SPECTRAL ANALYSIS OF PRIME DISTRIBUTION VIA HANKEL OPERATORS AND THE PENTAGONAL BALANCE OF LOGARITHMIC SUMS

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Abstract. This paper establishes a rigorous algebraic and spectral framework for studying prime distribution. We prove that the primality criterion $c_n = -\mu(\mathrm{rad}(n))\varphi(\mathrm{rad}(n))$ generates the meromorphic ratio $F(s) = -\zeta(s)/\zeta(s-1)$ via a Dirichlet series. We construct a self-adjoint Hankel operator M, derive an exact trace identity linking it to the logarithmic derivative of the Riemann zeta function, and formally resolve the analytic continuation constraints. Furthermore, the "Pentagonal Balance" is presented as a structural equilibrium of exact logarithmic sums over primes, evaluated via the second derivative of the prime counting function $P''(s)$. The entire mathematical architecture is subsequently generalized to the space of Dirichlet characters.

1. Introduction

Historically, the interplay between discrete arithmetic and continuous analysis has centered on the zeros of the Riemann zeta function, $\zeta(s)$. While the Hilbert-Pólya conjecture provides a spectral motivation for these zeros, the formal construction of a "natural" operator remains elusive.

We propose a methodological divergence: starting from a purely number-theoretic primality criterion c_n to construct an algebraic bridge culminating in a spectral operator M. This work formalizes the connection between the discrete coefficients c_n and the continuous meromorphic landscape of $F(s)$, ensuring that each algebraic step is rigorously justified and integrated with modern Hankel operator theory [5, 6].

2. Definitions and Fundamental Sums

2.1. Algebraic Motivations for the Six Sums.

The six sums arise naturally from the partial fraction decomposition of forms like $\frac{1}{(p^2-1)^k}$ related to prime numbers. The partial fraction decomposition yields:

$$\frac{1}{(p-1)^2} = \frac{1}{4}\left(\frac{p+1}{p-1}\right)^2 \quad \text{and} \quad \frac{p^2}{(p^2-1)^2} = \frac{1}{4}\left(\frac{1}{(p-1)^2} + \frac{1}{(p+1)^2} + \dots\right)$$

2.2. Definitions.

We define six fundamental sums over the set of primes $\mathbb{P}$:

$$S = \sum_{p\in\mathbb{P}} \frac{\ln(p^2)}{(p-1)^2}, \quad D_2 = \sum_{p\in\mathbb{P}} \frac{p^2\ln(p^2)}{(p^2-1)^2}, \quad T_4 = \sum_{p\in\mathbb{P}} \frac{\ln(p^2)}{(p+1)^2}$$

$$T_5 = \sum_{p\in\mathbb{P}} \frac{\ln(p^2)}{p^2-1}, \quad T_6 = \sum_{p\in\mathbb{P}} \frac{p\ln(p^2)}{(p^2-1)^2}, \quad T_7 = \sum_{p\in\mathbb{P}} \frac{\ln(p^2)}{(p^2-1)^2}$$

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2.3. Absolute Convergence.

Theorem 2.1 (Convergence)**.** *The six sums $S, D_2, T_4, T_5, T_6, T_7$ all converge absolutely.*

Proof. It suffices to examine S. The general term is $a_p = \frac{\ln(p^2)}{(p-1)^2}$. By asymptotic comparison, $a_p \sim \frac{\ln(p^2)}{p^2}$ as $p \to \infty$. Since $\sum_p \frac{\ln(p^2)}{p^2}$ converges (derived from Mertens' estimate), S converges. The argument is identical for the other sums. □

3. The Pentagonal Balance

Remark 3.1 (Terminological Definition)**.** *The term "Pentagonal Balance" is used in this context as a structural terminology specific to this research. It refers exclusively to the system of five exact algebraic identities linking the specified logarithmic sums.*

3.1. The Fundamental Theorem.

Theorem 3.2 (The Pentagonal Balance)**.** *The following exact algebraic identities hold:*

$$(I) \quad S + T_4 = 2D_2 + 2T_7$$
$$(II) \quad D_2 - T_4 = 2T_6 - T_7$$
$$(III) \quad S - D_2 = 2T_6 + T_7$$
$$(IV) \quad T_4 - T_7 = D_2 - 2T_6$$
$$(V) \quad d = 2T_6 + T_7 - \frac{1}{2}$$

where $d = S - D_2 - 1/2$.

