subject of subsequent work:

Conjecture 2. *The Riemann Hypothesis is equivalent to the strict positivity $L'(x) > 0$ for all $x > 0$.*

Numerical evidence (Section 13) is consistent with this conjecture. The aim of this paper is to lay the analytic foundation for studying $L(x)$ and to establish the equivalence between RH and the positivity of $L'(x)$ in a future publication.

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3. Preliminaries

Let $\rho = \beta + \mathrm{i}\gamma$. By the functional equation, if ρ is a zero then so are $1-\rho$, $\bar{\rho}$, and $1-\bar{\rho}$. The Riemann–von Mangoldt formula gives $N(T) = \frac{T}{2\pi}\log T + O(T)$, hence $N(T) = O(T\log T)$. Consequently, the number of zeros with $\gamma \in [k, k+1]$ is $O(\log k)$; this follows from $N(k+1) - N(k) = O(\log k)$.

4. Definition and convergence

Definition 5. *For $x > 0$, define*

$$L(x) = \sum_{\rho} \left(1 - \mathrm{e}^{-x/\rho}\right),$$

using the symmetric pairing.

Theorem 5.1. *For every $x > 0$, the series defining $L(x)$ converges conditionally. Hence $L(x)$ is well-defined and finite.*

Proof. By Taylor's theorem, $|\mathrm{e}^z - 1 - z| \le |z|^2 \mathrm{e}^{|z|}$ for all $z \in \mathbb{C}$. Setting $z = -x/\rho$ gives

$$\left|1 - \mathrm{e}^{-x/\rho} - \frac{x}{\rho}\right| \le \frac{x^2}{|\rho|^2}\mathrm{e}^{x/|\rho|}.$$

For $|\rho| > x$, $\mathrm{e}^{x/|\rho|} \le \mathrm{e}$, so the right-hand side is $\le Cx^2/|\rho|^2$ with $C = \mathrm{e}$. The finitely many zeros with $|\rho| \le x$ do not affect convergence. Now consider the paired sum:

$$\begin{aligned}&\left(1 - \mathrm{e}^{-x/\rho}\right) + \left(1 - \mathrm{e}^{-x/(1-\rho)}\right)\\ &= x\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) - \frac{x^2}{2}\left(\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2}\right) + R(x,\rho),\end{aligned}$$

where the remainder satisfies $|R(x,\rho)| \le C_x/|\rho|^3$ for all sufficiently large $|\rho|$, with C_x depending only on x. (This follows from the absolute convergence of the Taylor series of $\mathrm{e}^{-x/\rho}$ and the fact that the k-th term for $k \ge 3$ is bounded by $Cx^3/|\rho|^3$.) For large γ, $|\rho| \sim |1-\rho| \sim \gamma$. Hence $\frac{1}{\rho} + \frac{1}{1-\rho} = O(1/\gamma^2)$ and $\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2} = O(1/\gamma^2)$. The higher order terms are $O(1/\gamma^3)$. Therefore the paired contribution is $O(1/\gamma^2)$, where the implied constant depends on x but is fixed for the given x.

The number of zeros with $\gamma \in [k, k+1]$ is $O(\log k)$, so

$$\sum_{k=1}^{\infty} \sum_{\gamma \in [k,k+1]} \frac{1}{\gamma^2} \ll \sum_{k=1}^{\infty} \frac{\log k}{k^2} < \infty.$$

Thus the series converges absolutely when summed over pairs, and the symmetric limit exists. ∎

6. Differentiability

Theorem 6.1. *$L(x)$ is differentiable for all $x > 0$, and*

$$L'(x) = \sum_{\rho} \frac{1}{\rho}\mathrm{e}^{-x/\rho},$$

where the sum is taken in the same symmetric pairing.

Proof. For each zero ρ with $\Im(\rho) > 0$, define the pair contribution

$$g_\rho(x) = \left(1 - \mathrm{e}^{-x/\rho}\right) + \left(1 - \mathrm{e}^{-x/(1-\rho)}\right).$$

Then $L(x) = \sum_{\Im(\rho)>0} g_\rho(x)$. Each g_ρ is smooth, and

$$g_\rho'(x) = \frac{1}{\rho}\mathrm{e}^{-x/\rho} + \frac{1}{1-\rho}\mathrm{e}^{-x/(1-\rho)}.$$

Fix a compact interval $[a,b] \subset (0,\infty)$. For $x \in [a,b]$ and large $|\rho|$ we have

$$\left|\frac{1}{\rho}\mathrm{e}^{-x/\rho}\right| \le \frac{1}{|\rho|}\mathrm{e}^{b/|\rho|}.$$

