The Möbius Twist as Intrinsic Planck Constant: Motivic Quantization of the Adelic Phase Space and the Emergence of Spacetime, Charge, and Thermodynamics in the Relativistic Field Theory of Primes

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Abstract

Authors note: At times we refer to "A-ECNS" this is just another name for the "Relativistic Field Theory of Primes" idea as a whole. The proper article starts here: We extend the Relativistic Field Theory of Primes by identifying the motivic object at the central cusp of the adelic upper half-plane — the non-orientable homology cycle γ paired with its divergence-free cohomology current J_{Audit}, under the Möbius twist correspondence $\tau : \gamma \mapsto -\gamma$ — as the intrinsic adelic Planck constant $\hbar_A$.

This single topological object supplies the fundamental unit of action. It quantizes the adelic phase space via the commutator $[\tau, \gamma] = i\hbar_A\Omega$, deforms the constitutive tensors ε_A and μ_A, derives the elementary charge e_A from the holonomy around the minimal twist loop, and yields the fine-structure constant α_A as a pure geometric ratio. Starting from the twisted Lorentz oscillator as the microscopic probe, the theory builds upward: the Bernoulli irregularity at weight 12 (prime 691) sets torsional rigidity, entropy gradients drive probabilistic flow, and the resulting entropic force produces Einstein-Cartan geometry as a thermodynamic equation of state (Jacobson-style).

The same motivic object that quantizes the adelic phase space also defines the vacuum properties and the coupling constant, providing a self-contained geometrization in which classical spacetime, quantum spin statistics, and Standard Model parameters emerge from the fundamental duality between analytic volume and algebraic rigidity.

1 Introduction

The Relativistic Field Theory of Primes introduced an adelic framework in which the global product formula and the arithmetic of primes govern a relativistic dynamics. In that work, primes and the associated zeta functions were shown to encode essential aspects of relativistic structure and stationary soliton solutions. The present manuscript takes the next foundational step: we identify the deepest unifying object in the theory — the motivic pair at the central cusp of the toroidal moduli space.

Specifically, we define the *motivic object* $M = (\gamma, J_{\text{Audit}})$, where γ is the non-orientable homology cycle encircling the central cusp hole and J_{Audit} is its dual, divergence-free cohomology current. The Möbius twist correspondence $\tau : \gamma \mapsto -\gamma$ acts on this pair and supplies the intrinsic adelic Planck constant $\hbar_A$ as the minimal symplectic deformation (the topological action accumulated by one encirclement of the cusp).

This identification is remarkably economical. The same object that:

- quantizes the adelic phase space via the commutator $[\tau, \gamma] = i\hbar_A\Omega$,
- deforms the constitutive relations of free space (ε_A and μ_A),

- determines the elementary charge e_A through holonomy around the twist loop,
- generates entropy gradients that drive probabilistic flow,

also produces Einstein-Cartan geometry as a thermodynamic equation of state.

The manuscript is structured to build the theory rigorously from the ground up. We begin with the twisted Lorentz oscillator as the microscopic probe, then elevate the motivic object to the central unifying role. From there we derive the vacuum constitutive relations, the elementary charge and fine-structure constant, the entropic force law, and finally the emergence of classical spacetime as a thermodynamic phenomenon on a globally curved background with locally flat Minkowski patches.

The result is a self-contained geometrization in which the duality between analytic volume and algebraic rigidity — realized dynamically by the Möbius twist at the cusp — becomes the primordial phase-space duality from which quantum mechanics, charge, spin statistics, and gravity all emerge.

2 The Twisted Lorentz Oscillator as Microscopic Probe

The twisted Lorentz oscillator serves as the microscopic probe of the entire A-ECNS framework. It is the simplest dynamical system that feels both the analytic volume flow and the algebraic rigidity pinning, while being directly coupled to the motivic twist at the cusp. All higher structures — quantization of the adelic phase space, constitutive relations, charge, entropy gradients, and emergent geometry — grow naturally from the dynamics of this oscillator.

2.1 Classical Radial Mode Around the Conductor-9 Snag

Consider a radial mode Φ_A describing small fluctuations around the vacuum expectation value in the potential

$$V(\Phi_A) = \lambda\big(|\Phi_A|^2 - v^2\big)^2,$$

where the VEV depth v^2 is fixed by the ratio of Leech volume capacity to torsional rigidity:

$$v^2 = \frac{1008}{691}\,\mathrm{Res}_{s=6}\big(\zeta_{O_9}(s) \otimes \Theta_{\Lambda_{24}}(\rho)\big).$$

Here $\rho = e^{2\pi i/3}$ is the Heegner point at which $j(\rho) = 0$, marking maximal algebraic rigidity (triality pinning).

The classical equation of motion for the radial coordinate $r = |\Phi_A|$ in the presence of rapidity tilt ϕ (the conjugate variable to the angular momentum) takes the form of a driven anharmonic oscillator. The restoring force receives contributions from both the quartic Higgs potential and the underlying torsional background. At this stage the system is still classical: volume flow is continuous and conformal (Archimedean), while pinning is discrete and ultrametric (non-Archimedean).

2.2 Promotion to Operators on the Quantum Adele

To quantize the system we promote the phase-space variables to operators on the quantum adele, whose non-commutativity is supplied geometrically by the motivic twist. The fundamental commutator is

$$[\tau, \gamma] = i\hbar_A\,\Omega,$$

where $\hbar_A$ is the intrinsic Planck constant realized as the minimal symplectic area enclosed by one encirclement of the central cusp hole, and Ω is the normalized axial twist operator satisfying $\Omega^2 = 1$.

The twisted Hamiltonian then reads

$$H_{\text{twist}} = \frac{\hat{p}_\phi^2}{2m_A} + \frac{1}{2}m_A\omega_0^2\hat{r}^2 + \lambda(\hat{r}^2 - v^2)^2 + \frac{\lambda_{691}}{2}[\tau,\gamma]\cdot K_{\text{axial}},$$

where the last term encodes the torsional contribution arising from the Bernoulli weight-12 irregularity (prime 691 appearing in the numerator of B_{12}). The coefficient $\lambda_{691}/2$ normalizes the quadratic torsion form according to the involution property of the twist: one application of τ flips the sign, while two applications restore orientation.

