J. S. Bell's Inequality Postulate Reviewed By Direct Comparison Of Expectation Formulae

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Abstract

This paper identifies inconsistencies between the expression for the Kolmogorov expectation value of the product of two discrete random variables $A(\overrightarrow{a},\lambda)$, $B\left(\overrightarrow{b},\lambda\right)$ and the incorporation of such expectation values in Bell-type inequalities. Calculations related to Bell and Horne-Clauser-Shinomy-Holt inequalities are included.

1 Introduction

Bell's inequality postulate regarding a statistical expectation value $P\left(\overrightarrow{a},\overrightarrow{b}\right)$ is stated in [2, section IV. Contradiction] as:

> "the quantum mechanical expectation value cannot be represented, either accurately or arbitrarily closely, in the form":

$$P\left(\overrightarrow{a},\overrightarrow{b}\right) = \int d\lambda\rho(\lambda)\,A(\overrightarrow{a},\lambda)\,B\left(\overrightarrow{b},\lambda\right) \tag{1}$$

where:
$\overrightarrow{a}$ and $\overrightarrow{b}$ are unit vectors representing the settings of two detection devices,
λ is a unit vector random variable from the domain of surface vectors of a unit sphere,
$\rho(\lambda)$ represents the uniform distribution of vectors in the domain of surface vectors of a unit sphere.

The following statement appears in [2, section II. Formulation] with respect to [2, equations (2) and (3)]:

> equation (1) "should equal the quantum mechanical value, which for the singlet state is":

$$\left\langle\overrightarrow{\sigma}_1\cdot\overrightarrow{a}\,\overrightarrow{\sigma}_2\cdot\overrightarrow{b}\right\rangle = -\overrightarrow{a}\cdot\overrightarrow{b} \tag{2}$$

But this is not necessarily so as:

- equation (1) refers to the expectation value of the product of two discrete random variables $A(\overrightarrow{a},\lambda)$ and $B\left(\overrightarrow{b},\lambda\right)$, whereas:
- equation (2) refers to the expectation value of the product of two quantum mechanical observables $\overrightarrow{\sigma}_1\cdot\overrightarrow{a}$ and $\overrightarrow{\sigma}_2\cdot\overrightarrow{b}$ that operate on a continuous random variable λ.

$\overrightarrow{\sigma}_1$ and $\overrightarrow{\sigma}_2$ represent the vector of Pauli matrices.

A mathematically complete representation of the quantum mechanical expectation value (2) is:

$$\left\langle\lambda_b\lambda_a\middle|\overrightarrow{\sigma}_1\cdot\overrightarrow{a}\,\overrightarrow{\sigma}_2\cdot\overrightarrow{b}\middle|\lambda_b\lambda_a\right\rangle = -\overrightarrow{a}\cdot\overrightarrow{b} \tag{3}$$

where:
$\overrightarrow{\sigma}_1\cdot\overrightarrow{a}$ operates only on the random variable λ_a and
$\overrightarrow{\sigma}_2\cdot\overrightarrow{b}$ operates only on the random variable λ_b.

2 Expectation Analysis

In what follows, equation (1) is replaced by the equivalent equation:

$$E\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right) = \int \rho\left(\lambda\right)A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)d\lambda \tag{4}$$

For a uniform distribution of a unit three dimensional Euclidean vector λ over the surface of a unit sphere this equation becomes, by Lemma 1, Appendix (A):

$$E\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right) = \frac{1}{4\pi}\int A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)d\lambda \tag{5}$$

From the Expected Value Theorem of Appendix B, the solution to equation (1) is:

$$\mathbb{E}\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right) = \frac{2\theta}{\pi} - 1 \tag{6}$$

where;

θ radians represents the angle between the two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

This result agrees with the result of a more complex wholly three-dimensional treatment.

