Quantum Superposition from Octonionic Nonassociativity

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We propose a framework in which quantum superposition emerges from the nonassociative structure of the octonion algebra. A dynamical octonion-valued field is introduced whose self-interaction is governed by the octonionic associator. The imaginary octonions define a natural G_2 three-form which determines a metric structure, providing a geometric link between algebra and spacetime. The resulting field dynamics are summarized by a single master equation containing propagation, curvature coupling, and nonassociative interaction terms. Because octonion multiplication admits distinct ordering channels, the dynamics naturally generate independent algebraic evolution paths whose linear combination reproduces the superposition principle of quantum mechanics. Projection onto quaternionic subalgebras yields complex wavefunctions obeying the Schrödinger equation in the nonrelativistic limit. The framework suggests that the Hilbert-space structure of quantum mechanics may originate from the nonassociative nature of the underlying algebraic structure.

Introduction

Quantum mechanics is fundamentally characterized by the principle of superposition: physical states may exist in linear combinations whose interference produces characteristic quantum phenomena. In standard formulations this principle is postulated as an axiom of Hilbert-space quantum mechanics.

A natural question is whether the superposition principle could originate from deeper algebraic structures underlying physical theory. Among the possible mathematical candidates, the four normed division algebras — $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ — play a special role in theoretical physics.

The octonions $\mathbb{O}$ are particularly remarkable because they are nonassociative. While real numbers, complex numbers, and quaternions satisfy the associative law, octonions obey

$$(xy)z \neq x(yz). \tag{1}$$

The deviation from associativity is measured by the associator

$$A(x, y, z) = (xy)z - x(yz). \tag{2}$$

In this work we propose that quantum superposition arises from this nonassociative structure. The key idea is that nonassociativity naturally generates multiple algebraic composition channels which appear simultaneously in dynamical equations.

Octonion Algebra

The octonions $\mathbb{O}$ form an eight–dimensional real division algebra. An arbitrary octonion can be written as

$$x = x_0 e_0 + \sum_{i=1}^{7} x_i e_i, \tag{3}$$

where $e_0 = 1$ and e_i denote the imaginary basis elements. The multiplication rules are

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k, \tag{4}$$

where f_{ijk} are the completely antisymmetric structure constants determined by the Fano plane.

Geometry from Octonions

The imaginary octonions define a seven-dimensional vector space

$$V = \mathrm{Im}(\mathbb{O}). \tag{5}$$

Octonion multiplication induces a three-form

$$\phi(x, y, z) = \langle x, yz \rangle. \tag{6}$$

The stabilizer of this form is the exceptional Lie group G_2. A fundamental result of G_2 geometry is that the three-form uniquely determines a metric

$$g_{ij} = \frac{1}{6} \phi_{ikl} \phi_{jkl}. \tag{7}$$

Thus the algebraic structure of octonion multiplication generates a geometric structure.

Octonionic Field Dynamics

We introduce an octonion-valued field

$$\Psi(x) \in \mathbb{O} \tag{8}$$

defined on a four-dimensional spacetime manifold $(M, g_{\mu\nu})$.

The form of the action is determined by three guiding principles:

1. *General covariance.* The theory should be invariant under spacetime diffeomorphisms.

2. *Octonionic algebraic invariance.* The dynamics should depend only on invariant operations of the octonion algebra.
3. *Minimality.* Only the lowest-order scalar invariants constructed from the field and its derivatives are included.

Algebraic Invariants

The octonion algebra possesses two natural invariant structures.

The first is the norm

$$N(x) = x\bar{x}, \tag{9}$$

which induces the inner product

$$\langle x, y\rangle = \mathrm{Re}(x\bar{y}). \tag{10}$$

This structure allows the construction of a kinetic invariant for the field

$$\langle \nabla_\mu \Psi, \nabla^\mu \Psi\rangle. \tag{11}$$

The second characteristic invariant of the octonion algebra is the associator

$$A(x, y, z) = (xy)z - x(yz). \tag{12}$$

Since the octonions are the only normed division algebra that is nonassociative, the associator provides the natural algebraic quantity that measures deviations from associative multiplication.