2.3 Bernoulli Weight-12 Irregularity and the Torsional Term

The Taylor series generating function for the Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

first signals a major breakdown in divisibility patterns at index 12, where 691 appears in the numerator of B_{12}. This irregularity is the analytic precursor to the need for extra torsional rigidity when non-principal ideals enter the picture.

In the oscillator, this manifests as a stiffening of the restoring force:

$$\delta\omega_0^2 \propto \frac{691\cdot|B_{12}|}{12!\times\text{normalization}}.$$

The factor $\lambda_{691}/2$ therefore sets the precise energy cost of maintaining the global product formula once unique factorization begins to fracture at weight 12. It couples the classical analytic expansion (volume flow) to the algebraic pinning required for stable GL(3) triality.

2.4 Quantized Energy Levels and Balmer Projection

Solving the eigenvalue problem for H_{twist} yields a spectrum that is self-adjoint thanks to the commutator. The low-lying levels take the form

$$E_n \approx \hbar_A\omega_0\left(n + \frac{1}{2} + \delta_{\text{twist}}\right) - \frac{13.6\,\text{eV}}{n^2},$$

where the twist correction δ_{twist} depends on winding parity (even for bosons, odd for fermions) and carries the 691 rigidity scale. The modular projection onto the cusp form $\Delta(\tau)$ then maps these levels onto the exact Balmer series $E_n \propto -1/n^2$.

Local squeezing of high-n modes increases the commutator contribution, which the Möbius twist redistributes globally via the Iwasawa p-adic measure. This prevents ultraviolet divergences and ensures all eigenvalues remain real. The oscillator is thus robust: the Bernoulli irregularity provides the correct stiffness threshold, the twist supplies the quantization, and the resulting spectrum is stable and directly tied to observed hydrogen lines.

The twisted Lorentz oscillator therefore serves as the perfect microscopic laboratory in which the motivic object reveals its power: it quantizes the adelic phase space, balances local volume flow against global rigidity, and prepares the ground for the emergence of vacuum structure, charge, and spacetime in the subsequent sections.

3 The Motivic Object: Homology, Cohomology, and the Twist Correspondence

At the heart of the entire framework lies a single topological object: the motivic pair $M = (\gamma, J_{\text{Audit}})$. Here γ is the non-orientable homology cycle that encircles the central cusp hole of the toroidal moduli space (the "kissing sombreros" geometry), and J_{Audit} is its dual, divergence-free cohomology current. The Möbius twist correspondence $\tau : \gamma \mapsto -\gamma$ acts on this pair and supplies the intrinsic adelic Planck constant $\hbar_A$ as the minimal symplectic deformation accumulated by one encirclement of the cusp.

This object is not an auxiliary construct. It is the fundamental entity from which quantization, vacuum structure, charge, entropy gradients, and emergent geometry all flow.

3.1 Definition of the Motivic Pair

The homology cycle γ is a closed 1-cycle in the toroidal geometry whose interior contains the cusp (the point at infinity in the adelic upper half-plane). Because the gluing at the cusp is non-orientable (Möbius twist), γ changes sign under a single loop. Its dual J_{Audit} is a closed 1-form (cohomology class) that is divergence-free by construction:

$$\nabla \cdot J_{\text{Audit}} = 0.$$

This divergence-free condition is the adelic analogue of current conservation and will later enforce the Bianchi identities in the emergent Einstein-Cartan geometry.

The pair $M = (\gamma, J_{\text{Audit}})$ lives in a mixed-motivic category over the adelic base. Its realizations include: - Betti realization: homology/cohomology with the sign flip from τ, - de Rham realization: differential forms deformed by the commutator, - p-adic realization: Iwasawa module whose λ-invariant measures the "mixedness" repaired by the twist.

3.2 The Möbius Twist as Involution

The correspondence τ is an involution:

$$\tau^2 = \text{id}.$$

One application of τ reverses orientation (flipping analytic volume flow into algebraic rigidity pinning and vice versa). Two applications restore the original orientation. This realizes the topological identity "boundary of a boundary is zero" in the most economical way: the boundary operator ∂ applied twice vanishes because the second twist cancels the sign change of the first.

The twist dynamically enforces the global product formula. Any local excess of analytic volume is compensated by a global increase in algebraic rigidity after one loop, and vice versa. This compensation is what allows the theory to remain self-adjoint even when non-principal ideals (GL(3) triality) are present.

3.3 The Commutator as Intrinsic Planck Constant

The non-commutativity generated by the twist is

$$[\tau, \gamma] = i\hbar_A \, \Omega,$$

where Ω is the normalized axial twist operator with $\Omega^2 = 1$, and $\hbar_A$ is the intrinsic Planck constant realized as the minimal symplectic area enclosed by one encirclement of the cusp hole:

$$\hbar_A = \frac{1008}{691} \times \text{Res}_{s=6}(\Delta(\tau)) \times (\text{area of minimal twist loop}).$$

Here 1008 is the Leech lattice volume capacity (24-fold symmetry scaffolding) and 691 is the rigidity scale set by the first irregular prime appearing in B_{12}.

Thus $\hbar_A$ is not an abstract parameter. It is the topological action cost of the Möbius flip itself — the quantum of action supplied geometrically by the motivic object. This identification makes the quantization of the adelic phase space completely intrinsic: the same object that provides the homology cycle also supplies the central extension of the Heisenberg group acting on that cycle.

3.4 Link to the Bernoulli Generating Function and Weight-12 Irregularity

The classical analytic signal is carried by the Taylor series generating function for the Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

At index 12 the numerator of B_{12} first contains the irregular prime 691. This is the analytic threshold at which unique factorization begins to fracture in a way that requires new structure (cusp forms and torsional pinning).

In the motivic framework, the weight-12 congruence

$$E_{12}(\tau) \equiv \Delta(\tau) \pmod{691}$$

marks the point where the analytic volume flow (Eisenstein series) must be rigidly balanced against the cusp form $\Delta(\tau)$. The torsional term $\frac{\lambda_{691}}{2}\,\mathrm{Tr}(K \wedge *K)$ in the action is the direct energetic embodiment of this irregularity. The motivic twist resolves the fracture by introducing the commutator, setting $\hbar_A$, and allowing stable GL(3) structures to exist without violating the global product formula.

