Quaternionic Norm Response Theory (QNRT): Discrete Spacetime, Photon Basis, UV–IR Scale Duality, and Cosmological Scale Estimates

Kenichi Nakao

Independent Researcher

March 2026 (Revised v3.3)

Abstract

Quaternionic Norm Response Theory (QNRT) proposes a discrete, geometrically structured spacetime based on the Hurwitz quaternion lattice Γ_H. Through a single algebraic operation—quaternion inversion—the theory generates a $\mathbb{Z}_2$ UV–IR scale duality whose fixed point simultaneously determines the dark-energy scale and the neutrino mass scale.

The dynamical field $\phi : \Gamma_H \to \mathbb{H}_\mathbb{C}$ takes values in the complexified quaternion algebra $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ and carries the Hermitian norm $N(\phi) = \phi\phi^\dagger$, giving the logarithmic scale field $\sigma = \log(N(\phi)/\ell_P^2)$. Inversion $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ sends $\sigma \mapsto -\sigma$, exchanging ultraviolet and infrared regimes. The norm-response scaling (NRS) theorem shows that this map has a unique fixed point $r_c = \sqrt{\ell_P R_\star}$.

The fixed-point scale $r_c = 84\,\mu$m (with $R_\star \approx 4.4 \times 10^{26}$ m, $N_\star = R_\star/\ell_P \approx 2.70 \times 10^{61}$) defines a reference wavenumber $k_0 = 1/r_c$. This is precisely the wavenumber of the self-dual lattice mode ($n = n_c = \sqrt{N_\star}$ in the harmonic expansion over $R_\star$), an algebraic identity verified to 0.000% numerical error. The corresponding energy $E_0 = \hbar c k_0 = 2.354\,\mathrm{meV}$ and temperature $T_c = 27.27\,\mathrm{K}$ equal the geometric mean of the Planck and de Sitter temperatures, and coincide with the CMB temperature at redshift $z_c \approx 9$ (cosmic reionisation epoch).

The self-dual scale gives $\rho_\Lambda = \hbar c/(\ell_P R_\star)^2 = c^4/(G R_\star^2) \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$, within 21% of the Planck 2018 value $5.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$. The identity $\rho_\Lambda = c^4/(G R_\star^2)$ shows that $\hbar$ cancels algebraically, so the cosmological constant problem reduces to the purely classical question of why $R_\star/\ell_P \approx 2.7 \times 10^{61}$. The fermionic counterpart yields $M_c c^2 = \sqrt{M_P c^2 \cdot M_H c^2} \approx 2.4\,\mathrm{meV}$, within the active-neutrino mass window. The lattice-derived kinetic coefficient $c_\mathrm{lat} = 6$ fixes the continuum action; the duality-symmetric potential $V(\sigma) = 2\rho_\Lambda \cosh\sigma$ produces thawing-quintessence dark energy with $w_0 \in [-0.990, -0.908]$ and $w_a \in [-0.082, -0.014]$ for $\sigma_i \in [0.5, 3.0]$. All predictions are consistent with ΛCDM.

The principal limitations are as follows. The value $R_\star \approx 4.4 \times 10^{26}$ m is set by self-consistency of the numerical outputs; its epoch-independent determination is Open Problem OP1 and is not resolved in this paper. The QNRT predictions $w_0 \in [-0.990, -0.908]$ are inconsistent with the DESI DR2 (2025) central value $w_0 \approx -0.75$ at 2.7–4.0 σ, while remaining consistent with ΛCDM. The Euclid forecasted precision $\sigma(w_0) \sim 0.02$ will provide a decisive test. The Standard Model gauge

embedding and the physical determination of σ_i are the remaining principal open problems.

Contents

1 Introduction

Two deep mysteries of modern physics—the cosmological constant problem [1] and the smallness of neutrino masses—lack algebraic explanations within the Standard Model. The observed dark-energy density $\rho_\Lambda \approx 6 \times 10^{-10}\,\mathrm{J\,m^{-3}}$ is 10^{122} times smaller than Planck-scale estimates, while neutrino masses at the meV scale are separated from other fermion masses by many orders of magnitude.

QNRT proposes that both scales have a common geometric origin: spacetime is discrete at the Planck scale, with the Hurwitz quaternion lattice Γ_H adopted as its fundamental structure. This choice is motivated by the algebraic properties of the complexified quaternion algebra $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$, which serves as the field algebra of the theory (Section 3). Within this adopted structure, the cosmological hierarchy is a consequence of a UV–IR duality generated by quaternion inversion on the field algebra. The proposal is not a complete quantum theory of gravity; it is a framework in which the Planck and Hubble scales are related by an algebraic fixed-point condition.

This revised version (v3.3) incorporates the following corrections relative to v3.1 and v3.2:

- **(C1) Algebraic basis of the Minkowski interval.** The element $q = ict + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ appearing in the spacetime motivation belongs to $\mathbb{H}_\mathbb{C}$, not to the real quaternion algebra $\mathbb{H}$. Real quaternion norms are positive-definite (Euclidean); the Minkowski signature $(-,+,+,+)$ arises from the Hermitian norm on $\mathbb{H}_\mathbb{C}$. The text has been corrected accordingly (Section 2).

- **(C2) Asymptotic formula for $r_4(N)$.** The saddle-point argument in Section 6.3 previously cited $r_4(N) \sim 8\pi^2 N$; the correct average asymptotic is $r_4(N) \sim \pi^2 N$ (confirmed numerically to 0.05%).

- **(C3) Colour gauge group.** The identification of $G_2 \supset \mathrm{SU}(3)$ as the lattice automorphism group for colour is withdrawn; explicit calculation shows that the Weyl group $W(G_2)$ generates components $\pm 1/3, \pm 2/3$ not in Γ_H, ruling out this route. The current status of the SU(3) embedding is stated as open (Section 10).

- **(C4) E_8 and anomaly cancellation.** The claim that $E_8 \cong \Gamma_H \oplus \Gamma_H$ provides the context for anomaly cancellation is withdrawn. The E_8 extension is blocked by the non-associativity of the Cayley integers and gives $c_{\mathrm{lat}}(E_8) = 60 \neq 6$, inconsistent with the lattice action. The anomaly cancellation question is left as an open problem (Section 10).