A scalar invariant can therefore be constructed from the associator via

$$|A(x, y, z)|^2 = \langle A, A\rangle. \tag{13}$$

Minimal Covariant Action

Combining the geometric and algebraic invariants yields the simplest generally covariant action

$$S = \int d^4x\sqrt{-g}\left(R + \alpha\langle \nabla_\mu \Psi, \nabla^\mu \Psi\rangle - \lambda|A(\Psi, \Psi, \Psi)|^2\right). \tag{14}$$

The three contributions have the following interpretation:

- The Einstein–Hilbert term R describes spacetime curvature.
- The kinetic term governs the propagation of the octonion field.
- The associator potential encodes the nonassociative self-interaction of the field.

Higher-order invariants involving additional derivatives or higher powers of the associator may appear

Variation of the Action

To derive the field equation we vary the action with respect to the octonion field Ψ while keeping the metric fixed.

Kinetic Term

The variation of the kinetic term is

$$\delta S_{\mathrm{kin}} = \alpha \int d^4x\sqrt{-g}\,\mathrm{Re}\left(\nabla_\mu \delta\Psi\, \nabla^\mu \bar{\Psi}\right). \tag{15}$$

Using integration by parts and neglecting boundary terms we obtain

$$\delta S_{\mathrm{kin}} = -\alpha \int d^4x\sqrt{-g}\,\mathrm{Re}\left(\delta\Psi\, \nabla^2 \bar{\Psi}\right), \tag{16}$$

where the covariant Laplacian is

$$\nabla^2\Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu \Psi\right). \tag{17}$$

Associator Interaction

The interaction potential is

$$V(\Psi) = \lambda|A(\Psi, \Psi, \Psi)|^2 \tag{18}$$

with

$$|A|^2 = \langle A, A\rangle. \tag{19}$$

The variation yields

$$\delta V = 2\lambda\langle A(\Psi, \Psi, \Psi), \delta A\rangle. \tag{20}$$

The variation of the associator follows from

$$A(\Psi, \Psi, \Psi) = (\Psi\Psi)\Psi - \Psi(\Psi\Psi). \tag{21}$$

A direct computation gives

$$\delta A = (\delta\Psi\Psi)\Psi + (\Psi\delta\Psi)\Psi + (\Psi\Psi)\delta\Psi \tag{22}$$

$$- \delta\Psi(\Psi\Psi) - \Psi(\delta\Psi\Psi) - \Psi(\Psi\delta\Psi). \tag{23}$$

Each term corresponds to inserting $\delta\Psi$ into one position of the nonassociative product.

Thus the variation of the potential contributes a term proportional to the associator itself,

$$\delta S_{\text{int}} = \lambda \int d^4x \sqrt{-g}\, \langle \delta\Psi, \mathcal{A}(\Psi) \rangle, \tag{24}$$

where $\mathcal{A}(\Psi)$ is an octonion-valued function constructed from $A(\Psi, \Psi, \Psi)$.

Field Equation

Combining the kinetic and interaction contributions and requiring $\delta S = 0$ yields the Euler–Lagrange equation

$$\nabla^2\Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R\Psi = 0. \tag{25}$$

We refer to this equation as the *octonionic master equation.* It contains three contributions:

- propagation of the octonion field via the Laplace–Beltrami operator
- nonlinear self-interaction generated by the octonionic associator
- coupling to spacetime curvature

The presence of the associator term is crucial: it introduces distinct algebraic multiplication channels which lead to the superposition structure derived in the following sections.

Superposition from Nonassociativity

The central feature of the octonion algebra is its nonassociativity,

$$(xy)z \neq x(yz). \tag{26}$$

The deviation from associativity is quantified by the associator

$$A(x, y, z) = (xy)z - x(yz). \tag{27}$$

In the octonionic field theory introduced above the associator appears explicitly in the dynamical equation

$$\nabla^2\Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R\Psi = 0. \tag{28}$$

The presence of the associator term implies that different multiplication orderings of the field contribute independently to the dynamics.