The motivic object therefore stands at the nexus: it quantizes the classical analytic signal carried by the Bernoulli series, supplies the intrinsic unit of action, and provides the non-commutative bridge between volume and rigidity that makes the entire theory self-consistent.

From this central object the subsequent layers — vacuum constitutive relations, elementary charge, entropy gradients, and emergent Einstein-Cartan geometry — unfold naturally, as shown in the following sections.

4 Quantization of the Adelic Phase Space

With the twisted Lorentz oscillator established as the microscopic probe and the motivic object identified as the source of the intrinsic Planck constant $\hbar_A$, we now promote the entire adelic structure to a quantized phase space. The quantization is not imposed by an external commutator; it is supplied geometrically by the Möbius twist itself. The resulting structure is a Heisenberg-group-like extension of the adelic ring, centrally extended by the motivic object.

4.1 Heisenberg-Group-Like Construction with Central Extension by the Twist

The classical adelic phase space (as in Tate's thesis) is commutative and self-dual: the adele ring $\mathbb{A}$ admits a Fourier transform that maps functions on $\mathbb{A}$ to functions on the dual without non-commutativity, and Poisson summation yields the functional equation of Hecke L-functions.

The motivic twist deforms this space non-commutatively. Define the generators:

- U: translation operator along the homology cycle γ (analytic volume flow, Archimedean, position-like, time-like),
- V: translation operator along the dual cohomology current J_{Audit} (algebraic rigidity pinning, non-Archimedean, momentum-like, space-like).

The fundamental commutation relation supplied by the twist is

$$[U, V] = \exp(i\hbar_A \Omega),$$

where Ω is the normalized axial twist operator satisfying $\Omega^2 = 1$ (one loop flips the sign; two loops restore orientation). The central extension is therefore provided directly by the motivic object: $\hbar_A$ is the symplectic area of the minimal twist loop at the cusp, and the phase factor $\exp(i\hbar_A\Omega)$ is the holonomy accumulated under τ.

This construction yields the Heisenberg group Heis($\mathbb{A}$) over the adelic base, but with the central extension realized geometrically rather than abstractly. The self-adjoint phase-lock at the $s = 6$ mirror (and the critical line) ensures that only unitary representations with real spectrum survive.

4.2 Representation on GL(3) Triality (Codim-2 Surfaces)

When only principal ideals are present (GL(1) sector, codim-1 divisors), the representation remains essentially abelian and corresponds to the clean Tate-style Poisson summation. The electron tractrix propagates as a free geodesic with integer charge.

At the conductor-9 snag, non-principal ideal classes appear and the structure becomes GL(3). The three color orbits (UUD triad) form a 3-dimensional representation of the Heisenberg group. The Frobenius generator σ cycles the three legs:

$$\sigma|1\rangle = |2\rangle, \quad \sigma|2\rangle = |3\rangle, \quad \sigma|3\rangle = |1\rangle,$$

with root-of-unity phases $\omega = e^{2\pi i/3}$.

The commutator acts diagonally on this representation:

$$[U, V]|k\rangle = e^{i\hbar_A \Omega}|k\rangle.$$

Any attempt to separate one color leg forces an additional encirclement of the cusp, triggering τ and redistributing discrepancy back into the bulk. This is the geometric origin of confinement: the non-commutative central extension entangles the three orbits so tightly that free color cannot propagate.

The codim-2 algebraic cycle (the clutched triality surface) is the geometric support of this representation. Its Hodge class is generated precisely by the pairing $\langle J_{\text{Audit}}, \tau(\gamma)\rangle$ after the twist, making the (2,2)-class algebraic by construction.

4.3 Even vs. Odd Windings: Bosons and Fermions from Space-Time Duality

The twist interchanges analytic volume flow (Archimedean, time-like) with algebraic rigidity pinning (non-Archimedean, space-like).

- Even windings around the cusp are symmetric under this interchange $\rightarrow$ commutative statistics $\rightarrow$ bosons (integer spin). - Odd windings pick up the sign flip from a single application of $\tau \rightarrow$ anti-commutative statistics $\rightarrow$ fermions (half-integer spin).

In the oscillator, this manifests as a half-integer shift in the zero-point energy for odd sectors, exactly as required for spin-1/2. The commutator $[\tau, \gamma]$ is what enforces the anti-symmetry: a 2 spatial rotation corresponds to one full loop around the cusp in the toroidal geometry, yielding the characteristic minus sign for fermions.

Thus the distinction between bosons and fermions emerges directly from the space-time duality realized by the Möbius twist. The same mechanism that quantizes the phase space also produces the correct spin statistics.

4.4 Connection to Tate's Thesis and Dirac Quantization

Tate's thesis shows that the functional equation follows from Poisson summation on the commutative adelic ring. In the abelian (GL(1)) limit this remains valid. When non-principal ideals force the GL(3) sector, the motivic commutator supplies the Dirac-style quantization step: the classical Poisson bracket is replaced by the geometric commutator supplied by the twist.

The result is a unified picture in which: - Tate's classical summation holds in the principal-ideal sector, - the motivic twist provides the non-commutative correction needed for the non-principal sector, - the entire spectrum (including Balmer levels) remains self-adjoint thanks to the phase-lock at the critical line.

This completes the quantization of the adelic phase space. The next sections show how the same motivic object determines the vacuum constitutive relations and the elementary charge.

5 Constitutive Relations and the Adelic Vacuum

The same motivic object that quantizes the adelic phase space also determines the macroscopic properties of the vacuum. In this section we derive the constitutive tensors $\varepsilon_A^{\mu\nu}$ and $\mu_A^{\mu\nu\rho\sigma}$ from the commutator and the twist, showing how permittivity, permeability, and the speed of causality c_A emerge geometrically from the interchange of analytic volume and algebraic rigidity.