3 Comparison

As depicted in Figure 1, the linear function $\mathbb{E}\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right) = \frac{2}{\pi}\theta - 1$ is clearly different from the non-linear function $\left\langle \overrightarrow{\sigma}_1 \cdot \overrightarrow{a}\, \overrightarrow{\sigma}_2 \cdot \overrightarrow{b} \right\rangle = -\overrightarrow{a} \cdot \overrightarrow{b} = -Cos\left(\theta\right)$.

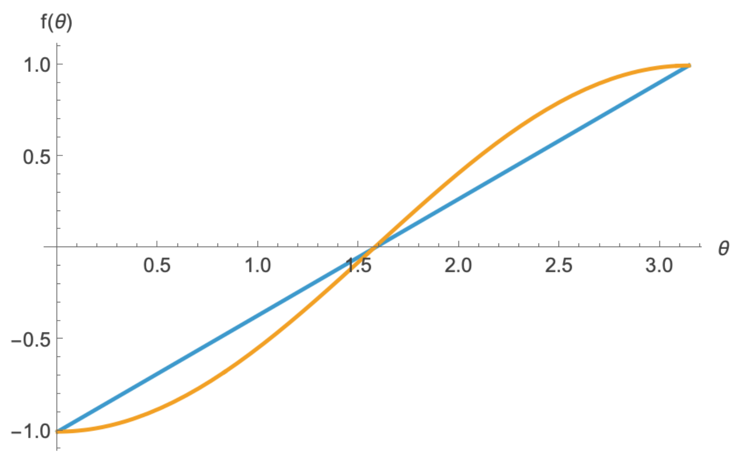

Figure 1: $\frac{2}{\pi}\theta - 1$ (blue), $-Cos\left(\theta\right)$ (orange)

4 Consideration Of Bell's 1964 Inequality

Define $\theta_{pq} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as the radian angle between two vectors vectors $\overrightarrow{p}$ and $\overrightarrow{q}$.

Bell's 1964 inequality [2, equation (15)] states:

$$1 + P\left(\overrightarrow{b},\overrightarrow{c}\right) \geq \left|P\left(\overrightarrow{a},\overrightarrow{b}\right) - P\left(\overrightarrow{a},\overrightarrow{c}\right)\right| \tag{7}$$

Therefore, substituting equation (6) for each of the expectation terms results in:

$$1 + \left(\frac{2}{\pi}\theta_{bc} - 1\right) \geq \left|\frac{2}{\pi}\theta_{ab} - 1 - \left(\frac{2}{\pi}\theta_{ac} - 1\right)\right| \quad (8)$$

$$\frac{2}{\pi}\theta_{bc} \geq \left|\frac{2}{\pi}\theta_{ab} - \frac{2}{\pi}\theta_{ac}\right| \quad (9)$$

$$\theta_{bc} \geq |\theta_{ab} - \theta_{ac}| \quad (10)$$

For $\theta_{bc} < 0$ equation 10 is clearly invalid.
For example, for detector settings $a, b, c = 0, \frac{\pi}{2}, \frac{\pi}{4}$:
$\theta_{ab} = \frac{\pi}{2}$
$\theta_{ac} = \frac{\pi}{4}$
$|\theta_{ab} - \theta_{ac}| = \frac{\pi}{4}$
Therefore:

$$\theta_{bc} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} < |\theta_{ab} - \theta_{ac}| \quad (11)$$

Representing the setting of $\overrightarrow{a}$ by $\theta_{\mathbf{a}} = \mathbf{0}$ and the relative angles of $\overrightarrow{b}$, and $\overrightarrow{c}$, by θ_b and θ_c respectively Table 1 illustrates this comparison.