Multiplication Channels

Consider the cubic products appearing in the associator. Two distinct algebraic compositions exist,

$$\Psi_A = (\Psi\Psi)\Psi, \tag{29}$$
$$\Psi_B = \Psi(\Psi\Psi). \tag{30}$$

The associator can therefore be written as

$$A(\Psi, \Psi, \Psi) = \Psi_A - \Psi_B. \tag{31}$$

Because the octonion algebra is nonassociative these two expressions are generically different.

Lemma. For generic octonions Ψ, the elements Ψ_A and Ψ_B are linearly independent.

Proof. If the algebra were associative one would have $\Psi_A = \Psi_B$. However octonions satisfy $A(\Psi, \Psi, \Psi) \neq 0$ for generic elements, implying that the two products cannot coincide. Thus they span a two–dimensional subspace of $\mathbb{O}$.

Dynamical Coupling of Channels

Substituting the channel decomposition into the master equation gives

$$\nabla^2\Psi + \lambda(\Psi_A - \Psi_B) + \kappa R\Psi = 0. \tag{32}$$

The field equation therefore couples two algebraically independent directions in the octonion algebra. As a consequence, the local solution space must contain both channels.

The general solution can therefore be written as a linear combination

$$\Psi = a\Psi_A + b\Psi_B. \tag{33}$$

Emergence of Superposition

The above structure is mathematically identical to the superposition principle. The octonionic field evolves simultaneously along multiple algebraic multiplication channels, and the physical state corresponds to their linear combination.

Thus the nonassociativity of octonion multiplication naturally generates a superposition structure,

$$\Psi = a\Psi_A + b\Psi_B. \tag{34}$$

After projection to a complex subspace the state becomes a conventional quantum wavefunction

$$\psi = a\psi_A + b\psi_B, \tag{35}$$

which reproduces the usual quantum-mechanical superposition principle.

Emergent Hilbert Structure

The octonion norm induces an inner product

$$\langle x, y\rangle = \mathrm{Re}(x\bar{y}). \tag{36}$$

Restricting this inner product to the solution space spanned by $\{\Psi_A, \Psi_B\}$ gives

$$||\Psi||^2 = |a|^2||\Psi_A||^2 + |b|^2||\Psi_B||^2 + 2\mathrm{Re}(a^* b\langle\Psi_A, \Psi_B\rangle). \tag{37}$$

After projection to a complex subspace

$$\mathbb{O} \to \mathbb{H} \to \mathbb{C} \tag{38}$$

the coefficients become complex numbers and the solution space becomes a Hilbert space.

Decoherence from Nonassociative Interactions

In realistic physical systems quantum superposition is often suppressed by interactions with the environment. This process, known as decoherence, can be naturally interpreted within the octonionic framework.

Consider a decomposition of the total octonionic field into a system and an environment contribution,

$$\Psi = \Psi_S + \Psi_E. \tag{39}$$

Because octonion multiplication is nonassociative, the associator generates additional mixed interaction terms

$$A(\Psi, \Psi, \Psi) = A(\Psi_S, \Psi_S, \Psi_S)+A(\Psi_S, \Psi_S, \Psi_E)+A(\Psi_S, \Psi_E, \Psi_E)+A(\Psi_E, \Psi_E, \Psi_E). \tag{40}$$

The mixed associator contributions couple the system degrees of freedom to the environment.

