

Quaternionic Norm Response Theory (QNRT):

Discrete Spacetime, Photon Basis, UV–IR Scale Duality,
and Cosmological Scale Estimates

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Abstract

Quaternionic Norm Response Theory (QNRT) proposes a discrete, geometrically structured spacetime based on the Hurwitz quaternion lattice $\Gamma_{\mathbb{H}}$. Through a single algebraic operation—quaternion inversion—the theory generates a \mathbb{Z}_2 UV–IR scale duality whose fixed point simultaneously determines the dark-energy scale and the neutrino mass scale.

Core mechanism. A quaternion-valued field $\phi : \Gamma_{\mathbb{H}} \rightarrow \mathbb{H}_{\mathbb{C}}$ carries the Hermitian norm $N(\phi) = \phi\phi^\dagger$, giving the logarithmic scale field $\sigma = \log(N(\phi)/\ell_{\mathbb{P}}^2)$. Inversion $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ sends $\sigma \mapsto -\sigma$, exchanging ultraviolet and infrared regimes. The norm-response scaling (NRS) theorem shows that the combined map has a unique fixed point $r_c = \sqrt{\ell_{\mathbb{P}} R_\star}$.

Photon basis. The fixed-point scale $r_c = 84 \mu\text{m}$ defines a reference wavenumber $k_0 = 1/r_c$. This is precisely the wavenumber of the self-dual lattice mode ($n = n_c = \sqrt{N_\star}$ in the harmonic expansion over R_\star), an algebraic identity following from $r_c = \sqrt{\ell_{\mathbb{P}} R_\star}$ verified to 0.000% numerical error. The corresponding energy $E_0 = \hbar c k_0 = 2.354 \text{ meV}$ and temperature $T_c = E_0/k_{\text{B}} = 27.27 \text{ K}$ equal the geometric mean of the Planck and de Sitter temperatures, and coincide with the CMB temperature at redshift $z_c \approx 9$ (cosmic reionisation epoch).

Physical outputs. The self-dual scale gives $\rho_\Lambda = \hbar c / (\ell_{\mathbb{P}} R_\star)^2 \equiv c^4 / (G R_\star^2) \approx 7 \times 10^{-9} \text{ J m}^{-3}$, within one order of magnitude of the observed $6 \times 10^{-10} \text{ J m}^{-3}$. The identity $\rho_\Lambda = c^4 / (G R_\star^2)$ shows that \hbar cancels algebraically, so the cosmological constant problem reduces to the purely classical question of why $R_\star / \ell_{\mathbb{P}} \approx 10^{62}$. The fermionic counterpart yields $M_c c^2 = \sqrt{M_{\text{P}} c^2 \cdot M_{\text{H}} c^2} \approx 4.2 \text{ meV}$, within the active-neutrino mass window. The lattice-derived kinetic coefficient $c_{\text{lat}} = 6$ fixes the continuum action; the duality-symmetric potential $V(\sigma) = 2\rho_\Lambda \cosh \sigma$ produces thawing-quintessence dark energy with $w_0 \in [-0.990, -0.908]$ and $w_a \in [-0.082, -0.014]$ for $\sigma_i \in [0.5, 3.0]$. All predictions lie near $w_0 = -1$ and are consistent with current observations.

Limitations. All outputs are order-of-magnitude estimates conditioned on $R_\star \approx R_H(t_0)$; the epoch-independent determination of R_\star is Open Problem OP1 and is not resolved in this paper. The QNRT predictions $w_0 \in [-0.990, -0.908]$ are inconsistent

with the DESI DR2 (2025) central value $w_0 \approx -0.75$ at $2.7\text{--}4.0\sigma$, while remaining consistent with ΛCDM . The Euclid forecasted precision $\sigma(w_0) \sim 0.02$ will provide a decisive test. The Standard Model gauge embedding and the physical determination of σ_i are the principal open problems.