θ_{ab}	θ_{ac}	$\theta_{ab} - \theta_{ac}$	$\|\theta_{ab} - \theta_{ac}\|$	$\theta_{bc} = \theta_{ac} - \theta_{ab}$	$\theta_{bc} \geq \|\theta_{ab} - \theta_{ac}\|$?
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$	No
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{\pi}{2} + \frac{\pi}{4} = -\frac{\pi}{4}$	No

Table 1: Bell 1964 Inequality Comparison With $\overrightarrow{a}$ Angle Set to 0

In fact, as:

$$\theta_{bc} = \theta_{ac} - \theta_{ab} \quad (12)$$

$$\frac{2}{\pi}\theta_{bc} = \frac{2}{\pi}\theta_{ac} - \frac{2}{\pi}\theta_{ab} \quad (13)$$

$$\frac{2}{\pi}\theta_{bc} - 1 = \frac{2}{\pi}\theta_{ac} - 1 - \left(\frac{2}{\pi}\theta_{ab} - 1\right) - 1 \quad (14)$$

which provides the equation:

$$\mathbb{E}\left(A\left(\overrightarrow{b}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) = \mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) - \mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{b}, \lambda\right)\right) - 1 \quad (15)$$

or, in the terminology of reference [2]:

$$1 + P\left(\overrightarrow{b}, \overrightarrow{c}\right) = P\left(\overrightarrow{a}, \overrightarrow{c}\right) - P\left(\overrightarrow{a}, \overrightarrow{b}\right) \quad (16)$$

5 Consideration Of The HCSH Inequality

The Horne-Clauser-Shinomy-Holt (HCSH) inequality [3, equation (154.5)] states:

$$\left|P\left(\overrightarrow{a}, \overrightarrow{b}\right) - P\left(\overrightarrow{a}, \overrightarrow{c}\right)\right| + \left|P\left(\overrightarrow{d}, \overrightarrow{c}\right) + P\left(\overrightarrow{d}, \overrightarrow{b}\right)\right| \leq 2 \quad (17)$$

Therefore, substituting equation (6) for each of the expectation terms:

$$\left|\frac{2}{\pi}\theta_{ab} - 1 - \left(\frac{2}{\pi}\theta_{ac} - 1\right)\right| + \left|\frac{2}{\pi}\theta_{dc} - 1 + \frac{2}{\pi}\theta_{db} - 1\right| \leq 2 \quad (18)$$

$$\left|\frac{2}{\pi}\theta_{ab} - \frac{2}{\pi}\theta_{ac}\right| + \left|\frac{2}{\pi}\theta_{dc} + \frac{2}{\pi}\theta_{db} - 2\right| \leq 2 \quad (19)$$

$$|\theta_{ab} - \theta_{ac}| + |\theta_{dc} + \theta_{db} - \pi| \leq \pi \quad (20)$$

Setting $\theta_{\mathbf{a}} = \mathbf{0}$ and $\theta_{\mathbf{d}} = \frac{\pi}{\mathbf{4}}$:

$$|\theta_{ab} - \theta_{ac}| + \left|1 - \theta_{dc} - \frac{\pi}{4} - \theta_{db} - \frac{\pi}{4} - \pi\right| = |\theta_{ab} - \theta_{ac}| + \left|1 - \frac{3\pi}{2} - \theta_{dc} - \theta_{db}\right| \quad (21)$$

$$= |\theta_{ab} - \theta_{ac}| + \left|\frac{1}{2}(3\pi - 2) + \theta_{db} + \theta_{dc}\right| \quad (22)$$

θ_{ab}	θ_{ac}	θ_{db}	θ_{dc}	$\frac{3\pi-2}{2} + \theta_{db} + \theta_{dc}$	$\|\theta_{ab} - \theta_{ac}\| + \left\|\frac{3\pi-2}{2} + \theta_{db} + \theta_{dc}\right\|$		
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi-2}{2} + \frac{3\pi}{4} = \frac{9\pi}{4} - 1$	$\frac{\pi}{4} + \frac{9\pi}{4} - 1 = \frac{5\pi}{2} - 1$	$> \pi - 1$	> 2
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi-2}{2} - \frac{\pi}{4} = \frac{5\pi}{4} - 1$	$\frac{\pi}{4} + \frac{5\pi}{4} - 1 = \frac{3\pi}{2} - 1$	$> \pi - 1$	> 2

Table 2: HCSH Inequality Comparison

The data shown in Table 2 reveal that the (HCSH) inequality, equation (17), is not satisfied for some settings of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ and $\overrightarrow{d}$.