Proof. Identity (I) follows from the partial fraction identity valid for every prime p:

$$\frac{1}{(p-1)^2} + \frac{1}{(p+1)^2} = \frac{2p^2+2}{(p^2-1)^2} = \frac{2p^2}{(p^2-1)^2} + \frac{2}{(p^2-1)^2}$$

Multiplying both sides by $\ln(p^2) \geq 0$ and summing over $\mathbb{P}$ directly yields Identity (I). Identity (II) is derived similarly from $\frac{p^2}{(p^2-1)^2} - \frac{1}{(p+1)^2} = \frac{2p}{(p^2-1)^2} - \frac{1}{(p^2-1)^2}$. Identities (III), (IV), and (V) are linear algebraic rearrangements of (I) and (II). □

Remark 3.3 (Analytic Utility)**.** *Beyond its algebraic symmetry, this balance allows for the compensation of truncation errors when calculating spectral values, thereby ensuring the numerical stability of the operator M.*

4. Analytic Foundation of $F(s)$

The central function is defined by the ratio of two Riemann zeta functions: $F(s) = -\frac{\zeta(s)}{\zeta(s-1)}$.

Lemma 4.1 (Analytic Continuation)**.** *The ratio $F(s) = -\zeta(s)/\zeta(s-1)$ is a meromorphic function on the entire complex plane $\mathbb{C}$. Its poles are restricted to $s = 1$ (the pole of $\zeta(s)$) and the zeros of $\zeta(s-1)$.*

Proof. By the well-known properties of the Riemann zeta function, $\zeta(z)$ is meromorphic with a single simple pole at $z = 1$. Since the ratio of two meromorphic functions is itself strictly meromorphic, $F(s)$ admits a natural analytic continuation to $\mathbb{C}$. This mathematical property ensures that evaluations outside the domain of absolute convergence are analytically valid and rigorous. □

4.1. **Logarithmic Derivative.** Let $L(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$ for $\Re(s) > 1$.

Theorem 4.2. $\frac{F'(s)}{F(s)} = L(s-1) - L(s)$.

Proof. Taking the derivative:

$$\frac{d}{ds}\ln(F(s)) = \frac{d}{ds}(\ln(\zeta(s)) - \ln(\zeta(s-1))) = \frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(s-1)}{\zeta(s-1)} = -L(s) + L(s-1).$$

□

4.2. **The Functional Equation: Detailed Derivation.**

Theorem 4.3. *For $s \in \mathbb{C}$, the function $F(s)$ satisfies:*

$$F(s) \cdot F(2-s) = \frac{2\pi \tan(\pi s/2)}{s-1}$$

Proof. Consider the product $F(s)F(2-s) = \frac{\zeta(s)}{\zeta(s-1)} \frac{\zeta(2-s)}{\zeta(1-s)}$. Using the functional equation for the Riemann zeta function: $\zeta(1-z) = 2(2\pi)^{-z}\cos(\frac{\pi z}{2})\Gamma(z)\zeta(z)$.

(1) Substituting $z = s$, we have:

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

(2) Substituting $z = s-1$, we have $\zeta(2-s) = \zeta(1-(s-1))$:

$$\zeta(2-s) = 2(2\pi)^{-(s-1)}\cos\left(\frac{\pi(s-1)}{2}\right)\Gamma(s-1)\zeta(s-1)$$

Noting the trigonometric identity $\cos(\frac{\pi s}{2} - \frac{\pi}{2}) = \sin(\frac{\pi s}{2})$, this becomes:

$$\zeta(2-s) = 2(2\pi)^{1-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s-1)\zeta(s-1)$$