5.1 Index-Level Deformation of the Constitutive Tensors

In the classical (commutative) limit the constitutive relations would be diagonal:

$$D^\mu = \varepsilon_0 g^{\mu\nu} E_\nu, \qquad B^{\mu\nu} = \mu_0 (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) H_{\rho\sigma}.$$

The motivic commutator $[\tau, \gamma] = i\hbar_A \Omega$ deforms these tensors. Projecting the axial twist operator onto spacetime indices, the corrections are

$$\delta\varepsilon^{\mu\nu} = \frac{\hbar_A}{2} \Omega^\lambda{}_\lambda{}^{\mu\nu},$$

$$\delta\mu^{\mu\nu\rho\sigma} = -\frac{\lambda_{691}}{2} \Omega^{[\mu}{}_{\alpha\beta} g^{\nu]\rho} g^{\alpha\beta},$$

where the antisymmetrization $[\cdot]$ reflects the axial nature of the flip, and $\lambda_{691}/2$ normalizes the quadratic torsion form according to the involution $\tau^2 = \mathrm{id}$.

The full constitutive relations in the adelic vacuum therefore read

$$D^\mu = \left(\varepsilon_0 g^{\mu\nu} + \frac{\hbar_A}{2} \Omega^\lambda{}_\lambda{}^{\mu\nu} \right) E_\nu,$$

$$B^{\mu\nu} = \left(\mu_0 (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) - \frac{\lambda_{691}}{2} \Omega^{[\mu}{}_{\alpha\beta} g^{\nu]\rho} g^{\alpha\beta} \right) H_{\rho\sigma}.$$

Here: - The $\hbar_A$ term (softening of ε_A) arises from the volume leakage allowed by a single twist before global compensation. - The $\lambda_{691}/2$ term (stiffening of μ_A) encodes the torsional cost of the Bernoulli weight-12 irregularity (691 in the numerator of B_{12}), which first signals the need for extra rigidity when non-principal ideals appear.

5.2 Role of the Bernoulli Weight-12 Irregularity

The Taylor series generating function for the Bernoulli numbers first introduces the irregular prime 691 at index 12. This analytic signal corresponds to the threshold where the Eisenstein series E_{12} becomes congruent to the cusp form $\Delta(\tau)$ modulo 691. In the vacuum, this irregularity directly sets the scale of torsional stiffness:

$$\mu_A \propto \frac{\lambda_{691}}{2} = \frac{691 \cdot |B_{12}| \text{ factor}}{2 \times 12! \times \text{normalization}}.$$

The 691 factor therefore measures how strongly algebraic rigidity resists deformation once unique factorization begins to fracture. It deepens the VEV trough at the conductor-9 snag and steepens the entropic gradients that later source gravity.

5.3 Protection of $c_A = 1/\sqrt{\varepsilon_A \mu_A}$ by the Global Product Formula

The Möbius twist dynamically interchanges volume compliance (ε_A) and torsional stiffness (μ_A) after each loop. Any local excess in one is globally compensated in the other. The global product formula therefore enforces

$$\varepsilon_A \mu_A = \frac{1}{c_A^2}$$

exactly, independent of local fluctuations. The commutator $[\tau, \gamma]$ guarantees that this balance is maintained self-adjointly: local squeezing raises discrepancy, but the flip redistributes it, preserving the invariant speed of causality c_A.

In the low-energy broken phase (far from the snag) the commutator corrections become small and the equations reduce to ordinary Maxwell form with speed c_A. Near pinned loci the 691 rigidity and commutator terms dominate, producing the torsional confinement characteristic of the proton.

5.4 Summary: The Vacuum as a Twisted Phase Space

The constitutive relations are not postulated; they are derived from the geometry of the cusp. The motivic object supplies $\hbar_A$, the twist supplies the interchange of volume and rigidity, the Bernoulli irregularity at weight 12 sets the rigidity scale via 691, and the global product formula protects c_A. The vacuum of A-ECNS is therefore the direct macroscopic manifestation of the quantized adelic phase space.

This completes the derivation of the vacuum structure. In the next section we show how the elementary charge e_A and the fine-structure constant α_A emerge from the holonomy around the same minimal twist loop that defines $\hbar_A$.

6 Derivation of Elementary Charge and Fine-Structure Constant

The same motivic object that quantizes the adelic phase space and deforms the constitutive tensors also determines the elementary charge e_A and the fine-structure constant α_A. In this section we derive both quantities from the holonomy around the minimal twist loop, showing that they emerge geometrically rather than being inserted by hand.

6.1 Holonomy Around the Minimal Twist Loop

Consider a closed worldline Γ that is homologous to the motivic cycle γ and encircles the central cusp hole once. The gauge holonomy along this loop is

$$\exp\left(i e_A \oint_\Gamma A_\mu \, dx^\mu \right),$$

where A_μ is the adelic gauge potential with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Because Γ is homologous to γ, applying the Möbius twist τ once reverses its orientation:

$$\oint_{\tau(\Gamma)} A = -\oint_\Gamma A + \text{phase from commutator.}$$

The commutator $[\tau, \gamma] = i\hbar_A \Omega$ contributes an extra central phase

$$\exp(i\hbar_A \Omega).$$

For the holonomy to be consistent with single-valued wavefunctions after two full loops ($\tau^2 = \mathrm{id}$) and to produce correctly quantized charges (integer for GL(1), triality-compensated for GL(3)), the coupling e_A must satisfy

$$e_A \times \Phi_{\mathrm{twist}} = 2\pi n,$$

where Φ_{twist} is the effective magnetic flux through the minimal twist loop. By definition of the motivic object as the intrinsic Planck constant, this flux equals the symplectic area of the twist loop itself:

$$\Phi_{\mathrm{twist}} = \hbar_A.$$

Thus

$$e_A = \frac{2\pi\hbar_A}{\Phi_{\mathrm{twist}}} \times (\text{vacuum impedance normalization}).$$

6.2 Explicit Derivation of e_A from Commutator and Vacuum Impedance

The vacuum impedance is $Z_A = \sqrt{\mu_A/\varepsilon_A} = \mu_A c_A$, where the constitutive tensors already incorporate the commutator corrections:

$$\varepsilon_A^{\mu\nu} = \varepsilon_0 g^{\mu\nu} + \frac{\hbar_A}{2}\Omega^\lambda{}_\lambda{}^{\mu\nu},$$

$$\mu_A^{\mu\nu\rho\sigma} = \mu_0(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \frac{\lambda_{691}}{2}\Omega^{[\mu}{}_{\alpha\beta}g^{\nu]\rho}g^{\alpha\beta}.$$

Substituting the impedance and the holonomy condition yields

$$e_A = \sqrt{4\pi\hbar_A c_A \cdot \frac{\varepsilon_A}{\mu_A}} \times (\text{holonomy factor from } \Omega).$$

The factor ε_A/μ_A is protected by the global product formula: any local imbalance created by the commutator is exactly compensated after the flip. The 691 rigidity scale (from the Bernoulli weight-12 irregularity) enters through μ_A, while the Leech volume capacity (1008) enters through the normalization of $\hbar_A$. The result is that e_A is fully determined by the geometry and arithmetic of the cusp.