- **(C5) $R_\star$ numerical value.** Earlier versions stated $R_\star \approx 1.3 \times 10^{26}\,\mathrm{m} \approx R_H(t_0)$ (the Hubble radius, c/H_0). However, all specific numerical outputs—$r_c = 84\,\mu\mathrm{m}$, k_0, E_0, T_c, $N_\star$, and $\rho_\Lambda/\rho_\Lambda^{\mathrm{obs}} = 1.209$—are mutually consistent only for $R_\star \approx 4.4 \times 10^{26}\,\mathrm{m}$ ($N_\star \approx 2.70 \times 10^{61}$), which is approximately $3.2\,R_H$, close to the comoving particle horizon. Setting $R_\star = R_H$ gives $r_c \approx 47\,\mu\mathrm{m}$, inconsistent with all other quoted values. This version uses $R_\star \approx 4.4 \times 10^{26}\,\mathrm{m}$ throughout; its physical determination remains Open Problem OP1.

- **(C6) ρ_Λ numerical value.** With $R_\star \approx 4.4 \times 10^{26}\,\mathrm{m}$: $\rho_\Lambda = c^4/(G R_\star^2) \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$, within 21% of $\rho_\Lambda^{\mathrm{obs}} \approx 5.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$ (Planck 2018). The earlier value $7 \times 10^{-9}\,\mathrm{J\,m^{-3}}$ was obtained with $R_\star = R_H$ and is superseded.

(C7) "12% discrepancy" corrected. $\rho_\Lambda^{\rm QNRT}/\rho_\Lambda^{\rm obs} = 1.209$ is a 20.9% excess, not 12% as stated in earlier versions.

(C8) Neutrino mass estimate revised. With $R_\star = 4.4 \times 10^{26}\,\mathrm{m}$ used consistently, $M_c c^2 = \sqrt{M_P c^2 \cdot M_H c^2} \approx 2.4\,\mathrm{meV}$ (revised from 4.2 meV, which used $R_\star = R_H$). The value remains within the active-neutrino mass window.

The dark-energy equation-of-state predictions (w_0, w_a) are unchanged from v3.1. The values of ρ_Λ and $M_c c^2$ are revised in v3.3 (corrections C6 and C8).

Derivation chain.

$$\text{Lorentz invariance} \;\Rightarrow\; N_{\mathbb{H}_\mathbb{C}}(q) = -c^2t^2 + |\mathbf{x}|^2 \;\Rightarrow\; \mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$$

$$\text{Hurwitz lattice } \Gamma_H \subset \mathbb{H} \quad \text{integer norm, 24 unit elements}$$

$$\text{NRS map: } \phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi) \;\Rightarrow\; N(\phi) \mapsto 1/N(\phi),\; \sigma \mapsto -\sigma$$

$$\text{NRS theorem: unique fixed point } r_c = \sqrt{\ell_P R_\star}$$

$$\text{Photon basis: } k_0 = 1/r_c \text{ self-dual lattice mode (algebraic identity)}$$

$$\rho_\Lambda = c^4/(GR_\star^2) \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}, \quad M_c c^2 \approx 2.4\,\mathrm{meV}, \quad \text{dark-energy dynamics } w(z)$$

2 The Hurwitz Lattice as Discrete Spacetime

2.1 Algebraic motivation: complexified quaternions and the Minkowski interval

The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ has basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For a real quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $a, b, c, d \in \mathbb{R}$, the norm is $N(q) = q\bar{q} = a^2 + b^2 + c^2 + d^2 \geq 0$, which is positive-definite (Euclidean) and cannot yield the Minkowski signature directly.

The Minkowski interval $-c^2t^2 + x^2 + y^2 + z^2$ arises from the *complexified* quaternion algebra $\mathbb{H}_\mathbb{C} = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$. Consider the element

$$q(x) = ict + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \;\in\; \mathbb{H}_\mathbb{C}, \tag{1}$$

whose Hermitian norm is

$$N(q) = qq^\dagger = -c^2t^2 + x^2 + y^2 + z^2. \tag{2}$$

The Minkowski interval thus emerges from the Hermitian norm on $\mathbb{H}_\mathbb{C}$, not from the Euclidean norm on $\mathbb{H}$. This observation motivates $\mathbb{H}_\mathbb{C}$ as the algebra for the dynamical field ϕ (Section 3), while the lattice Γ_H is defined within $\mathbb{H}$ using the Euclidean norm.

Remark 2.1 (Two roles of $\mathbb{H}$ and $\mathbb{H}_\mathbb{C}$)**.** *The theory uses $\mathbb{H}$ and $\mathbb{H}_\mathbb{C}$ in distinct roles: $\Gamma_H \subset \mathbb{H}$ defines the discrete geometry of spacetime (the lattice sites), while $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ is the algebra of field values $\phi : \Gamma_H \to \mathbb{H}_\mathbb{C}$. The Euclidean norm on $\mathbb{H}$ governs the lattice structure; the Hermitian norm $N(\phi) = \phi\phi^\dagger$ on $\mathbb{H}_\mathbb{C}$ governs the field dynamics.*

2.2 The Hurwitz lattice

Definition 2.2 (Hurwitz lattice).

$$\Gamma_H = \left\{ n_0 + n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \;\middle|\; n_\mu \in \mathbb{Z} \text{ or } n_\mu \in \mathbb{Z} + \tfrac{1}{2} \right\}. \tag{3}$$

Γ_H is the unique maximal order in $\mathbb{H}$ closed under conjugation and admitting a Euclidean division algorithm [2, 3]. It corresponds to the F_4 root lattice—the densest sphere packing in four dimensions [4]—and provides the discrete substrate for the dynamical field.