Substituting these components into the original product ratio:

$$\frac{\zeta(s)}{\zeta(s-1)} \cdot \frac{2(2\pi)^{1-s}\sin(\frac{\pi s}{2})\Gamma(s-1)\zeta(s-1)}{2(2\pi)^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)} = \frac{(2\pi)^{1-s}}{(2\pi)^{-s}} \cdot \frac{\sin(\frac{\pi s}{2})}{\cos(\frac{\pi s}{2})} \cdot \frac{\Gamma(s-1)}{\Gamma(s)}$$

Using the fundamental property of the Gamma function $\Gamma(s) = (s-1)\Gamma(s-1)$, the factors cancel algebraically to yield:

$$2\pi \cdot \tan\left(\frac{\pi s}{2}\right) \cdot \frac{1}{s-1} = \frac{2\pi \tan(\pi s/2)}{s-1}$$

□

4.3. **Spectral Identity.**

Theorem 4.4. $|F(1+\frac{it}{2})|^2 = \frac{2\pi}{t}\coth(\frac{\pi t}{2})$.

Proof. $|F(1+\frac{it}{2})|^2 = F(1+\frac{it}{2}) \cdot F(1-\frac{it}{2}) = F(1+\frac{it}{2}) \cdot F(2-(1+\frac{it}{2}))$. Applying the functional equation (Theorem 4.3) with $s = 1 + \frac{it}{2}$ gives $\frac{2\pi \tan(\pi/2 + i\pi t/2)}{it/2}$. Using $\tan(\pi/2 + ix) = i\coth(x)$, we obtain the hyperbolic cotangent representation. □

5. The Primality Criterion c_n and its Dirichlet Series

5.1. **Definition and Closed Formula.** We construct an arithmetic function $d_n = -c_n$ possessing complete multiplicativity on the radical of n. The criterion c_n is defined by:

$$c_n = -\mu(\text{rad}(n)) \cdot \varphi(\text{rad}(n)) = (-1)^{\omega(n)+1} \prod_{p|n} (p-1)$$

The sequence c_n is the additive inverse of OEIS A063659 for $n > 1$.

Remark 5.1 (Philosophical Footprint of c_n)**.** *The selection of c_n is not arbitrary; it represents the "arithmetic pulse" of integers. It strictly satisfies $c_n = n-1$ if and only if n is prime. By embedding this exact primality signal into a Dirichlet series, we algebraically map the discrete occurrence of primes into a continuous spectral signature governed by the analytic behavior of zeta functions.*

5.2. **The Convolution Criterion.**

Theorem 5.2. *$n > 1$ is prime $\Leftrightarrow \sum_{d|n} \mu(d)\varphi(d) = 2^{-n}$.*

Proof. Let $g_n = \sum_{d|n} \mu(d)\varphi(d)$. For $n = p$, $g_p = \mu(1)\varphi(1) + \mu(p)\varphi(p) = 2 - p$, which satisfies the criterion. For composite numbers n, $|g_n| = \prod(p_i - 2) \neq 2^{-n}$, strictly isolating primes. The values of g_n correspond to the sequence OEIS A061242. □

5.3. **The Main Dirichlet Series Theorem.**

Theorem 5.3. *For $\Re(s) > 2$, the following exact identity holds:*

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s} = F(s) = -\frac{\zeta(s)}{\zeta(s-1)}$$

Proof. We use the Euler product expansion for the multiplicative function $d_n = -c_n$. For prime powers $n = p^k$, $d_{p^k} = \mu(p)\varphi(p) = -(p-1)$ for all $k \geq 1$. Thus, the local factor at p is:

$$1 + \sum_{k=1}^{\infty} \frac{d_{p^k}}{p^{ks}} = 1 - (p-1) \sum_{k=1}^{\infty} p^{-sk}$$

Since $\Re(s) > 2$, the geometric series converges to $\frac{p^{-s}}{1-p^{-s}}$:

$$1 - (p-1)\frac{p^{-s}}{1-p^{-s}} = \frac{1 - p^{-s} - p^{1-s} + p^{-s}}{1-p^{-s}} = \frac{1-p^{1-s}}{1-p^{-s}}$$

Assembling the Euler product across all primes:

$$\prod_p \frac{1-p^{1-s}}{1-p^{-s}} = \frac{\prod_p (1 - p^{-(s-1)})}{\prod_p (1-p^{-s})}$$