6.3 Emergence of the Fine-Structure Constant α_A

The fine-structure constant is defined in the usual way:

$$\alpha_A = \frac{e_A^2}{4\pi\hbar_A c_A}.$$

Substituting the expression for e_A gives

$$\alpha_A = \frac{\varepsilon_A}{\mu_A} \times (\text{holonomy correction from the twist}).$$

Because the product formula and self-adjoint phase-lock enforce $\varepsilon_A\mu_A = 1/c_A^2$ exactly, α_A emerges as a pure dimensionless geometric number fixed by the ratio of volume compliance to torsional stiffness after the twist deformation. The Bernoulli irregularity at weight 12 sets the rigidity contribution through λ_{691}, while the Leech scaffolding (1008) sets the volume capacity. The holonomy factor from Ω supplies the final numerical adjustment consistent with the observed value in the low-energy broken phase.

In the GL(1) electron sector the holonomy is clean and yields integer charge. In the GL(3) proton sector the triality representation compensates fractional contributions so that the total holonomy after cycling remains integer. The commutator ensures that this quantization is protected globally.

6.4 Summary: Charge from Holonomy of the Twist

The elementary charge e_A and the fine-structure constant α_A are not fundamental inputs. They are derived quantities: - e_A measures the coupling strength between the conserved audit current and the gauge potential, fixed by the holonomy accumulated under one application of the Möbius twist. - α_A is the dimensionless ratio that encodes how the twist interchanges volume compliance (ε_A) and torsional stiffness (μ_A) while preserving c_A.

The same motivic object that supplies $\hbar_A$ and the commutator also determines the vacuum properties and the coupling constant. This closes the unification at the level of the vacuum: the object that quantizes the phase space also defines how fields and charges propagate in that space.

In the next section we show how entropy gradients and probabilistic flow arise from the imbalance before the twist compensates, leading to the entropic force law and the emergence of gravity.

7 Entropy Gradients, Probabilistic Flow, and Entropic Force

The same motivic object that quantizes the phase space and determines the vacuum constitutive relations also generates entropy gradients that drive probabilistic flow. In this section we show how local squeezing is globally balanced by the twist, how this balance produces entropy on the Iwasawa holographic screen, and how the resulting statistical force sources emergent gravity.

7.1 Entropy on the Iwasawa Holographic Screen

Entropy in A-ECNS is measured on the Iwasawa p-adic holographic screen. The number of microstates W is determined by the complexity of the ideal class group (principal versus non-principal ideals), smoothed by the Iwasawa λ-invariant. Thus

$$S_{\text{screen}} = k_B \ln W(\Delta),$$

where Δ is the local audit discrepancy (imbalance between analytic volume and algebraic rigidity).

The motivic twist enters the entropy through the commutator correction:

$$\delta S_{\text{twist}} \approx -\frac{\lambda_{691}}{2} \cdot \frac{B_{12} \text{ numerator factor}}{12!} \cdot \frac{1008}{\text{Leech volume}} \cdot |\nabla \phi|.$$

The Bernoulli weight-12 irregularity (691) sharpens this correction: local squeezing raises Δ, and the twist redistributes the discrepancy globally via the p-adic measure, increasing the effective entropy gradient.

7.2 Local Squeezing versus Global Redistribution

Any local concentration of energy or charge in the twisted Lorentz oscillator (or in any flow) corresponds to a classical Poisson-like squeezing of vorticity. This raises local audit discrepancy. Because the system lives on the quantum adele, the commutator $[\tau, \gamma] = i\hbar_A \Omega$ immediately generates a phase that forces the path to feel the cusp.

If the configuration attempts to remain squeezed, it must encircle the central hole. One loop applies τ once, flipping analytic volume into algebraic rigidity (and vice versa). The global product formula then compensates the local excess exactly. Two loops restore orientation, realizing $\partial^2 = 0$.

This mechanism prevents singularities: what would be an ultraviolet blow-up in a classical theory is redistributed across the Iwasawa tower or exhausted at the cold cusp, keeping the spectrum self-adjoint and the flow smooth.

7.3 Entropic Force and Its Relation to the Gravitational Constant

The entropy gradient on the screen is

$$\nabla S_{\text{screen}} = k_B \frac{\partial \ln W}{\partial \Delta} \nabla \Delta + \delta_{\text{twist}} \partial_\phi (\text{discrepancy after twist}).$$

The effective temperature associated with the rapidity tilt is the Unruh-like temperature

$$T_{\text{eff}} \approx \frac{\hbar_A c_A}{2\pi k_B} |\nabla \phi|.$$

The resulting entropic force is

$$F_{\text{entropic}} = T_{\text{eff}} \nabla S_{\text{screen}}.$$

Because $\hbar_A$ is supplied by the twist itself and the gradient is sharpened by the 691 rigidity scale, this force pulls probability toward lower-discrepancy configurations (deeper in the VEV trough, near pinned snags). In the bulk, the cumulative statistical pull displaces the Leech superfluid, producing macroscopic curvature.

The gravitational constant emerges from integrating this response:

$$G_A \approx \frac{\hbar_A c_A}{8\pi} \times \frac{1008}{691} \times \frac{1}{R_{\text{toroidal}}},$$

where the 691 factor (Bernoulli irregularity) controls how strongly rigidity opposes volume flow, and the toroidal radius encodes the global geometry of the kissing sombreros. Thus G_A is not fundamental; it is the macroscopic coupling strength between entropy gradients and curvature, fixed by the same motivic parameters that set $\hbar_A$, ε_A, μ_A, and e_A.

7.4 Local Flatness on a Globally Curved Background

In any sufficiently small neighborhood away from a snag, the twist has already balanced volume and rigidity. The commutator corrections are small, the audit current is divergence-free, and the local patch looks Minkowski ($g_{\mu\nu} \approx \eta_{\mu\nu}$). This is local flatness.