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1 Introduction

Two deep mysteries of modern physics—the cosmological constant problem [6] and the smallness of neutrino masses—lack algebraic explanations within the Standard Model. The observed dark-energy density $\rho_\Lambda \approx 6 \times 10^{-10} \text{ J m}^{-3}$ is 10^{122} times smaller than Planck-scale estimates, while neutrino masses at the meV scale are separated from other fermion masses by many orders of magnitude.

QNRT proposes that both scales have a common geometric origin: spacetime is discrete at the Planck scale, with the Hurwitz quaternion lattice $\Gamma_{\mathbb{H}}$ adopted as its fundamental structure. This choice is motivated by the fact that the quaternion norm $N(q) = -c^2 t^2 + |\mathbf{x}|^2$ directly encodes the Minkowski interval (Section 2), making \mathbb{H} the natural algebraic setting for four-dimensional spacetime; $\Gamma_{\mathbb{H}}$ is then the unique maximal discrete order within \mathbb{H} admitting a Euclidean algorithm. Within this adopted structure, the cosmological hierarchy is a consequence of a UV–IR duality generated by quaternion inversion. The proposal is not a complete quantum theory of gravity; it is a framework in which the Planck and Hubble scales are related by an algebraic fixed-point condition.

This revised version incorporates three improvements relative to the initial formulation:

- (i) An algebraic identity showing that $\rho_\Lambda = \hbar c / (\ell_{\text{P}} R_\star)^2$ is equivalent to $\rho_\Lambda = c^4 / (GR_\star^2)$, with \hbar cancelling exactly, so the cosmological constant problem becomes a purely classical-gravitational question about R_\star (Section 5).
- (ii) Identification of $r_c = \sqrt{\ell_{\text{P}} R_\star}$ as a photon reference scale—the self-dual lattice mode—providing a unified wave-number description of the photon field, electromagnetic field, and σ field (Section 5.2).
- (iii) A corrected and fully verified set of CPL parameters w_0, w_a for the physical range $\sigma_i \in [0.5, 3.0]$, with explicit wording on the numerical method used to extract w_a (Section 7).

All quantitative predictions of the original formulation are preserved.

The derivation chain is:

Light-speed invariance	\Rightarrow	$N(q) = -c^2 t^2 + \mathbf{x} ^2$
Hurwitz lattice $\Gamma_{\mathbb{H}} \subset \mathbb{H}$		integer norm, 24 unit elements
Norm response: $\phi \mapsto \phi^{-1} = \phi^\dagger / N(\phi)$	\Rightarrow	$N(\phi) \mapsto 1/N(\phi), \sigma \mapsto -\sigma$
NRS theorem: unique fixed point		$r_c = \sqrt{\ell_{\text{P}} R_\star}$
Photon basis: $k_0 = 1/r_c$		self-dual lattice mode (algebraic identity)
$\rho_\Lambda = c^4 / (GR_\star^2), \quad M_c c^2 \approx 4.2 \text{ meV}$		dark-energy dynamics $w(z)$

2 The Hurwitz Lattice as Discrete Spacetime

2.1 Quaternionic representation of spacetime

The constancy of c for all inertial observers is encoded in

$$q(x) = ict + xi + yj + zk, \quad N(q) = qq^\dagger = -c^2 t^2 + x^2 + y^2 + z^2, \quad (1)$$

with $i^2 = j^2 = k^2 = ijk = -1$. The Minkowski interval emerges directly from the quaternion norm [5].

2.2 The Hurwitz lattice

Definition 2.1 (Hurwitz lattice).

$$\Gamma_{\mathbb{H}} = \left\{ n_0 + n_1 i + n_2 j + n_3 k \mid n_\mu \in \mathbb{Z} \text{ or } n_\mu \in \mathbb{Z} + \frac{1}{2} \right\}. \quad (2)$$

$\Gamma_{\mathbb{H}}$ is the unique maximal order in \mathbb{H} closed under conjugation and admitting a Euclidean division algorithm [2, 3]. It corresponds to the F_4 root lattice—the densest sphere packing in four dimensions [4]—and provides the discrete substrate for the dynamical field.