Suppression of Interference

Suppose the system is prepared in a superposition state

$$\Psi_S = a\Psi_A + b\Psi_B. \tag{41}$$

After projection onto a complex subspace the corresponding wavefunction is

$$\psi = a\psi_A + b\psi_B. \tag{42}$$

The probability density contains the interference term

$$|\psi|^2 = |a|^2|\psi_A|^2 + |b|^2|\psi_B|^2 + 2\,\mathrm{Re}(a^* b\psi_A^*\psi_B). \tag{43}$$

Coupling to environmental degrees of freedom introduces fluctuating phases generated by the mixed associator terms,

$$\psi_A^*\psi_B \to \psi_A^*\psi_B\, e^{i\theta_E}. \tag{44}$$

Averaging over many environmental states yields

$$\langle e^{i\theta_E}\rangle \approx 0. \tag{45}$$

The interference contribution therefore vanishes and the probability density reduces to

$$|\psi|^2 = |a|^2|\psi_A|^2 + |b|^2|\psi_B|^2. \tag{46}$$

Classical Limit

In macroscopic systems the number of environmental degrees of freedom is extremely large, and the mixed associator interactions produce rapidly fluctuating phases that effectively suppress all interference effects.

Thus the same nonassociative algebraic mechanism that generates quantum superposition also provides a natural explanation for the emergence of classical behavior through decoherence.

Experimental Signature

A direct experimental test of nonassociative quantum dynamics arises in multi-path interferometry.

Define the third-order interference quantity

$$I_{123} = P_{123} - P_{12} - P_{23} - P_{13} + P_1 + P_2 + P_3. \tag{47}$$

In standard quantum mechanics one finds

$$I_{123} = 0. \tag{48}$$

Within the octonionic framework the associator generates corrections

$$I_{123} \sim \lambda A(\Psi_1, \Psi_2, \Psi_3). \tag{49}$$

A nonzero value of I_{123} in high-precision triple-path interferometry would therefore signal the presence of nonassociative quantum dynamics.

Conclusion

We have proposed a framework in which the superposition principle of quantum mechanics arises from the nonassociative structure of the octonion algebra. The octonionic master equation contains an associator term that generates independent algebraic multiplication channels, whose linear combination produces the superposition structure of quantum states. After projection onto complex subspaces this structure reproduces the conventional quantum-mechanical wavefunction. In this picture superposition is not a fundamental postulate but a consequence of the nonassociative algebraic dynamics of the underlying field.

Appendix

Emergence of Standard Physical Equations from the Octonionic Master Equation

The framework proposed in this work is governed by the *octonionic master equation*

$$\nabla^2\Psi + \lambda A(\Psi,\Psi,\Psi) + \kappa R\Psi = 0, \tag{50}$$

where $\Psi(x)$ is an octonion-valued field and

$$A(x,y,z) = (xy)z - x(yz) \tag{51}$$

is the octonionic associator, which measures the nonassociativity of the algebra.

The covariant Laplacian is

$$\nabla^2\Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\Psi\right). \tag{52}$$

Different physical theories arise as limits or projections of this equation.

Einstein Field Equations

The octonionic master equation contains an explicit coupling of the field Ψ to the Ricci scalar R. This indicates that the energy density of the octonionic field contributes to spacetime curvature.

To make this relation explicit we consider the energy–momentum tensor associated with the octonionic field. Using the inner product induced by the octonion norm

$$\langle x,y\rangle = \mathrm{Re}(x\bar{y}), \tag{53}$$

the kinetic energy density of the field can be written as

$$|\nabla\Psi|^2 = g^{\mu\nu}\langle\nabla_\mu\Psi,\nabla_\nu\Psi\rangle. \tag{54}$$

The interaction potential generated by the associator is

$$V(\Psi) = \lambda|A(\Psi,\Psi,\Psi)|^2. \tag{55}$$

The stress–energy tensor of the octonionic field therefore takes the form

$$T_{\mu\nu} = \nabla_\mu\Psi\nabla_\nu\bar{\Psi} - \frac{1}{2}g_{\mu\nu}|\nabla\Psi|^2 + g_{\mu\nu}V(\Psi). \tag{56}$$

This tensor represents the local energy and momentum carried by the octonionic field.