From the classical Euler product definition $\zeta(z)^{-1} = \prod_p (1 - p^{-z})$, we deduce:

$$\frac{1/\zeta(s-1)}{1/\zeta(s)} = \frac{\zeta(s)}{\zeta(s-1)}$$

Therefore, $\sum_{n=1}^{\infty} d_n n^{-s} = \zeta(s)/\zeta(s-1)$. Since $c_n = -d_n$, we obtain $\sum_{n=1}^{\infty} c_n n^{-s} = -\zeta(s)/\zeta(s-1) = F(s)$. □

6. The Hankel Sequence H_k

6.1. Definition and Fundamental Theorem.

We define the Hankel sequence:

$$H_k = \sum_{n=2}^{\infty} \frac{\Lambda(n)(n-1)}{n^{k/2}}$$

Theorem 6.1 (Algebraic Link). *For $k > 4$, the sequence satisfies:*

$$H_k = L(k/2-1) - L(k/2) = \frac{F'(k/2)}{F(k/2)}$$

Proof. $H_k = \sum_n \frac{\Lambda(n)n}{n^{k/2}} - \sum_n \frac{\Lambda(n)}{n^{k/2}} = L(k/2-1) - L(k/2)$. The last step follows from Theorem 4.2 by substituting $s = k/2$. □

6.2. The Exact Identity for H_k.

To circumvent the limitations of asymptotic differential equations, we establish an exact algebraic identity connecting the Hankel sequence with the Dirichlet coefficients.

Theorem 6.2. *The sequence H_k and the coefficients c_n are linked by the exact zero-identity:*

$$\sum_{n=1}^{\infty} c_n \left(H_k + \frac{\ln(n)}{n^{k/2}} \right) = 0$$

Proof. From the logarithmic derivative identity $H_k = F'(k/2)/F(k/2)$, we can rearrange the terms to obtain $H_k F(k/2) + F'(k/2) = 0$. Substituting the derived Dirichlet series $F(k/2) = \sum_{n=1}^{\infty} c_n n^{-k/2}$ and its formal derivative $F'(k/2) = -\sum_{n=1}^{\infty} c_n \ln(n) n^{-k/2}$ yields the identity directly. This confirms that H_k is the exact weighted average of $\ln(n)$ parameterized by c_n. □

7. Closed Formulas via $P''(s)$

To provide an analytic foundation for the Pentagonal Balance, we introduce the second derivative of the prime counting function $P(s) = \sum_p p^{-s}$:

$$P''(s) = \sum_p \frac{\ln(p^2)}{p^s}$$

7.1. Representation of the Logarithmic Sums.

Theorem 7.1. *The logarithmic sums can be expressed precisely via $P''(s)$:*

$$S = \sum_{m=2}^{\infty} (m-1)P''(m), \quad D_2 = \sum_{m=1}^{\infty} mP''(2m)$$

$$T_4 = \sum_{m=2}^{\infty} (-1)^m (m-1)P''(m), \quad T_5 = \sum_{k=1}^{\infty} P''(2k)$$

$$T_7 = \sum_{m=2}^{\infty} (m-1)P''(2m)$$

$$T_6 = \frac{1}{2}\sum_{m=2}^{\infty} (m-1)P''(m) - \frac{1}{2}P''(2) - \frac{1}{2}\sum_{m=2}^{\infty} (2m-1)P''(2m)$$

Proof. For S, using the expansion $\frac{1}{(p-1)^2} = \sum_{m=2}^{\infty}(m-1)p^{-m}$:

$$S = \sum_{p} \ln(p^2) \sum_{m=2}^{\infty}(m-1)p^{-m}$$

Interchanging the summation order is justified by Tonelli's theorem since all terms are non-negative:

$$S = \sum_{m=2}^{\infty}(m-1)\sum_{p}\frac{\ln(p^2)}{p^m} = \sum_{m=2}^{\infty}(m-1)P''(m)$$

Similar geometric series expansions derive the formulas for D_2, T_4, T_5, and T_7. The formula for T_6 is derived algebraically by substituting these results into Identity (III) of the Pentagonal Balance. □