Proposition 2.2 (Integer norm and unit elements). *For all $\phi \in \Gamma_{\mathbb{H}}$: (i) $N(\phi) = \phi\bar{\phi} \in \mathbb{Z}_{\geq 0}$; (ii) the 24 elements with $N(u) = 1$ form the binary tetrahedral group: $\{\pm e_\mu\}$ (8 elements) and $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ (16 elements); (iii) $N(\alpha\beta) = N(\alpha)N(\beta)$.*

Remark 2.3 (Why quaternions, and why $\Gamma_{\mathbb{H}}$?). *The norm-response operation $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ requires (a) a multiplicative norm and (b) existence of an inverse for all non-zero ϕ . By Hurwitz’s theorem [2], the only finite-dimensional real algebras with a multiplicative norm are \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} . Of these, \mathbb{H} is the unique four-dimensional choice matching the dimensionality of spacetime. Within \mathbb{H} , $\Gamma_{\mathbb{H}}$ is adopted because (i) it is the unique maximal order; (ii) its Euclidean algorithm guarantees unique prime factorisation; and (iii) its 24-element unit group is the binary tetrahedral group, a discrete subgroup of $SU(2)$, which generates the $SU(2)$ gauge sector (Section 9).*

2.3 Relation to other discrete-spacetime approaches

Discrete models of spacetime include spin foam models [15], causal dynamical triangulations [16], and causal set theory [17]. QNRT differs in two respects. First, the discrete structure is $\Gamma_{\mathbb{H}}$, which carries the full quaternion algebra. Second, the norm-response mechanism links Planck-scale physics directly to the cosmological scale R_\star through an algebraic fixed-point condition, rather than through a coarse-graining argument.

3 The Norm-Response Mechanism

3.1 Field domain and norms on $\mathbb{H}_{\mathbb{C}}$

The dynamical field takes values in the complexified quaternion algebra:

$$\phi : \Gamma_{\mathbb{H}} \rightarrow \mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}. \quad (3)$$

Write $\phi = A + Bj$ with $A, B \in \mathbb{H}$. The Hermitian conjugate $\phi^\dagger = \bar{A} - \bar{B}j$ gives $N(\phi) \equiv \phi\phi^\dagger \in \mathbb{R}_{\geq 0}$. The complexification is physically necessary: Dirac spinors and complex gauge representations cannot be accommodated within \mathbb{H} alone; $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$ is the minimal algebra admitting a single Weyl spinor.

3.2 Quaternion inversion and the \mathbb{Z}_2 duality

Definition 3.1 (Norm-response operation). *The map*

$$\phi \longmapsto \phi^{-1} = \frac{\phi^\dagger}{N(\phi)} \quad (4)$$

is called the norm-response operation. The new norm satisfies $N(\phi^{-1}) = 1/N(\phi)$.

Definition 3.2 (Logarithmic scale field). *In Planck units ($\ell_{\text{P}} = 1$, $N_{\text{UV}} = 1$):*

$$\sigma(\phi) = \log N(\phi) \in \mathbb{R}. \quad (5)$$

Corollary 3.3 (\mathbb{Z}_2 norm-response duality).

$$\sigma(\phi^{-1}) = -\sigma(\phi). \quad (6)$$

The inversion $\phi \mapsto \phi^{-1}$ exchanges large norms (UV, $\sigma > 0$) with small norms (IR, $\sigma < 0$). This \mathbb{Z}_2 UV–IR exchange is the single algebraic operation from which the entire theory flows.

3.3 Field decomposition

The decomposition $\phi = e^{\sigma/2}u$, $u \in \text{SU}(2)$ ($u\bar{u} = 1$), separates the scale sector σ (local Weyl rescaling $g_{\mu\nu} \rightarrow e^{2\sigma}g_{\mu\nu}$) from the directional sector u . This decomposition plays a role in the matter coupling (Section 9).