From equation (15), for θ_{bc}:

$$1 + \mathbb{E}\left(A\left(\overrightarrow{b}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) = \mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) - \mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{b}, \lambda\right)\right) \quad (23)$$

For θ_{bc}:

$$\theta_{bc} = \theta_{dc} - \theta_{db} \quad (24)$$

$$\frac{2}{\pi}\theta_{bc} = \frac{2}{\pi}\theta_{dc} - \frac{2}{\pi}\theta_{db} \quad (25)$$

$$\frac{2}{\pi}\theta_{bc} - 1 = \frac{2}{\pi}\theta_{dc} - 1 - \left(\frac{2}{\pi}\theta_{db} - 1\right) - 1 \quad (26)$$

$$1 + \left(\frac{2}{\pi}\theta_{bc} - 1\right) = \frac{2}{\pi}\theta_{dc} - 1 - \left(\frac{2}{\pi}\theta_{db} - 1\right) \quad (27)$$

which provides the equality:

$$1 + \mathbb{E}\left(A\left(\overrightarrow{b}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) = \mathbb{E}\left(A\left(\overrightarrow{d}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) - \mathbb{E}\left(A\left(\overrightarrow{d}, \lambda\right) B\left(\overrightarrow{b}, \lambda\right)\right) \quad (28)$$

Therefore:

$$\mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) - \mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B\left(\overrightarrow{b}, \lambda\right)\right) \quad (29)$$

$$= \mathbb{E}\left(A\left(\overrightarrow{d}, \lambda\right) B\left(\overrightarrow{c}, \lambda\right)\right) - \mathbb{E}\left(A\left(\overrightarrow{d}, \lambda\right) B\left(\overrightarrow{b}, \lambda\right)\right) \quad (30)$$

6 Conclusion

Substitution of equation (6) into the Bell 1964 and HCSH inequalities indicates that neither are valid for some combinations of detector settings. This conclusion appears to have been supported by experimental evidence [1].

The discrepancy may be due to [2, equation (15)] being the result of a mathematical transformation rather than a direct evolution of equalities and comparisons as indicated by equation (15) which shows $\mathbb{E}\left(A\left(\overrightarrow{b},\lambda\right)B\left(\overrightarrow{c},\lambda\right)\right)$ as a function of $\mathbb{E}\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right)$ and $\mathbb{E}\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{c},\lambda\right)\right)$.

7 Acknowledgements

The formulation of the Theorem in Appendix B was developed withe help of chatGPT and is the culmination of over a decade's collaboration with Gordon Watson.

Appendices

A Lemma

The uniform distribution $\rho(\lambda)$ for a unit three dimensional Euclidean vector λ over the surface of a unit sphere is given by:

$$\rho(\lambda) = \frac{1}{4\pi} \tag{31}$$

Proof

$$\begin{aligned}
\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\rho(\lambda)\,sin(\theta)\,d\theta d\phi &= \rho(\lambda)\int_0^{2\pi}\int_0^{\pi}sin(\theta)\,d\theta d\phi && (32)\\
&= \rho(\lambda)\int_0^{2\pi}\left[-cos(\theta)\right]_0^{\pi}d\phi && (33)\\
&= \rho(\lambda)\int_0^{2\pi}\left(-cos(\pi)-(-cos(0))\right)d\phi && (34)\\
&= \rho(\lambda)\int_0^{2\pi}\left(1-(-1)\right)d\phi && (35)\\
&= 2\rho(\lambda)\left[\phi\right]_0^{2\pi} && (36)\\
&= 4\pi\rho(\lambda) && (37)
\end{aligned}$$

$$\begin{aligned}
4\pi\rho(\lambda) &= 1 && (38)\\
\rho(\lambda) &= \frac{1}{4\pi} && (39)
\end{aligned}$$