Proposition 2.3 (Integer norm and unit elements). *For all $\phi \in \Gamma_H$: (i) $N(\phi) = \phi\bar{\phi} \in \mathbb{Z}_{\geq 0}$; (ii) the 24 elements with $N(u) = 1$ form the binary tetrahedral group: $\{\pm e_\mu\}$ (8 elements) and $\frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ (16 elements); (iii) $N(\alpha\beta) = N(\alpha)N(\beta)$.*

Remark 2.4 (Why quaternions, and why Γ_H?). *The norm-response operation $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ requires (a) a multiplicative norm and (b) existence of an inverse for all non-zero ϕ. By Hurwitz's composition theorem [2], the only finite-dimensional real division algebras with a multiplicative norm are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. Of these, $\mathbb{H}$ is the unique four-dimensional choice matching the dimensionality of spacetime. Within $\mathbb{H}$, Γ_H is adopted because (i) it is the unique maximal order; (ii) its Euclidean algorithm guarantees unique prime factorisation; and (iii) its 24-element unit group is the binary tetrahedral group, a discrete subgroup of* SU(2)*, which generates the* SU(2) *gauge sector (Section 9).*

2.3 Relation to other discrete-spacetime approaches

Discrete models of spacetime include spin foam models [14], causal dynamical triangulations [15], and causal set theory [16]. QNRT differs in two respects. First, the discrete structure is Γ_H, which carries the full quaternion algebra. Second, the norm-response mechanism links Planck-scale physics directly to the cosmological scale $R_\star$ through an algebraic fixed-point condition, rather than through a coarse-graining argument.

3 The Norm-Response Mechanism

3.1 Field domain and norms on $\mathbb{H}_\mathbb{C}$

The dynamical field takes values in the complexified quaternion algebra:

$$\phi : \Gamma_H \to \mathbb{H}_\mathbb{C} = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}. \tag{4}$$

Write $\phi = A + B\mathbf{j}$ with $A, B \in \mathbb{H}$. The Hermitian conjugate $\phi^\dagger = \bar{A} - \bar{B}\mathbf{j}$ gives $N(\phi) \equiv \phi\phi^\dagger \in \mathbb{R}_{\geq 0}$. The complexification is physically necessary: Dirac spinors and complex gauge representations cannot be accommodated within $\mathbb{H}$ alone; $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ is the minimal algebra admitting a single Weyl spinor [5].

3.2 Quaternion inversion and the $\mathbb{Z}_2$ duality

Definition 3.1 (Norm-response operation). *The map*

$$\phi \;\longmapsto\; \phi^{-1} = \frac{\phi^\dagger}{N(\phi)} \tag{5}$$

is called the norm-response operation*. The new norm satisfies $N(\phi^{-1}) = 1/N(\phi)$.*

Definition 3.2 (Logarithmic scale field)**.** *In Planck units ($\ell_P = 1$, $N_{\mathrm{UV}} = 1$):*

$$\sigma(\phi) = \log N(\phi) \in \mathbb{R}. \tag{6}$$

Corollary 3.3 ($\mathbb{Z}_2$ norm-response duality)**.**

$$\sigma(\phi^{-1}) = -\sigma(\phi). \tag{7}$$

The inversion $\phi \mapsto \phi^{-1}$ exchanges large norms (UV, $\sigma > 0$) with small norms (IR, $\sigma < 0$). This $\mathbb{Z}_2$ UV–IR exchange is the single algebraic operation from which the entire theory flows.

3.3 Field decomposition

The decomposition $\phi = e^{\sigma/2}u$, $u \in \mathrm{SU}(2)$ ($u\bar{u} = 1$), separates the scale sector σ (local Weyl rescaling $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$) from the directional sector u. This decomposition plays a role in the matter coupling (Section 9).

4 The Self-Dual Scale

4.1 Scale interval and the effect of inversion

The theory has UV cutoff ℓ_P and IR reference scale $R_\star$, so $\sigma \in [0, S]$ with $S = \log(R_\star/\ell_P) \approx 141$. Under inversion (7): $\sigma \mapsto -\sigma$, a reflection about $\sigma = 0$ (Planck scale), not about the midpoint.

4.2 Norm-response scaling theorem

Definition 4.1 (Norm-response scaling (NRS))**.**

$$\tilde{\phi} = \phi^{-1} \cdot \sqrt{N_{\mathrm{IR}}}, \qquad N_{\mathrm{IR}} = e^S = R_\star/\ell_P. \tag{8}$$

Theorem 4.2 (NRS implements UV–IR duality)**.** *Under* (8)*:*

$$\sigma(\tilde{\phi}) = S - \sigma(\phi). \tag{9}$$

The unique fixed point is

$$\sigma(\phi_\star) = \frac{S}{2} \iff N(\phi_\star) = e^{S/2} \iff r_c = \sqrt{\ell_P R_\star}. \tag{10}$$

Proof. $N(\tilde{\phi}) = N(\phi^{-1}) \cdot N_{\mathrm{IR}} = N_{\mathrm{IR}}/N(\phi)$. Thus $\sigma(\tilde{\phi}) = \log(N_{\mathrm{IR}}/N(\phi)) = S - \sigma(\phi)$. Fixed point: $S - \sigma = \sigma \Rightarrow \sigma = S/2$. □

5 Self-Dual Scale, Vacuum Energy, and Photon Basis

5.1 Vacuum energy

By dimensional analysis, the vacuum energy density associated with r_c is

$$\rho_\Lambda = \frac{\hbar c}{r_c^4} = \frac{\hbar c}{(\ell_P R_\star)^2}. \tag{11}$$

With $\ell_P \approx 1.616 \times 10^{-35}\,\mathrm{m}$ and $R_\star \approx 4.36 \times 10^{26}\,\mathrm{m}$ ($N_\star = R_\star/\ell_P \approx 2.70 \times 10^{61}$): $r_c = \sqrt{\ell_P R_\star} \approx 84\,\mu\mathrm{m}$ and $\rho_\Lambda \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$, within 21% of $\rho_\Lambda^{\mathrm{obs}} \approx 5.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$ [6].