The dynamics of spacetime curvature are then governed by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \tag{57}$$

Thus gravity emerges as the geometric sector of the octonionic dynamics.

Dirac Equation

The octonion algebra contains many quaternionic subalgebras

$$\mathbb{H} \subset \mathbb{O} \tag{58}$$

generated by sets of the form

$$\{1, e_i, e_j, e_ie_j\}. \tag{59}$$

Quaternion algebras are closely related to Clifford algebras. In particular

$$\mathbb{H} \cong Cl(0,2). \tag{60}$$

Using a quaternionic basis, the octonionic field can be decomposed into a pair of complex components

$$\Psi = \psi_1 + \psi_2 e_k. \tag{61}$$

These components can be arranged into a two-component spinor

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{62}$$

Introducing gamma matrices satisfying

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \tag{63}$$

the projected field obeys the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{64}$$

Thus relativistic fermionic dynamics arise from quaternionic projections of the octonionic field.

Schrödinger Equation

To obtain the nonrelativistic limit we consider the regime in which the nonassociative interaction term is small,

$$|A(\Psi, \Psi, \Psi)| \ll 1. \tag{65}$$

In this case the master equation reduces approximately to a relativistic wave equation

$$\Box\psi + m^2c^2\psi = 0, \tag{66}$$

where

$$\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2. \tag{67}$$

We separate the rapidly oscillating rest-energy contribution

$$\psi(x,t) = \exp\left(-\frac{imc^2t}{\hbar}\right)\phi(x,t). \tag{68}$$

Computing the time derivatives gives

$$\partial_t\psi = e^{-imc^2t/\hbar}\left(\partial_t\phi - \frac{imc^2}{\hbar}\phi\right) \tag{69}$$

and

$$\partial_t^2\psi = e^{-imc^2t/\hbar}\left(\partial_t^2\phi - \frac{2imc^2}{\hbar}\partial_t\phi - \frac{m^2c^4}{\hbar^2}\phi\right). \tag{70}$$

Substituting into the relativistic wave equation and cancelling the dominant rest-energy terms yields

$$-\frac{2im}{\hbar}\partial_t\phi - \nabla^2\phi = 0. \tag{71}$$

Multiplying by $\hbar^2/(2m)$ gives

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\nabla^2\phi, \tag{72}$$

which is the Schrödinger equation.

Born Rule

The octonion algebra possesses a natural multiplicative norm

$$N(x) = x\bar{x}. \tag{73}$$

For

$$x = x_0 + \sum_i x_i e_i \tag{74}$$

this norm becomes

$$N(x) = x_0^2 + \sum_i x_i^2. \tag{75}$$

The norm induces an inner product

$$\langle x, y\rangle = \mathrm{Re}(x\bar{y}). \tag{76}$$

After projection onto a complex subspace

$$\mathbb{O} \to \mathbb{H} \to \mathbb{C}, \tag{77}$$

the octonionic field reduces to a complex wavefunction $\psi(x,t)$.

The norm then becomes

$$N(\psi) = \psi^*\psi = |\psi|^2. \tag{78}$$

Expanding the wavefunction in orthonormal basis states

$$\psi = \sum_i c_i\phi_i \tag{79}$$

gives

$$\langle\psi, \psi\rangle = \sum_{ij} c_i^* c_j \langle\phi_i, \phi_j\rangle. \tag{80}$$

Using orthonormality

$$\langle\phi_i, \phi_j\rangle = \delta_{ij} \tag{81}$$

one obtains

$$\langle\psi, \psi\rangle = \sum_i |c_i|^2. \tag{82}$$

Thus the probability of measuring state ϕ_i is

$$P_i = |c_i|^2. \tag{83}$$

Summary

Starting from the octonionic master equation

$$\nabla^2\Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R\Psi = 0, \tag{84}$$

different projections and limits yield

- Einstein field equations

- Dirac equation
- Schrödinger equation
- Born rule

suggesting that these structures may originate from a common nonassociative algebraic framework.

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