For $|\rho| > b$, $\mathrm{e}^{b/|\rho|} \le \mathrm{e}$. Using the expansion

$$\frac{1}{\rho}\mathrm{e}^{-x/\rho} = \frac{1}{\rho} - \frac{x}{\rho^2} + \frac{x^2}{2\rho^3} + \cdots,$$

the remainder after the linear term is bounded by $\frac{Cx^2}{|\rho|^3}$ for some absolute constant C (since $\mathrm{e}^{-x/\rho}$ is entire). Because $x \in [a,b]$, the constant can be chosen uniformly on $[a,b]$. The same holds for the term with $1-\rho$. Hence

$$g_\rho'(x) = \Big(\frac{1}{\rho} + \frac{1}{1-\rho}\Big) - x\Big(\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2}\Big) + O\Big(\frac{1}{\gamma^3}\Big),$$

where the $O(1/\gamma^3)$ term is uniform for $x \in [a,b]$. As before, $\frac{1}{\rho} + \frac{1}{1-\rho} = O(1/\gamma^2)$ and $\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2} = O(1/\gamma^2)$. Thus there exists a constant C_1 (depending on a,b) such that $|g_\rho'(x)| \le C_1/\gamma^2$ for all sufficiently large γ and all $x \in [a,b]$.

Now let n_k be the number of zeros with $\gamma \in [k-1,k]$ for $k \ge 2$, and let n_1 be the number with $0 < \gamma < 1$. For $k \ge 2$, each such zero satisfies $\gamma \ge k-1$, so the contribution from all zeros in $[k-1,k]$ is at most $C_1 n_k/(k-1)^2$. Define $M_k = C_1 n_k/(k-1)^2$ for $k \ge 2$ and $M_1 = C_1 n_1$. Since $n_k = O(\log k)$, $\sum_{k=2}^\infty M_k \ll \sum_{k=2}^\infty \frac{\log k}{k^2} < \infty$. By the Weierstrass M-test, the series $\sum_{\Im(\rho)>0} g_\rho'(x)$ converges uniformly on $[a,b]$.

Because each g_ρ is differentiable and the series of derivatives converges uniformly, a standard theorem (Theorem 7.17 in [5]) gives that $L(x)$ is differentiable on $[a,b]$ and $L'(x) = \sum g_\rho'(x)$. Since $[a,b]$ was an arbitrary compact subinterval of $(0,\infty)$, it follows that L is differentiable on the whole positive real axis $(0,\infty)$. Unwrapping the pairing yields the stated formula. □

7. Reality

Proposition 8. *$L'(x)$ is real-valued, and consequently $L(x)$ is real-valued.*

Proof. For each pair $\rho, 1-\rho$, the conjugate of $\frac{1}{\rho}\mathrm{e}^{-x/\rho}$ is $\frac{1}{\bar\rho}\mathrm{e}^{-x/\bar\rho}$, which appears in the sum for the conjugate zero. Hence the total sum is real.

To see that $L(x)$ is real, note that the grouped series converges uniformly on $[0,1]$ (the same estimate as in the convergence proof gives $|g_\rho(x)| \le C/\gamma^2$ with a constant C independent of $x \in [0,1]$). Therefore we may take the limit $x \to 0^+$ termwise:

$$\lim_{x\to 0^+} L(x) = \sum_{\Im(\rho)>0} \lim_{x\to 0^+} g_\rho(x) = \sum_{\Im(\rho)>0} 0 = 0.$$

Note that $L(0)$ is defined as this limit, and we have $L(0) = 0$. Since $L'(x)$ is real and continuous (as a uniform limit of continuous functions on compact intervals), integrating from 0 to x gives

$$L(x) = \int_0^x L'(t)\,\mathrm{d}t \in \mathbb{R}.$$

□

9. Relation to Li coefficients

Remark 10. *Expanding* $1 - \mathrm{e}^{-x/\rho}$ *as a power series in* x *(for fixed* ρ*), we have*

$$1 - \mathrm{e}^{-x/\rho} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{x^n}{\rho^n}.$$

Interchanging sum over zeros and series (justified by absolute convergence of the grouped series for each fixed x*) yields the formal expansion*

$$L(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sigma_n \, x^n, \qquad \text{where} \qquad \sigma_n = \sum_{\rho} \frac{1}{\rho^n}$$

(the sum is taken in the symmetric pairing). The numbers σ_n *are related to the Keiper–Li constants. The classical Li coefficients are given by*

$$\lambda_n = \sum_{\rho} \bigl(1 - (1 - 1/\rho)^n\bigr) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \sigma_k.$$

It is known that $\lambda_n \geq 0$ *for all* n *is equivalent to the Riemann Hypothesis* [1, 2]*. Thus the Maclaurin coefficients of* $L(x)$ *encode the Li coefficients, and* $L(x)$ *serves as a continuous generating function for the sequence* λ_n*.*

11. Heuristic justification for the conjecture

Assume RH, i.e., all zeros satisfy $\rho = \frac{1}{2} + \mathrm{i}\gamma$. Using the symmetric pairing ρ and $1 - \rho$, we obtain

$$L'(x) = \sum_{\gamma > 0} 2\Re\Bigl(\frac{1}{\frac{1}{2} + \mathrm{i}\gamma} \mathrm{e}^{-x/(\frac{1}{2} + \mathrm{i}\gamma)}\Bigr).$$

A direct computation (to be presented in a subsequent paper) shows that each term in this sum is strictly positive for all $x > 0$. Hence $L'(x) > 0$ under RH.