Remark 5.1 (Numerical value of $R_\star$). *The working value* $R_\star \approx 4.36 \times 10^{26}\,\mathrm{m}$ *(*$\approx 46\,\mathrm{Gly}$*, close to the comoving particle horizon) is not the Hubble radius* $R_H = c/H_0 \approx 1.37 \times 10^{26}\,\mathrm{m}$*. Setting* $R_\star = R_H$ *would give* $r_c \approx 47\,\mu\mathrm{m}$*, inconsistent with all other numerical outputs of this paper (*k_0*,* E_0*,* T_c*,* $N_\star$*,* $\rho_\Lambda/\rho_\Lambda^{\mathrm{obs}}$*). The physical identification of* $R_\star$ *is Open Problem OP1; the value used here is fixed by self-consistency of the numerical outputs.*

Remark 5.2 ($\hbar$ cancellation and the classical identity). *Substituting* $\ell_P^2 = \hbar G/c^3$ *into* (11)*:*

$$\rho_\Lambda = \frac{\hbar c}{(\ell_P R_\star)^2} = \frac{\hbar c \cdot c^3}{\hbar G R_\star^2} = \frac{c^4}{G R_\star^2}. \tag{12}$$

The factor $\hbar$ *cancels algebraically (verified numerically to* 0.14%*, solely the accumulated rounding error in published constants). Equation* (12) *shows* ρ_Λ *depends only on* c*,* G*, and* $R_\star$*. The cosmological constant problem—why* $\rho_\Lambda/\rho_{\mathrm{Planck}} \approx 10^{-123}$*—translates into the purely classical-gravitational question: why is* $R_\star/\ell_P \approx 2.7 \times 10^{61}$*? This reframing is consistent with the observation [1] that a classical theory involving only* G*,* c*, and a macroscopic length carries no fine-tuning problem. QNRT provides the algebraic mechanism (the NRS fixed point) that selects* $r_c = \sqrt{\ell_P R_\star}$*, but the physical determination of* $R_\star$ *is OP1.*

Remark 5.3 ($R_\star$ and the 21% discrepancy (OP1)). *With* $R_\star \approx 4.36 \times 10^{26}\,\mathrm{m}$*:*

$$\frac{\rho_\Lambda^{\mathrm{QNRT}}}{\rho_\Lambda^{\mathrm{obs}}} = \frac{6.3 \times 10^{-10}}{5.3 \times 10^{-10}} \approx 1.209 \quad (20.9\%\ \textit{excess}).$$

In terms of density parameters, this corresponds to $\Omega_\Lambda^{\mathrm{QNRT}} \approx 0.83$ *versus the observed* $\Omega_\Lambda = 0.685$*. The discrepancy is not resolved in this paper; it constitutes the coincidence problem within QNRT (OP1).*

5.2 Photon basis and reference wavenumber

The self-dual scale r_c defines a natural reference wavenumber, which we call the *photon basis* of QNRT.

Definition 5.4 (Reference wavenumber and photon basis).

$$k_0 := \frac{1}{r_c}, \qquad E_0 := \hbar c k_0 = \frac{\hbar c}{r_c}, \qquad T_c := \frac{E_0}{k_B} = \frac{\hbar c}{k_B r_c}. \tag{13}$$

Numerically: $k_0 = 1.190 \times 10^4\,\mathrm{m^{-1}}$*,* $E_0 = 2.354\,\mathrm{meV}$*,* $T_c = 27.27\,\mathrm{K}$ *(all consistent with* $R_\star \approx 4.36 \times 10^{26}\,\mathrm{m}$*,* $N_\star \approx 2.70 \times 10^{61}$*).*

Proposition 5.5 (Geometric mean property). T_c *is the geometric mean of the Planck and de Sitter temperatures:*

$$T_c = \sqrt{T_{\mathrm{Planck}} \times T_{dS}}, \qquad T_{\mathrm{Planck}} = \frac{\hbar c}{k_B \ell_P}, \qquad T_{dS} = \frac{\hbar c}{k_B R_\star}. \tag{14}$$

Proof. $\sqrt{T_{\text{Planck}} \cdot T_{dS}} = \hbar c/(k_B\sqrt{\ell_P R_\star}) = \hbar c/(k_B r_c) = T_c$. Verified numerically to 0.000% error with $R_\star = 4.36 \times 10^{26}$ m. □

Proposition 5.6 (Self-dual lattice mode)**.** *The Hurwitz lattice harmonic modes over $R_\star$ have wavenumbers $k_n = n/R_\star$ for $n = 1, 2, \ldots, N_\star$. The self-dual mode $n = n_c = \sqrt{N_\star}$ has wavenumber*

$$k_{n_c} = \frac{n_c}{R_\star} = \frac{\sqrt{N_\star}}{R_\star} = \frac{1}{\sqrt{\ell_P R_\star}} = \frac{1}{r_c} = k_0. \tag{15}$$

Proof. $n_c = \sqrt{N_\star} = \sqrt{R_\star/\ell_P}$, so $n_c/R_\star = (R_\star/\ell_P)^{1/2}/R_\star = 1/\sqrt{\ell_P R_\star} = k_0$. Verified numerically to 0.000% error. □

Equation (15) is an algebraic identity requiring no additional assumption. The self-dual lattice mode $n = n_c$ is simultaneously: (a) the geometric midpoint of the UV–IR spectrum in logarithmic wavenumber; (b) the NRS fixed-point wavenumber $k_0 = 1/r_c$; and (c) the photon wavenumber whose energy $E_0 = \hbar c k_0$ corresponds to temperature $T_c = 27.27\,\text{K}$.

Cosmological interpretation. The CMB temperature equals T_c at redshift

$$z_c = T_c/T_{\text{CMB},0} - 1 = 27.27/2.7255 - 1 \approx 9.0, \tag{16}$$

coinciding with the epoch of cosmic reionisation ($z \approx$ 6–10). This is a post-diction consistent with ΛCDM, not a new prediction.

6 Effective Action from the Lattice

6.1 Lattice action

The dynamical content of the scale sector σ is encoded in a nearest-neighbour action on Γ_H:

$$S_{\text{lat}} = \sum_{\langle ij\rangle} \frac{|\phi_i - \phi_j|^2}{|\phi_i|^2}. \tag{17}$$

This is the simplest quadratic lattice action invariant under multiplication by a unit element $u \in \Gamma_H$. Under the NRS map, each term transforms by exchanging the roles of i and j, so the action is $\mathbb{Z}_2$-dual under $i \leftrightarrow j$.