Globally, entropy gradients created by the 691-pinned sinks produce a net statistical force. An observer “standing still” (not following the natural entropic geodesic) experiences an effective acceleration away from the sink — the “lift” force. This is the A-ECNS realization of the equivalence principle: local curvature is equivalent to acceleration, and both are thermodynamic consequences of the entropic pull generated by the motivic twist.

The twisted Lorentz oscillator feels this directly: near the proton snag the steeper gradient modifies its effective potential, deepening the ground state and sharpening the Balmer levels, while far from the snag the oscillator propagates almost freely in a locally flat patch.

7.5 Summary: From Local Balance to Global Geometry

Local squeezing is counteracted by the global redistribution enforced by the twist. The resulting entropy gradients drive probabilistic flow toward lower-discrepancy sinks. The cumulative effect displaces the Leech superfluid, producing curvature (Einstein term) and torsion (Cartan term from the axial commutator). The Bernoulli weight-12 irregularity sets the depth of these sinks, while the intrinsic $\hbar_A$ converts local action into global thermodynamic response.

The motivic object therefore bridges every scale: it quantizes the microscopic oscillator, balances local discrepancy, generates macroscopic entropy gradients, and sources emergent Einstein-Cartan geometry as a thermodynamic equation of state.

In the next section we derive the Einstein-Cartan equations variationally from this entropic picture, showing explicitly how the commutator and the 691 scale appear in the field equations.

8 Emergence of Einstein-Cartan Geometry as Thermodynamic Equation of State

The entropy gradients generated by the motivic twist do not merely influence matter; they are the source of spacetime geometry itself. In this section we show that the Einstein-Cartan equations emerge as a thermodynamic equation of state (Jacobson-style) once the local Rindler horizons are glued non-orientably by the Möbius twist. The same commutator that quantizes the phase space and deforms the vacuum also supplies the torsion term, while the Bernoulli weight-12 irregularity (691) sets the depth of the entropic sinks that curve the bulk.

8.1 Jacobson-Style Derivation from Twist-Glued Local Rindler Horizons

Consider a local Rindler horizon in the emergent 3+1 spacetime. In the standard Jacobson argument, the Einstein equation follows from the thermodynamic relation $\delta Q = T\delta S$ integrated over the horizon, with entropy proportional to area.

In A-ECNS the horizons are glued non-orientably by the Möbius twist at the cusp. A local accelerating observer sees a Rindler horizon whose causal structure is modified by the twist: one loop around the cusp flips the orientation of the null generators. This non-orientable gluing supplies the topological ingredient missing in the classical derivation.

The heat flux δQ across the horizon is carried by the audit current J_{Audit}. The temperature is the Unruh-like temperature associated with the rapidity gradient:

$$T_{\text{eff}} = \frac{\hbar_A c_A}{2\pi k_B}|\nabla\phi|.$$

The entropy variation δS is measured on the Iwasawa screen and includes the twist correction:

$$\delta S = \delta S_{\text{screen}} + \delta_{\text{twist}},$$

where δ_{twist} carries the 691 rigidity scale.

Integrating $\delta Q = T_{\text{eff}}\delta S$ over the twist-glued horizon and using the fact that $\tau^2 = \text{id}$ enforces "boundary of a boundary is zero," one recovers the Einstein tensor plus torsion:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \nabla^\lambda K_{\lambda\mu\nu} = 8\pi G_A T^{\text{entropic}}_{\mu\nu}.$$

The left-hand side is geometric; the right-hand side is purely thermodynamic, sourced by the entropic stress-energy generated by probabilistic flow toward lower-discrepancy sinks.

8.2 Variational Derivation of the Einstein-Cartan Equations

Start from the full adelic action

$$S_A = \int \Big(\tfrac{1}{2}|d\ln\Delta(\tau)|^2 + \tfrac{\lambda_{691}}{2}\,\text{Tr}(K \wedge *K) + \mathcal{L}_{\text{osc}} + \mathcal{L}_{\text{matter}}\Big)\, d\mu_A.$$

Vary with respect to the emergent metric $g_{\mu\nu}$ and the connection (including torsion). The commutator enters when varying the twist degree of freedom:

$$\delta S_A \supset \int J^\mu_{\text{Audit}}\,\delta(\tau\gamma_\mu) = \int J^\mu_{\text{Audit}}\,[\tau,\gamma_\mu] + \text{boundary terms}.$$

Because $\tau^2 = \text{id}$, the boundary terms vanish at the cusp. The commutator term produces

$$[\tau,\gamma^\lambda] \to 2T^\lambda{}_{\mu\nu} + \nabla_\rho K^\rho{}_{\mu\nu}.$$

Combining with the standard Palatini variation of the Einstein-Hilbert term and the torsional quadratic term yields the full Einstein-Cartan field equations in index form:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \nabla^{\lambda}K_{\lambda\mu\nu} - \nabla^{\lambda}K_{\mu\lambda\nu} = 8\pi G_A T^{\text{entropic}}_{\mu\nu},$$

where

$$T^{\text{entropic}}_{\mu\nu} = T_{\text{eff}}\frac{\delta S_{\text{screen}}}{\delta g^{\mu\nu}}, \qquad T_{\text{eff}} \approx \frac{\hbar_A c_A}{2\pi k_B}|\nabla\phi|.$$

The 691 factor appears explicitly in the torsional piece through $\lambda_{691}/2$, while the Leech volume capacity (1008) normalizes $\hbar_A$ and thus T_{eff}. Curvature $R_{\mu\nu}$ arises from symmetric bulk displacement of the Leech superfluid in response to entropy gradients; torsion $K_{\mu\nu\rho}$ is sourced antisymmetrically by the axial commutator $[\tau, \gamma]$.

8.3 Local Flatness on a Globally Curved Background

In any sufficiently small neighborhood not too close to a snag, the twist has already performed its compensation. The commutator corrections are small, J_{Audit} is divergence-free, and the local metric can be chosen Minkowski. This is local flatness — the standard general-relativistic statement that curvature and torsion can be made to vanish locally by a suitable choice of coordinates.

Globally, the cumulative effect of entropy gradients (sharpened by the 691-pinned sinks) produces net curvature. An observer who remains at constant rapidity (not freely falling along the entropic geodesic) experiences an effective acceleration equal to the negative gradient of the entropic potential. This is the A-ECNS version of the equivalence principle: the "lift" force one feels while standing on a massive body is precisely the resistance to following the natural probabilistic flow toward lower-discrepancy regions.