QED

B Expected Value Theorem

For two functions:

$$A(\mathbf{a}, \lambda) = sgn(\mathbf{a} \cdot \lambda) \tag{40}$$

$$B(\mathbf{b}, \lambda) = -A(\mathbf{b}, \lambda) \tag{41}$$

$$= -sgn(\mathbf{b} \cdot (\lambda)) \tag{42}$$

The expectation value $\mathbb{E}\left(A\left(\overrightarrow{a}, \lambda\right) B(\mathbf{b}, \lambda)\right)$ of the product of the two functions $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ for λ uniformly distributed over the surface of a unit sphere is given by:

$$\mathbb{E}\left(A(\mathbf{a}, \lambda) B(\mathbf{a}, \lambda)\right) = \frac{2\theta}{\pi} - 1 \tag{43}$$

where:
a and **b** are unit vectors in the $x - z$ plane of Euclidean three-space $[x, y, z]$,
θ radians is the angle between the vectors **a** and **b**,
$\lambda \in S^2$ is uniformly distributed,
sgn is the signum function defined as:

$$sgn(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \tag{44}$$

B.1 Assumptions

$$sgn(X)sgn(Y) = sgn((X)(Y)) \tag{45}$$

B.2 Proof

$$\mathbb{E}\left(A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda)\right) = \int_{S^2} \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \, d\lambda \tag{46}$$

$$= \int_{S^2} \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \, d\lambda \tag{47}$$

$$= \int_{S^2} \rho(\lambda) \, sign\left((\mathbf{a} \cdot \lambda)(\mathbf{b} \cdot \lambda)\right) d\lambda \tag{48}$$

Using Appendix A, for a uniform distribution $\rho(\lambda)$ of the unit vector λ over the surface of a unit sphere:

$$\rho(\lambda) = \frac{1}{4\pi} \tag{49}$$

Therefore:

$$\mathbb{E}\left(A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda)\right) = -\frac{1}{4\pi} \int_{S^2} sign\left((\mathbf{a} \cdot \lambda)(\mathbf{b} \cdot \lambda)\right) d\lambda \tag{50}$$

Let θ represent the angle between **a** and **b**, therefore:

$$\cos\theta = \mathbf{a} \cdot \mathbf{b} \tag{51}$$

By rotational symmetry of the unit sphere in the circular plane spanned by the unit vectors **a** and **b**, the distribution of λ is uniform and depends only on its angle relative to a reference axis in that plane.

Transform the three-dimensional description into a two-dimensional description by defining a plane through the **a** and **b** vectors with the reference direction of a polar coordinate system being defined by the direction of the **a** vector as 0 radians.

The **b** vector direction is therefor θ radians with the projection of any λ vector being defined as $\lambda_P\,(\phi \in [0, 2\pi))$.

$$
\begin{aligned}
sgn\,(\mathbf{a}\cdot\lambda) &= sgn\,(\mathbf{a}\cdot\lambda_{\mathbf{P}}) = sgn\,(\cos\phi)) && (52)\\
sgn\,(\mathbf{b}\cdot(\lambda)) &= sgn\,(\mathbf{b}\cdot(\lambda_{\mathbf{P}})) = sgn\,(\cos(\phi-\theta)) && (53)
\end{aligned}
$$

The corresponding expectation value of $sgn\,(\mathbf{a}\cdot\lambda)\,sgn\,(\mathbf{b}\cdot(\lambda))$ over the plane is given by:

$$
\begin{aligned}
\mathbb{E}\,(sgn\,(\mathbf{a}\cdot\lambda)\,sgn\,(\mathbf{b}\cdot(\lambda))) &= \frac{1}{2\pi}\int_0^{2\pi} sgn\,(\mathbf{a}\cdot\lambda)\,sgn\,(\mathbf{b}\cdot(\lambda))\,d\phi && (54)\\
&= \frac{1}{2\pi}\int_0^{2\pi} sgn\,(\cos\phi)\,sgn\,(\cos(\phi-\theta))\,d\phi && (55)
\end{aligned}
$$

$$
sgn\,(\cos\phi)\,sgn\,(\cos(\phi-\theta)) = \begin{cases} 1 & sgn\,(\cos\phi) = sgn\,(\cos(\phi-\theta)) \\ -1 & sgn\,(\cos\phi) = -sgn\,(\cos(\phi-\theta)) \end{cases}
$$