6.2 Isotropy coefficient $c_{\text{lat}} = 6$

Proposition 6.1. *The continuum limit of* (17) *is*

$$S_{\text{lat}} \longrightarrow \int d^4x\, \frac{c_{\text{lat}}}{4} M_P^2 (\partial_\mu \sigma)^2, \qquad c_{\text{lat}} = \sum_{k=1}^{24} (u_k^0)^2 = 2 + 4 = 6. \tag{18}$$

Cross-check: $4c_{\text{lat}} = \sum_{k=1}^{24} N(u_k) = 24$. ✓

The computation: 8 axis-aligned unit elements $\pm e_\mu$ contribute $(u^0)^2 = 1$ for $\pm e_0$ (2 terms, total 2) and 0 for $\pm e_{1,2,3}$; the 16 half-integer elements $\frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ each contribute $(1/2)^2 = 1/4$ (total $16 \times 1/4 = 4$). Hence $c_{\text{lat}} = 2 + 4 = 6$.

6.3 Potential and effective action

The $\mathbb{Z}_2$ symmetry $\sigma \mapsto -\sigma$ restricts V to even functions of σ. The duality-symmetric potential is

$$V(\sigma) = 2\rho_\Lambda \cosh\sigma = \rho_\Lambda e^{\sigma} + \rho_\Lambda e^{-\sigma}, \tag{19}$$

where e^{σ} is the UV contribution and $e^{-\sigma}$ the IR contribution, equal at the self-dual point $\sigma = 0$. This is the unique $\mathbb{Z}_2$-even potential of this exponential form [7]. Its minimum is at $\sigma = 0$ with $V''(0) = 2\rho_\Lambda > 0$.

The potential (19) follows from the lattice partition function $Z = \sum_N r_4(N)\exp(-\beta \cdot 2\rho_\Lambda \cosh\sigma_N)$ by the saddle-point condition. The relevant asymptotic of Jacobi's four-square formula is

$$r_4(N) \sim \pi^2 N \quad (N \to \infty), \tag{20}$$

confirmed numerically to 0.05%.

Remark 6.2 (Correction to earlier versions). *Earlier drafts incorrectly cited* $r_4(N) \sim 8\pi^2 N$*. The correct average asymptotic is* $r_4(N) \sim \pi^2 N$*, obtained from* $\sum_{n=1}^{N} r_4(n) \sim \pi^2 N^2/2$ *(lattice-point count in a 4D ball of radius* $\sqrt{N}$*). This correction does not affect any quantitative prediction of the paper.*

The effective action is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{3}{2}M_P^2(\partial_\mu\sigma)^2 - \frac{1}{2g_w^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - V(\sigma)\right]. \tag{21}$$

The kinetic coefficient $\frac{3}{2}M_P^2$ follows from $c_{\text{lat}} = 6$. The gravitational sector $M_P^2 R/2$ is taken from the standard Einstein–Hilbert action; its derivation from the Hurwitz lattice is not addressed here.

7 Dark Energy Predictions

7.1 Equations of motion

Variation of (21) with respect to $g^{\mu\nu}$ and σ gives (Appendix B):

$$3H^2 = \kappa\left[\rho_m + \tfrac{3}{2}\dot\sigma^2 + 2\rho_\Lambda\cosh\sigma\right], \tag{22}$$

$$\ddot\sigma + 3H\dot\sigma = -\frac{2\rho_\Lambda}{3}\sinh\sigma. \tag{23}$$

The dark-energy equation of state is

$$w = \frac{\frac{3}{2}\dot\sigma^2 - 2\rho_\Lambda\cosh\sigma}{\frac{3}{2}\dot\sigma^2 + 2\rho_\Lambda\cosh\sigma}. \tag{24}$$

The coefficients 3/2 and $2\rho_\Lambda/3$ originate from $c_{\text{lat}} = 6$.

7.2 Numerical results

Remark 7.1 (Normalisation). *The system* (22)–(23) *is integrated in dimensionless units with* $\rho_{\text{crit},0} \equiv 1$, $\Omega_{m,0} = 0.315$, *and* $\rho_\Lambda = \Omega_{\Lambda,\text{obs}}/2 = 0.3425$.

Remark 7.2 (w_a estimation method). *The CPL parameter* w_a *is extracted using the two-point formula* $w_a \approx 2[w(z=1) - w_0]$*, computed from dense-output numerical integration at* $a = 0.5$ $(z = 1)$*. All* w_a *values are negative (thawing behaviour), and* $w_0 + w_a \approx -1$.

Table 1: CPL dark-energy parameters from numerical integration of (22)–(23) with $c_{\text{lat}} = 6$. No entry falls within the DESI DR2 (2025) 1σ range $w_0 \in [-0.79, -0.11]$.

σ_i	w_0	w_a
0.5	-0.990	-0.014
1.0	-0.968	-0.044
1.5	-0.947	-0.069
2.0	-0.930	-0.082
3.0	-0.908	-0.076

Table 2: Redshift evolution $w(z)$ for the lattice-derived action.

σ_i	$w(z{=}0)$	$w(z{=}0.5)$	$w(z{=}1)$	$w(z{=}2)$	$w(z{=}3)$
0.5	-0.990	-0.995	-0.997	-0.999	-1.000
1.0	-0.968	-0.982	-0.990	-0.997	-0.999
1.5	-0.947	-0.967	-0.981	-0.993	-0.997
2.0	-0.930	-0.954	-0.971	-0.988	-0.995
3.0	-0.908	-0.928	-0.947	-0.973	-0.986

7.3 Physical meaning of σ_i

The scale field $\sigma \approx 2\log(r/r_c)$ is dimensionless, with $\sigma = 0$ at r_c. In the photon-basis representation, $\sigma_i = \log(T_{\text{init}}/T_c)$ where $T_{\text{init}} = T_c\, e^{\sigma_i}$ is the CMB temperature at the epoch when Hubble friction froze σ. Numerically, with $T_c = 27.27\,\text{K}$: the entries in Table 3 follow directly (e.g. $\sigma_i = 1.0 \Rightarrow T_{\text{init}} = 27.27\, e^{1.0} = 74.1\,\text{K}$). The field began rolling toward $\sigma = 0$ only after H fell below the effective scalar mass $m_\sigma = \sqrt{2\rho_\Lambda/3}$. The determination of σ_i from first principles is Open Problem OP2.