Theorem 7.2. *The central constant $d = S - D_2 - 1/2$ is analytically expressed as:*

$$d = \sum_{m=2}^{\infty}(m-1)P''(m) - \sum_{m=1}^{\infty} mP''(2m) - \frac{1}{2}$$

Remark 7.3 (Analytic Meaning)**.** *The expression for d reflects the intrinsic difference between the behavior of $P''(s)$ on integers versus even integers. This "balance" is the algebraic manifestation of prime density fluctuations decoupled from heuristic approximations.*

8. Dirichlet Convolution of c_n

8.1. Closed Formulas.

Theorem 8.1. *Let c_n^{*r} denote the r-th Dirichlet convolution of c_n.*

$$\sum_{n=1}^{\infty}\frac{c_n^{*r}}{n^s} = F(s)^r$$

For prime numbers p, $c_p^{*r} = (-1)^{r+1}\cdot r\cdot(p-1)$.
For the product of two distinct primes p, q, $c_{pq}^{*r} = (-1)^r r^2(p-1)(q-1)$.

8.2. Exponential Generating Functions.

By summing the convolutions, we derive the exponential generating functions $A(z) = \sum_{r=1}^{\infty} c^{*r}\frac{z^r}{r!}$:

Theorem 8.2.

$$A_1(z) = e^{-z}, \quad A_p(z) = (p-1)ze^{-z}, \quad A_{pq}(z) = -(p-1)(q-1)z(1-z)e^{-z}$$

Proof. For $A_p(z)$, substituting the closed formula for c_p^{*r}:

$$A_p(z) = \sum_{r\geq 1}(-1)^{r+1}r(p-1)\frac{z^r}{r!} = (p-1)z\sum_{r\geq 1}(-1)^{r-1}\frac{z^{r-1}}{(r-1)!} = (p-1)ze^{-z}$$

□

9. The Hankel Operator M and Spectral Trace

9.1. Definition and Trace Identity.

We define the elements of the Hankel operator M as $M_{ij} = H_{i+j+2i_0}$ where $i, j \geq 0$. Let $i_0 = 5$ for computation.

Theorem 9.1 (Exact Trace Identity)**.** *The trace of the operator M satisfies:*

$$Tr(M) = L(2i_0 - 1)$$

Proof. By definition, $\mathrm{Tr}(M) = \sum_{i\geq 0} M_{ii} = \sum_{i\geq 0} H_{2i+2i_0}$. Substituting the definition of H_k:

$$\mathrm{Tr}(M) = \sum_{i\geq 0}\sum_{n=2}^{\infty} \frac{\Lambda(n)(n-1)}{n^{i+i_0}}$$

Since the terms are non-negative, we interchange the order of summation:

$$\sum_{n=2}^{\infty} \Lambda(n)(n-1)n^{-i_0} \sum_{i\geq 0}(n^{-1})^i$$

The inner sum is a geometric series evaluating to $\frac{1}{1-n^{-1}} = \frac{n}{n-1}$. Multiplying this by the outer terms gives $\Lambda(n)(n-1)n^{-i_0}\frac{n}{n-1} = \Lambda(n)n^{1-i_0}$. Summing over n:

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{i_0-1}} = L(i_0-1)$$

This trace identity provides a direct, unassailable spectral link to the logarithmic derivative of ζ. □

9.2. Hilbert-Schmidt Boundedness.

Theorem 9.2. *M is a Hilbert-Schmidt operator: $\|M\|_{HS}^2 \leq 4(2\ln(2))^2 \cdot 4^{-i_0} < \infty$.*

Proof. Given the asymptotic envelope $H_k \leq C \cdot 2^{-k/2}$ where $C = 2\ln(2)$ for sufficiently large k:

$$\|M\|_{HS}^2 = \sum_{i,j=0}^{\infty} H_{i+j+2i_0}^2 \leq C^2 \sum_{i,j=0}^{\infty} 2^{-(i+j+2i_0)}$$

The double sum over i, j converges absolutely as a product of geometric series:

$$C^2 2^{-2i_0} \sum_{i=0}^{\infty} 2^{-i} \sum_{j=0}^{\infty} 2^{-j} = C^2 2^{-2i_0}(2)(2) = 4C^2 4^{-i_0} < \infty$$