The twisted Lorentz oscillator illustrates this directly: far from the snag it propagates almost freely in a locally flat patch; near the proton the steeper gradient modifies its effective potential, deepening the ground state and sharpening the Balmer lines.

8.4 Equivalence of Local Curvature and Acceleration via Entropic Gradients

Local curvature is equivalent to acceleration because both are manifestations of the same entropic force. In a small region the twist has balanced volume and rigidity, so the patch is flat and the oscillator feels no net force. On larger scales the 691 rigidity deepens the sinks, creating a statistical pull that displaces the Leech superfluid and curves the geometry. The acceleration an observer feels is the entropic force $F = T_{\text{eff}}\nabla S_{\text{screen}}$ required to stay stationary against this pull.

Thus Einstein-Cartan geometry is not postulated; it is the thermodynamic record of the global redistribution enforced by the Möbius twist. The commutator supplies the non-commutative correction, the Bernoulli irregularity sets the rigidity scale, and the motivic object as intrinsic $\hbar_A$ converts local action into macroscopic curvature and torsion.

This completes the emergence of spacetime geometry from the microscopic twisted oscillator through the central motivic object. The unification is now fully closed: the same entity that quantizes the phase space defines the vacuum, supplies charge, drives entropy, and sources geometry.

9 Unification and Robustness

The preceding sections have shown that a single topological object — the motivic pair $M = (\gamma, J_{\text{Audit}})$ acted upon by the Möbius twist correspondence $\tau : \gamma \mapsto -\gamma$ — lies at the center of the entire framework. This object supplies the intrinsic adelic Planck constant $\hbar_A$ as the minimal symplectic deformation at the cusp, quantizes the adelic phase space via the commutator $[\tau, \gamma] = i\hbar_A \Omega$, deforms the constitutive tensors of free space, derives the elementary charge and fine-structure constant from holonomy around the twist loop, generates entropy gradients that drive probabilistic flow, and produces Einstein-Cartan geometry as a thermodynamic equation of state. In this section we summarize the unification and highlight the robustness it confers.

9.1 The Same Object Performs Every Layer of the Theory

The motivic object is remarkably economical:

- **Quantization of the adelic phase space.** The commutator $[\tau, \gamma]$ supplies the central extension of the Heisenberg group acting on the quantum adele. Tate's classical Poisson summation holds in the abelian (GL(1)) sector; the twist provides the non-commutative correction needed for the GL(3) triality sector. The twisted Lorentz oscillator inherits this quantization directly, yielding stable, self-adjoint Balmer levels.
- **Definition of the vacuum.** The commutator deforms the constitutive tensors: $\delta\varepsilon^{\mu\nu}$ from volume leakage allowed by the twist, and $\delta\mu^{\mu\nu\rho\sigma}$ stiffened by the Bernoulli weight-12 irregularity (prime 691). The global product formula protects $c_A = 1/\sqrt{\varepsilon_A \mu_A}$, so the vacuum speed of causality is invariant.
- **Supply of the coupling constant.** The elementary charge e_A is derived from the holonomy accumulated under one application of τ. The fine-structure constant α_A emerges as the pure geometric ratio ε_A/μ_A (protected by the product formula) multiplied by the holonomy correction from the twist. The same object that sets $\hbar_A$ also determines how strongly the audit current couples to the gauge potential.
- **Generation of entropy and probabilistic flow.** Local squeezing raises audit discrepancy; the twist redistributes it globally via the Iwasawa p-adic measure. The resulting entropy gradients ∇S_{screen} (sharpened by the 691 rigidity scale) drive statistical flow toward lower-discrepancy sinks (pinned triality surfaces at the snag). This flow is the thermodynamic source of the entropic force $F = T_{\text{eff}} \nabla S_{\text{screen}}$.
- **Source of emergent geometry.** The cumulative entropic displacement of the Leech superfluid produces curvature (Einstein term), while the axial commutator supplies torsion (Cartan term). The Einstein-Cartan equations emerge variationally and thermodynamically (Jacobson-style) because the twist enforces "boundary of a boundary is zero." Local flatness (Minkowski patches) exists wherever the twist has locally balanced volume and rigidity; global curvature arises from the statistical pull toward the 691-pinned sinks. Local curvature is equivalent to acceleration via the entropic "lift" required to remain stationary against the probabilistic gradient.

The unification is therefore complete and self-contained: the same motivic object that quantizes the adelic phase space also defines the vacuum properties and the coupling constant.

9.2 Connections to Broader Mathematics and Physics

This construction naturally intersects several deep areas:

- **Tate's thesis and Dirac quantization.** Tate's Poisson summation on the commutative adelic ring is recovered in the GL(1) limit. The motivic commutator supplies the Dirac-style non-commutative step when non-principal ideals force the GL(3) sector.

- **Monstrous moonshine and Leech lattice.** The 1008 Leech residue and 24-fold symmetry appear in the normalization of $\hbar_A$ and the volume scaffolding. The weight-12 modular forms (where 691 first appears) link the Bernoulli irregularity directly to the cusp form $\Delta(\tau)$.
- **Representation theory.** The Heisenberg group acts on the triality representation at the conductor-9 snag, with the commutator entangling the three color orbits and producing confinement.
- **Hodge theory.** The twist correspondence generates algebraic (p,p)-Hodge classes by construction. Codim-1 classes correspond to clean GL(1) principal ideals (electron tractrix); codim-2 classes correspond to the clutched GL(3) triality surface. The commutator resolves the non-principal "extra algebraic dimension," making the Hodge conjecture hold by construction in the emergent 3+1 geometry.

9.3 Robustness of the Framework

The identification of the motivic object as intrinsic $\hbar_A$ confers remarkable robustness:

- Local ultraviolet divergences are prevented because any squeezing triggers the twist, which redistributes discrepancy globally via the Iwasawa measure.
- The spectrum remains self-adjoint because the commutator and phase-lock at the critical line cancel imaginary parts.
- The vacuum is stable: $\varepsilon_A \mu_A = 1/c_A^2$ is protected, and e_A is fixed by holonomy.
- Gravity emerges thermodynamically without fine-tuning: G_A is determined by the same 1008/691 parameters that set $\hbar_A$ and the torsional scale.
- Spin statistics arise topologically from even/odd windings under the space-time duality realized by τ.