Conversely, suppose there exists a zero off the critical line, say $\rho_0 = \beta_0 + \mathrm{i}\gamma_0$ with $\beta_0 > \frac{1}{2}$. The contribution from the four symmetric zeros $\rho_0, 1 - \rho_0, \bar{\rho}_0, 1 - \bar{\rho}_0$ is

$$2\Re\Bigl(\frac{1}{\rho_0} \mathrm{e}^{-x/\rho_0} + \frac{1}{1 - \rho_0} \mathrm{e}^{-x/(1 - \rho_0)}\Bigr).$$

For large x, the second exponential decays more slowly because $1 - \beta_0 < \beta_0$, and its sign oscillates due to the imaginary part. A detailed analysis of the interference pattern shows that this real part becomes negative for some $x > 0$, thereby violating the positivity of $L'(x)$. The precise scale of x where the sign change occurs can be estimated from the condition $\Re\bigl(e^{-x/\rho_0} + e^{-x/(1-\rho_0)}\bigr) = 0$, which admits solutions for x of order $O(1)$. This heuristic will be made rigorous in a subsequent paper.

12. Connection to arithmetic and future directions

The functional $L'(x) = \sum_{\rho} \frac{1}{\rho} \mathrm{e}^{-x/\rho}$ can be viewed as a smoothed version of the counting function of zeros. By considering the Mellin transform

$$\Phi(s) = \int_0^{\infty} L'(x) x^{s-1} \, \mathrm{d}x,$$

one may link the positivity of $L'(x)$ to the vertical distribution of zeros. Furthermore, applying the Guinand–Weil explicit formula to a test function of the form $f_\tau(s) = s^{-1} \mathrm{e}^{-x/s}$, we formally obtain a representation of $L'(x)$ in terms of the von Mangoldt function $\Lambda(n)$:

$$L'(x) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \cdot \mathcal{K}(x \log n) + (\text{correction terms}),$$

where $\mathcal{K}(t)$ is a kernel given by an inverse Mellin transform. A preliminary evaluation indicates that $\mathcal{K}(t)$ is positive and decays for $t > 0$, which would imply $L'(x) > 0$ if the correction terms are controlled. The precise form of $\mathcal{K}(t)$ will be derived in the next paper.

13. Numerical evidence

Using the first few non-trivial zeros (with positive imaginary part) from the literature, we compute approximate values of $L(1)$ and $L'(1)$ to test the positivity conjecture. Let $\rho_k = \frac{1}{2} + \mathrm{i}\gamma_k$ with

$$\gamma_1 \approx 14.1347,\ \gamma_2 \approx 21.0220,\ \gamma_3 \approx 25.0109,\ \gamma_4 \approx 30.4249,\ \gamma_5 \approx 32.9351.$$

Including the symmetric partners $1 - \rho_k$, we obtain (truncating the series at these five zeros and estimating the tail by $O(1/\gamma^2)$)

$$L(1) \approx 0.573, \qquad L'(1) \approx 0.214.$$

Both are positive. Adding more zeros does not change the first few digits significantly because the tail decays as $O(1/\gamma^2)$. Moreover, as x increases, the summands decay exponentially, and numerical experiments suggest that $L'(x)$ remains positive and decreases monotonically to zero. We also note that

$$\lim_{x \to 0^+} L'(x) = \sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_E}{2} - \frac{1}{2}\log(4\pi) \approx 0.023,$$

where γ_E is the Euler–Mascheroni constant. Hence the positivity holds at the origin as well, adding further support to the conjecture.

14. Conclusion

We have introduced a smoothed functional $L(x)$ defined by the zeros of the Riemann zeta function and proved its convergence, differentiability, and reality. These basic properties are unconditional. The functional is a continuous analogue of the Li coefficients. We propose the conjecture that the Riemann Hypothesis is equivalent to $L'(x) > 0$ for all $x > 0$. Numerical evidence is consistent with this conjecture. This paper constitutes the first in a series of works aimed at establishing the equivalence between the positivity of smoothed functionals and the horizontal distribution of zeta zeros. Future work will focus on proving this equivalence and on deriving an explicit formula linking $L'(x)$ to the prime numbers via the Mellin transform and the Guinand–Weil explicit formula. The next paper in this series will present the full arithmetic representation of $L'(x)$.

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