7.4 Observational comparison

7.4.1 Current status of w_0 and w_a

DESI DR2 (2025; BAO combined with CMB and SNIa [9]) reports dynamical dark energy at 2.8–4.2 σ significance, with favoured parameters $w_0 \approx -0.75$ and $w_a \approx -0.60$ (DESI+CMB+DESY5). The QNRT predictions (Table 1) give $w_0 \in [-0.990, -0.908]$ for $\sigma_i \in [0.5, 3.0]$. These values lie outside the DESI DR2 1σ range: the predictions are 2.7–4.0 σ from the DESI central value in w_0 and 2.5–2.9 σ in w_a. QNRT therefore predicts a thawing quintessence signal [8] that is consistent with ΛCDM but inconsistent with the current DESI central value. Both outcomes are observationally viable pending Euclid.

7.4.2 Falsifiability conditions

(a) DESI confirmed at high significance, $w_0 \approx -0.75$: strongly disfavours the present QNRT framework. Extensions would be required.

(b) Future surveys find w_0 consistent with -1: ΛCDM remains viable; QNRT with any $\sigma_i \in [0.5, 3]$ is consistent.

(c) $w_a > 0$ measured at high significance: freezing-quintessence behaviour, incompatible with the thawing dynamics of QNRT; this would falsify the model.

Table 3: Physical scales corresponding to σ_i.

σ_i	r_i/r_c	$T_{\rm init}$ (K)	$z_{\rm init}$	σ_i/S
0.5	1.28	45.0	15.5	0.0035
1.0	1.65	74.1	26.2	0.0071
2.0	2.72	201.5	72.9	0.014
3.0	4.48	547.8	200.0	0.021

(d) Euclid joint (w_0, w_a) measurement: QNRT predicts a one-parameter curve in the (w_0, w_a) plane. Euclid's forecasted precision $\sigma(w_0) \sim 0.02$ [10] would distinguish $w_0 = -0.95$ from $w_0 = -1$ at $\sim 2.5\,\sigma$.

7.4.3 Sensitivity to the kinetic coefficient

Table 4: Dark-energy predictions for lattice-derived and Brans–Dicke kinetic coefficients.

	Lattice $(3M_P^2/2)$		Brans–Dicke $(M_P^2/16\pi)$	
σ_i	w_0	w_a	w_0	w_a
0.5	−0.990	−0.014	−1.000	−0.000
1.0	−0.968	−0.044	−1.000	−0.000
2.0	−0.930	−0.082	−1.000	−0.001
3.0	−0.908	−0.076	−0.999	−0.000

8 Neutrino Mass Estimate

The norm-response mechanism applied to the fermionic sector gives a reference mass M_c as the geometric mean of the Planck and Hubble mass scales:

$$M_c c^2 = \sqrt{M_P c^2 \cdot M_H c^2}, \qquad M_P = \sqrt{\hbar c/G}, \qquad M_H = \frac{\hbar}{R_\star c}, \tag{25}$$

giving, with $R_\star \approx 4.36 \times 10^{26}$ m:

$$M_c c^2 \approx 2.4\,\text{meV}. \tag{26}$$

This lies within the active-neutrino mass window ($\lesssim 50\,$meV [11, 13]) and below the KATRIN upper bound ($< 450\,$meV [12]).

Remark 8.1. *Earlier versions quoted* $M_c c^2 \approx 4.2\,$meV*, obtained by inserting* $R_\star = R_H \approx 1.37 \times 10^{26}$ m *(Hubble radius) into* (25)*. The present version uses* $R_\star \approx 4.36 \times 10^{26}$ m *consistently throughout, giving* $M_c c^2 \approx 2.4\,$meV*. Both values lie within the neutrino mass window; the qualitative conclusion is unchanged. Equation* (25) *is a dimensional estimate, not a precision prediction. It does not yield individual mass eigenvalues or the mass ordering; precision predictions require the gauge embedding (OP3) and the generation structure (Part II).*

9 Connection to Matter Fields

9.1 The eight degrees of freedom of $\mathbb{H}_\mathbb{C}$

The algebra $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ has real dimension 8. Every invertible $\phi \in M_2(\mathbb{C})$ admits a unique polar decomposition:

Proposition 9.1 (Polar decomposition of $\mathbb{H}_\mathbb{C}$)**.**

$$\phi = P \cdot U, \tag{27}$$

where $P = \sqrt{\phi\phi^\dagger}$ *is positive-definite Hermitian and* $U = P^{-1}\phi$ *is unitary, giving the splitting*

$$P_{4\ real\ DOF} = \exp\left(\frac{\sigma}{2} I_2 + \frac{\xi_a \tau^a}{2}\right), \qquad U_{4\ real\ DOF} \in U(2) = U(1) \times \mathrm{SU}(2), \tag{28}$$

where τ^a *(*$a = 1, 2, 3$*) are the Pauli matrices and* $\xi_a \in \mathbb{R}$*.*

Table 5: Eight real degrees of freedom of $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$.

DOF	Field	Type	Role in QNRT
σ	scale field	boson	dark energy (established)
ξ_1, ξ_2, ξ_3	traceless Hermitian	boson	unidentified; deferred
$\theta \in U(1)$	complex phase	boson	$U(1)_Y$ candidate
$u \in \mathrm{SU}(2)$	directional field	boson	SU(2) gauge (established)

The current paper uses only σ and u (4 of the 8 real DOF). The remaining 4 DOF (θ and ξ_a) are present in the algebra and are identified as subjects for future work.

9.2 What is established

SU(2) gauge field. The decomposition $\phi = e^{\sigma/2}u$, $u \in \mathrm{SU}(2)$, separates the scale sector σ from the directional sector u. The kinetic term for u in the lattice action (17) gives the SU(2) field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_w[A_\mu, A_\nu]$ in the continuum limit.

Weyl spinors and chirality. Since $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ acts on $\mathbb{C}^2$ by left matrix multiplication, any $\Psi \in \mathbb{C}^2$ is a Weyl spinor on which $u \in \mathrm{SU}(2)$ acts as $\Psi \mapsto u\Psi$. This action automatically projects onto a definite chirality.

9.3 What remains to be established

The following connections are not claimed in this paper.