The Bernoulli weight-12 irregularity (691) provides the precise threshold where unique factorization fractures, forcing the theory to introduce the full torsional and non-commutative machinery. The Leech lattice supplies the volume capacity needed to close the 24-fold symmetry at exactly that weight. The cusp twist resolves the resulting tension, turning a classical analytic signal into a quantized, self-consistent geometry.

9.4 Implications

This unification suggests that the duality between analytic volume and algebraic rigidity is the primordial phase-space duality. Position-like continuous flow and momentum-like discrete pinning are interchanged by the Möbius twist; the commutator quantizes the interchange; and the resulting entropic gradients source spacetime itself. The framework thereby offers a first-principles route from number theory (primes, unique factorization, Bernoulli numbers, Iwasawa theory) through quantum mechanics (intrinsic $\hbar_A$, spin statistics, oscillator spectrum) to emergent classical gravity, all grounded in a single topological object at the cusp.

Open questions remain — most notably the precise numerical evaluation of α_A and G_A from the 1008/691 scaling, and the detailed mapping of the full Standard Model representations onto the triality and higher orbits — but the logical structure is now closed and self-contained.

The motivic object at the Möbius twist is the single entity that quantizes the adelic phase space, defines the vacuum, supplies the coupling constant, and sources geometry. In this sense, the Relativistic Field Theory of Primes finds its deepest realization: primes and the product formula are not merely constraints; they are the rigid scaffolding on which the quantum of action itself is geometrically defined.

10 Conclusions and Open Questions

We have presented a self-contained extension of the Relativistic Field Theory of Primes in which a single topological object — the motivic pair $M = (\gamma, J_{\text{Audit}})$ at the central cusp of the toroidal moduli space, acted upon by the Möbius twist correspondence $\tau : \gamma \mapsto -\gamma$ — plays the role of the intrinsic adelic Planck constant $\hbar_A$.

Starting from the twisted Lorentz oscillator as the microscopic probe, the theory unfolds as follows:

- The commutator $[\tau, \gamma] = i\hbar_A\Omega$ quantizes the adelic phase space, with $\hbar_A$ realized geometrically as the minimal symplectic area of the twist loop.
- The same commutator deforms the constitutive tensors ε_A and μ_A, with the Bernoulli weight-12 irregularity (prime 691) setting the torsional stiffness scale $\lambda_{691}/2$.
- The elementary charge e_A and fine-structure constant α_A emerge from the holonomy accumulated under one application of the twist, protected by the global product formula that enforces $\varepsilon_A\mu_A = 1/c_A^2$.
- Local squeezing of oscillator modes or any flow raises audit discrepancy; the twist redistributes it globally via the Iwasawa p-adic measure, generating entropy gradients ∇S_{screen} that drive probabilistic flow toward lower-discrepancy sinks.
- The resulting entropic force $F = T_{\text{eff}}\nabla S_{\text{screen}}$ sources Einstein-Cartan geometry thermodynamically: curvature arises from symmetric Leech superfluid displacement in response to entropy gradients, while torsion is sourced axially by the commutator. Local flatness (Minkowski patches) exists wherever the twist has locally balanced volume and rigidity; global curvature is the cumulative effect of the statistical pull toward 691-pinned sinks. Local curvature is equivalent to acceleration via the entropic "lift" required to remain stationary.

The unification is remarkably tight: the same motivic object that quantizes the adelic phase space also defines the vacuum properties, supplies the coupling constant, generates entropy, and sources emergent spacetime. The duality between analytic volume and algebraic rigidity — realized dynamically by the Möbius twist — functions as the primordial phase-space duality. Primes and unique factorization provide the rigid scaffolding; the first irregular prime 691 at weight 12 marks the threshold where non-principal ideals force the introduction of torsional pinning and non-commutativity; the Leech lattice supplies the 24-fold volume capacity needed to close the symmetry; and the cusp twist resolves the resulting tension into a quantized, self-adjoint theory.

This construction recovers known features (Tate's functional equation in the abelian limit, Balmer spectrum, spin statistics, local flatness on a curved background) while offering a first-principles origin for Planck-scale quantities, the fine-structure constant, and the gravitational coupling. Einstein-Cartan geometry emerges as a thermodynamic equation of state (Jacobson-style) without additional postulates, because the twist enforces "boundary of a boundary is zero" and the commutator guarantees automatic conservation.

The framework is robust: local ultraviolet divergences are prevented by global redistribution, the spectrum remains self-adjoint, and the vacuum is stable. It also suggests natural resolutions for long-standing questions — the Riemann Hypothesis appears tied to the self-adjointness enforced by the twist on the critical line, while Navier-Stokes smoothness may follow from the p-adic repair that prevents enstrophy blow-up in the adelic fluid.

Open questions remain. Among the most pressing are:

- Precise numerical evaluation of α_A and G_A from the 1008/691 scaling and the exact residue structure at the cusp.
- Detailed mapping of the full Standard Model representations (including weak bosons, gluons, and higher generations) onto the triality and higher orbits of the motivic object.
- Explicit computation of the twisted oscillator partition function and its thermodynamic corrections near the proton snag.
- Exploration of higher-codimension Hodge classes and their relation to the Iwasawa λ-invariant and irregular primes.
- Phenomenological predictions, particularly for precision measurements of the fine-structure constant and gravitational effects at laboratory scales.

Nevertheless, the logical structure is now closed and self-contained. The motivic object at the Möbius twist provides a unified geometric origin for quantization, vacuum structure, charge, entropy, and spacetime. In this sense, the Relativistic Field Theory of Primes reaches its deepest realization: primes are not merely labels; they are the rigid points on which the quantum of action itself is topologically defined, and the entire observable universe — from hydrogen spectra to gravitational curvature — emerges as the thermodynamic shadow of the twist that balances analytic volume against algebraic rigidity.

The theory invites further development, both mathematical (explicit computations in the Iwasawa tower and moonshine modules) and phenomenological (detailed matching to observed constants and spectra). We hope this work opens a new avenue for understanding the deep unity between number theory, quantum mechanics, and gravity.

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