Therefore the expectation value of λ being observed at angle ϕ is:

$$
\begin{aligned}
E(\theta)_\phi &= 1\times P(\text{same sign}) + (-1\times P(\text{opposite sign})) && (56)\\
&= (1 - P(\text{opposite sign})) - P(\text{opposite sign}) && (57)\\
&= 1 - 2P(\text{opposite sign}) && (58)
\end{aligned}
$$

The set of angles ϕ and $\phi-\theta$ that are of opposite sign is given by:

$$
\cos\phi\cdot\cos(\phi-\theta) < 0 \qquad (59)
$$

which happens when the two cosines have opposite signs:

- $\cos\phi = 0$ at $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$,

$\cos(\phi-\theta) = 0$ at $\frac{\pi}{2}+\theta, \frac{3\pi}{2}+\theta$.

The four points at $\frac{\pi}{2}, \frac{3\pi}{2}, \theta+\frac{\pi}{2}$ and $\theta+\frac{3\pi}{2}$ divide the circle into four regions:

* two disjoint wedge shaped intervals where the signs are the same:

$$
\begin{aligned}
\phi &\le \frac{\pi}{2} && (60)\\
\phi &\ge \frac{3\pi}{2}+\theta && (61)\\
\frac{\pi}{2}+\theta \ge \phi &\le \frac{3\pi}{2} && (62)
\end{aligned}
$$

* two disjoint wedge shaped intervals where the signs differ:

$$
\begin{aligned}
\frac{\pi}{2} < \phi &< \frac{\pi}{2}+\theta && (63)\\
\frac{3\pi}{2} > \phi &< \frac{3\pi}{2}+\theta && (64)
\end{aligned}
$$

* each disagreement interval is θ radians.

So the total sign disagreement measure is 2θ radians.

As ϕ is uniformly distributed on $[0, 2\pi]$, the probability of λ being observed with opposite signs at two detectors is:

$$P(\text{opposite sign}) = \frac{2\theta}{2\pi} = \frac{\theta}{\pi} \tag{65}$$

Therefore, using equation (58), the expectation value for λ being observed:

$$E(\theta) = 1 - 2P(\text{opposite sign}) \tag{66}$$

$$= 1 - \frac{2\theta}{\pi} \tag{67}$$

Therefore:

$$\frac{1}{4\pi}\int_{S^2} sgn(\mathbf{a}\cdot\lambda)sgn(\mathbf{b}\cdot\lambda)d\Omega = 1 - \frac{2\theta}{\pi} \tag{68}$$

Therefore, by the definition of expectation value:

$$\mathbb{E}\left(A\left(\overrightarrow{a},\lambda\right)B\left(\overrightarrow{b},\lambda\right)\right) = \mathbb{E}\left(sgn(\mathbf{a}\cdot\lambda)\left(-sgn(\mathbf{b}\cdot\lambda)\right)\right) \tag{69}$$

$$= \int_{S^2}\rho\left(\lambda\right)sgn(\mathbf{a}\cdot\lambda)\left(-sgn(\mathbf{b}\cdot\lambda)\right)d\Omega \tag{70}$$

$$= -\frac{1}{4\pi}\int_{S^2} sgn(\mathbf{a}\cdot\lambda)sgn(\mathbf{b}\cdot\lambda)d\Omega \tag{71}$$

$$= \frac{2\theta}{\pi} - 1 \tag{72}$$

QED

References

[1] Testing bell's inequalities. In *Europhysics News*, volume 22, pages 73, 75. European Physical Society, 1991.

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