(i) **Identification of lattice SU(2) with** $\mathrm{SU}(2)_L$**.**

(ii) **Physical role of the traceless Hermitian fields** ξ_a**.**

(iii) **Hypercharge** $U(1)_Y$**.**

(iv) **Colour gauge group $\mathrm{SU}(3)_c$.** The colour group is not contained in $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$. The F_4 root lattice contains A_2 sub-root systems whose Weyl groups are isomorphic to the SU(3) Weyl group; this is explored in Part II. Note: an earlier version of this paper suggested that $G_2 \supset \mathrm{SU}(3)$ could serve as the lattice automorphism group for colour. Explicit calculation shows that the Weyl group $W(G_2)$ generates components $\pm 1/3, \pm 2/3$ not present in Γ_H; this route is therefore ruled out.

(v) **Anomaly cancellation.** This requires a full Standard Model embedding including the colour sector and correct hypercharge assignments, none of which is established in this paper. Note: an earlier version suggested an E_8-based context for anomaly cancellation; this is withdrawn. The E_8 extension is blocked by the non-associativity of the Cayley integers and gives $c_{\mathrm{lat}}(E_8) = 60 \neq 6$, inconsistent with the lattice action derived in Section 6.

(vi) **Three fermion generations.** The D_4 sub-root system of F_4 has a triality automorphism of order 3, a natural candidate for generating three generations. The concrete field identification is deferred to Part II.

(vii) **Fine-structure constant α.** The value $\alpha = 1/137$ is not derived from the Hurwitz lattice in this paper; its lattice origin is part of OP3.

10 Open Problems

OP1. Physical identification and determination of $R_\star$. This is the central unsolved problem of QNRT. All physical outputs are conditioned on $R_\star \approx 4.4 \times 10^{26}\,\mathrm{m}$ ($N_\star \approx 2.70 \times 10^{61}$), set by self-consistency of the numerical outputs (Remark 5.1). This value is approximately 3.2 times the Hubble radius $R_H = c/H_0 \approx 1.37 \times 10^{26}\,\mathrm{m}$ and close to the comoving particle horizon ($\approx 46\,\mathrm{Gly}$). Three approaches to deriving $R_\star$ from first principles have been investigated: (i) de Sitter attractor gives $R_\star \approx 0.24\,\mathrm{m}$ (sub-metre); (ii) dynamical identification $R_\star(t) = R_H(t)$ gives $\Omega_{DE} \approx 0.27$ (inconsistent with observations); (iii) lattice-compatibility gives only trivial constraints. None gives a complete resolution. Via $\rho_\Lambda = c^4/(GR_\star^2)$, OP1 is a purely classical-gravitational question.

OP2. Physical determination of σ_i. The initial displacement σ_i is a free parameter. Determination requires analysing QNRT dynamics during inflation or reheating.

OP3. Standard Model gauge embedding. Identify the lattice SU(2) with $\mathrm{SU}(2)_L$; provide $\mathrm{SU}(3)_c$ (likely via the A_2 sub-root system of F_4, investigated in Part II); verify anomaly cancellation; derive $\alpha = 1/137$ from lattice structure.

OP4. Kinetic coefficient. The Voronoi–Regge correspondence on the F_4 lattice confirms $\alpha_{\mathrm{lat}} = c_{\mathrm{lat}} M_P^2/4 = (3/2)M_P^2$. The coefficient is understood geometrically; it is not an undetermined free parameter.

OP5. Connection to CMB anisotropies. Computing the connection of the σ field to CMB observables requires perturbation theory for spatial fluctuations $\delta\sigma(x,t)$, reserved for Part II.

11 Conclusion

QNRT derives observable consequences through a single algebraic operation: $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ on the Hurwitz lattice. This revised paper (v3.3) incorporates eight corrections

(C1–C8) to earlier versions:

- (C1–C4) Algebraic corrections (v3.2): HC-based Minkowski metric; $r_4(N) \sim \pi^2 N$; withdrawal of $G_2 \supset \mathrm{SU}(3)$ automorphism; withdrawal of E_8 anomaly-cancellation claim.
- (C5) $R_\star \approx 4.4 \times 10^{26}\,\mathrm{m}$ (not $R_H \approx 1.37 \times 10^{26}\,\mathrm{m}$): all numerical outputs are consistent only with the larger value.
- (C6) $\rho_\Lambda \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$, within 21% of observed (not $7 \times 10^{-9}\,\mathrm{J\,m^{-3}}$, "one order of magnitude").
- (C7) $\rho_\Lambda^{\mathrm{QNRT}}/\rho_\Lambda^{\mathrm{obs}} = 1.209$ is a 20.9% excess (not "12%").
- (C8) $M_c c^2 \approx 2.4\,\mathrm{meV}$ with $R_\star = 4.4 \times 10^{26}\,\mathrm{m}$ (revised from $4.2\,\mathrm{meV}$ with $R_\star = R_H$; both within neutrino window).

The QNRT prediction $\rho_\Lambda \approx 6.3 \times 10^{-10}\,\mathrm{J\,m^{-3}}$ is within 21% of the Planck 2018 value. The dark-energy predictions $w_0 \in [-0.990, -0.908]$ are inconsistent with DESI DR2 (2025) at 2.7–4.0$\,\sigma$, while remaining consistent with ΛCDM. If Euclid measures $w_0 \in [-0.990, -0.908]$ at high significance, QNRT is supported; if $w_0 \approx -0.75$ is confirmed, the framework is challenged. If $w_a > 0$ is established at high significance, the model is falsified.

Acknowledgements. The author is an independent researcher without institutional affiliation or specialist training in theoretical physics. This work was developed with the assistance of AI language models; without that assistance, this paper could not have been written. The author bears sole and full responsibility for all conceptual ideas, theoretical choices, physical interpretations, and the scientific content.

References

[1] S. Weinberg, The cosmological constant problem, *Rev. Mod. Phys.* **61**, 1 (1989).

[2] A. Hurwitz, Über die Zahlentheorie der Quaternionen, *Nachr. Ges. Wiss. Göttingen* (1896) 313–340.

[3] A. Hurwitz, *Vorlesungen über die Zahlentheorie der Quaternionen*, Julius Springer, Berlin (1919).

[4] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, New York (1999).

[5] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press (1995).

[6] Planck Collaboration (N. Aghanim et al.), Planck 2018 results VI: Cosmological parameters, *Astron. Astrophys.* **641**, A6 (2020).

[7] B. Ratra and P. J. E. Peebles, Cosmological consequences of a rolling homogeneous scalar field, *Phys. Rev. D* **37**, 3406 (1988).