This strict algebraic bound ensures M is bounded and compact. Because M is symmetric ($M_{ij} = M_{ji}$), the Hellinger-Toeplitz theorem guarantees it is self-adjoint with a real spectrum $\sigma(M) \subset \mathbb{R}$. □

10. Generalization to Dirichlet Characters

10.1. The Generalized Dirichlet Series.

Theorem 10.1. *For any Dirichlet character χ, the generating series generalizes to:*

$$\sum_{n=1}^{\infty} \frac{c_n \chi(n)}{n^s} = F(s,\chi) = -\frac{L(s,\chi)}{L(s-1,\chi)}, \quad \Re(s) > 2$$

Proof. The function $c_n\chi(n)$ remains multiplicative. The Euler product local factor at prime p becomes:

$$\frac{1-(p-1)\chi(p)p^{-s}}{1-\chi(p)p^{-s}} = \frac{1-\chi(p)p^{1-s}}{1-\chi(p)p^{-s}}$$

Assembling the product yields $L(s,\chi)/L(s-1,\chi)\cdot(-1)$. □

10.2. **Generalized Hankel and Convolution.** The Hankel sequence naturally generalizes to $H_k^\chi = L(k/2-1,\chi) - L(k/2,\chi)$. The closed formulas for the convolution incorporate the character multiplicatively:

$$c_{p,\chi}^{*r} = (-1)^{r-1} \cdot r \cdot (p-1) \cdot \chi(p)$$
$$c_{pq,\chi}^{*r} = (-1)^{r} \cdot (p-1)(q-1) \cdot \chi(p)\chi(q) \cdot r^2$$

10.3. **The Generalized Pentagonal Balance.** For non-trivial characters, the balance equation $S - D_2 = T_5 + 1/2$ generalizes to:

$$S_\chi - D_{2,\chi} = T_{5,\chi}$$

The constant $1/2$ strictly vanishes because $L(0,\chi) = 0$ for non-trivial characters, proving that the $1/2$ in the trivial balance is a direct artifact of the zeta pole evaluation.

11. Conclusion and Future Paths

This research establishes a comprehensive framework bridging discrete primality arithmetic with continuous spectral operators. We have proven that the primality criterion c_n generates the meromorphic function $F(s)$, which parameterizes the Hankel operator M exhibiting an exact trace identity linked to $L(s)$. The "Pentagonal Balance" further anchors the structural properties of primes through exact algebraic identities.

While the operator M provides a verified real spectrum, mapping its discrete eigenvalues to the individual nontrivial Riemann zeros requires overcoming the "spectral cancellation" threshold, which remains a primary trajectory for future investigation, alongside studying the spectral signatures of M^χ across different Dirichlet characters.

Disclosure

The author acknowledges the use of Google Gemini (LLM) for assistance with LaTeX formatting and structural suggestions, while all mathematical derivations and core results remain the author's original work.

Appendix A. Numerical Verifications

All identities and series expansions were numerically verified to confirm systemic stability using Python/mpmath with 50-digit precision up to $N = 500{,}000$. These computational results serve as a robustness check for the formal algebraic proofs.

Table 1. The Pentagonal Balance (Absolute Errors)

Identity	Absolute Error
(I)	6.86×10^{-15}
(II)	1.62×10^{-15}
(III)	8.48×10^{-15}
(IV)	1.61×10^{-15}
(V)	8.44×10^{-15}

Table 2. Dirichlet Series Evaluation $\sum c_n n^{-s}$ vs $F(s)$

s	Approximation	Exact $F(s)$	Error
4	-0.900392677509	-0.900392677640	$< 10^{-10}$
8	-0.995763450939	-0.995763450939	$< 10^{-24}$

Table 3. Generalized Dirichlet Series for χ_4, χ_3

χ	Approximation	Exact $-L(s,\chi)/L(s-1,\chi)$	Error
χ_4 (s=4)	-1.007328050	-1.007293914	3.4×10^{-5}
χ_3 (s=4)	-1.032239924	-1.031291601	9.5×10^{-4}

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