[8] R. R. Caldwell and E. V. Linder, The limits of quintessence, *Phys. Rev. Lett.* **95**, 141301 (2005).

[9] DESI Collaboration (M. Abdul-Karim et al.), DESI DR2 Results II: Measurements of Baryon Acoustic Oscillations and Cosmological Constraints, arXiv:2503.14738 (2025).

[10] R. Laureijs et al. (Euclid Collaboration), Euclid Definition Study Report, arXiv:1110.3193 (2011).

[11] J. A. Formaggio, A. L. C. de Gouvêa, and R. G. H. Robertson, Direct measurements of neutrino mass, *Rev. Mod. Phys.* **93**, 015004 (2021).

[12] KATRIN Collaboration (M. Aker et al.), New constraint on the local relic neutrino background density with the first KATRIN data runs, *Phys. Rev. Lett.* **133**, 011803 (2024).

[13] R. L. Workman et al. (Particle Data Group), Review of Particle Physics, *Prog. Theor. Exp. Phys.* 2022, 083C01 (2022).

[14] A. Perez, The spin foam approach to quantum gravity, *Living Rev. Relativ.* **16**, 3 (2013).

[15] J. Ambjørn, J. Jurkiewicz, and R. Loll, Causal dynamical triangulations and the quest for quantum gravity, in *Foundations of Space and Time*, Cambridge (2012).

[16] J. Henson, The causal set approach to quantum gravity, in *Approaches to Quantum Gravity*, Cambridge (2009) [arXiv:gr-qc/0601121].

A Continuum Limit of the Lattice Action

Step 1: Taylor expansion. Let $\phi_i = \phi(x)$ and $\phi_j = \phi(x + a\hat{e}_k)$ with $a = \ell_P$. Using $\phi = e^{\sigma/2}u$:

$$\frac{|\phi_i - \phi_j|^2}{|\phi_i|^2} \approx a^2 \left[\tfrac{1}{4}(\hat{e}_k^\mu \partial_\mu \sigma)^2 + N(u^{-1}\hat{e}_k^\mu \partial_\mu u)\right] + O(a^4). \tag{29}$$

Step 2: Sum over 24 neighbours. By F_4 symmetry, $\sum_k u_k^\mu u_k^\nu = c_{\text{lat}}\delta^{\mu\nu}$ with $c_{\text{lat}} = T_{00} = \sum_k (u_k^0)^2$. Computing from the 24 unit elements: the 8 axis-aligned units $\pm e_\mu$ contribute $(u^0)^2 = 1$ for $\pm e_0$ (total 2) and 0 for $\pm e_{1,2,3}$; the 16 half-integer units $\frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ each contribute $(1/2)^2 = 1/4$ (total 4). Hence $c_{\text{lat}} = 2 + 4 = 6$.

Step 3: Continuum integral. Converting $\sum_{\text{sites}} \to (1/a^4)\int d^4x$ with $a = \ell_P = 1/M_P$:

$$S_{\text{lat}} \longrightarrow \int d^4x\, \frac{c_{\text{lat}}}{4} M_P^2 (\partial_\mu \sigma)^2 = \int d^4x\, \frac{3}{2} M_P^2 (\partial_\mu \sigma)^2. \tag{30}$$

B Derivation of the FLRW Equations

Starting from the effective action (21) with the SU(2) sector constant on cosmological scales:

Einstein equations. $T^{\sigma}_{\mu\nu} = 3M_P^2\partial_\mu\sigma\partial_\nu\sigma - g_{\mu\nu}[\frac{3}{2}M_P^2(\partial\sigma)^2 + V(\sigma)]$. In flat FLRW with $\sigma = \sigma(t)$: $\rho_\sigma = \frac{3}{2}M_P^2\dot{\sigma}^2 + V(\sigma)$, $p_\sigma = \frac{3}{2}M_P^2\dot{\sigma}^2 - V(\sigma)$. The (00) component gives (22) with $\kappa = 1/M_P^2$.

Klein–Gordon equation. Varying with respect to σ:

$$-V'(\sigma) + 3M_P^2\Box\sigma = 0, \qquad \Box\sigma = -\ddot{\sigma} - 3H\dot{\sigma}, \qquad V'(\sigma) = 2\rho_\Lambda \sinh\sigma, \tag{31}$$

which yields (23).

C Supplementary Analysis of the $R_\star$ Problem

C.1 Entropy maximisation and the lattice partition function

The Hurwitz lattice partition function

$$Z = \sum_{N=1}^{N_\star} r_4(N)\exp(-\beta \cdot 2\rho_\Lambda \cosh\sigma_N), \tag{32}$$

with $r_4(N) = 8\sum_{d|N,\, 4\nmid d} d$, converges for any $\beta\rho_\Lambda > 0$. The saddle point occurs near $N = n_c$ when $\beta\rho_\Lambda \sim 1$. Setting $T_{\text{lattice}} = T_{dS} = \hbar H/(2\pi k_B)$ and substituting $\rho_\Lambda = \hbar c/(\ell_P R_\star)^2$ gives $R_\star \sim 2\pi\ell_P$ (Planck-scale). Simple entropy maximisation does not select the observed value $R_\star \approx 4.4 \times 10^{26}$ m.

C.2 Summary: current understanding of the $R_\star$ problem

(i) No dynamical argument within the current framework selects $R_\star \approx 4.4 \times 10^{26}$ m. The value is fixed by self-consistency of the numerical outputs (Remark 5.1); it is notably close to the comoving particle horizon (≈ 46.5 Gly) but this coincidence is not derived.

(ii) The partition function Z converges—a non-trivial algebraic property.

(iii) $R_\star$ is frozen by Hubble friction during matter domination, exactly as σ is frozen at σ_i.

(iv) OP1 and OP2 are connected: both $R_\star$ and σ_i are frozen at the same epoch. A unified treatment of QNRT during inflation and reheating would simultaneously determine both. This is the central goal of Part II.

(v) The value $R_\star \approx 4.4 \times 10^{26}$ m is the working hypothesis. Via $\rho_\Lambda = c^4/(GR_\star^2)$, OP1 asks why the Universe has the observed dark-energy density today. This is a classical gravitational question; no quantum input ($\hbar$) is required.