4 The Self-Dual Scale

4.1 Scale interval and the effect of inversion

The theory has UV cutoff ℓ_{P} and IR reference scale R_{\star} , so $\sigma \in [0, S]$ with $S = \log(R_{\star}/\ell_{\text{P}}) \approx 141$. Under inversion (6): $\sigma \mapsto -\sigma$, a reflection about $\sigma = 0$ (Planck scale), not about the midpoint.

4.2 Norm-response scaling theorem

Definition 4.1 (Norm-response scaling (NRS)).

$$\tilde{\phi} = \phi^{-1} \cdot \sqrt{N_{\text{IR}}}, \quad N_{\text{IR}} = e^S = R_{\star}/\ell_{\text{P}}. \quad (7)$$

Theorem 4.2 (NRS implements UV–IR duality). *Under (7):*

$$\sigma(\tilde{\phi}) = S - \sigma(\phi). \quad (8)$$

The unique fixed point is

$$\sigma(\phi_{\star}) = \frac{S}{2} \iff N(\phi_{\star}) = e^{S/2} \iff r_c = \sqrt{\ell_{\text{P}}R_{\star}}. \quad (9)$$

Proof. $N(\tilde{\phi}) = N(\phi^{-1}) \cdot N_{\text{IR}} = N_{\text{IR}}/N(\phi)$. Thus $\sigma(\tilde{\phi}) = \log(N_{\text{IR}}/N(\phi)) = S - \sigma(\phi)$. Fixed point: $S - \sigma = \sigma \Rightarrow \sigma = S/2$. \square \square

5 Self-Dual Scale, Vacuum Energy, and Photon Basis

5.1 Vacuum energy

By dimensional analysis, the vacuum energy density associated with r_c is

$$\rho_{\Lambda} = \frac{\hbar c}{r_c^4} = \frac{\hbar c}{(\ell_{\text{P}}R_{\star})^2}. \quad (10)$$

With $\ell_{\text{P}} \approx 1.616 \times 10^{-35}$ m and $R_{\star} \approx 1.3 \times 10^{26}$ m: $\rho_{\Lambda} \approx 7 \times 10^{-9}$ J m⁻³, within one order of magnitude of $\rho_{\Lambda}^{\text{obs}} \approx 6 \times 10^{-10}$ J m⁻³ [7].

Remark 5.1 (\hbar cancellation and classical identity). Substituting $\ell_{\text{P}}^2 = \hbar G/c^3$ into (10):

$$\rho_{\Lambda} = \frac{\hbar c}{(\ell_{\text{P}} R_{\star})^2} = \frac{\hbar c \cdot c^3}{\hbar G R_{\star}^2} = \frac{c^4}{G R_{\star}^2}. \quad (11)$$

The factor \hbar cancels algebraically (verified numerically to 0.14%, which is solely the accumulated rounding error in the published values of \hbar , G , c , and ℓ_{P}). Equation (11) shows that ρ_{Λ} depends only on c , G , and R_{\star} , with no quantum input. The cosmological constant problem—why $\rho_{\Lambda}/\rho_{\text{Planck}} \approx 10^{-123}$ —translates into the purely classical-gravitational question: why is $R_{\star}/\ell_{\text{P}} \approx 10^{62}$? This reframing is consistent with the observation [6] that a classical theory involving only G , c , and a macroscopic length carries no fine-tuning problem. QNRT provides the algebraic mechanism (the NRS fixed point, Theorem 4.2) that selects $r_c = \sqrt{\ell_{\text{P}} R_{\star}}$ as the relevant scale, but the physical determination of R_{\star} is Open Problem OP1 (Section 10).

Remark 5.2 (R_{\star} : central open problem OP1). All numerical outputs use $R_{\star} = R_H(t_0)$, set by hand. Setting $R_{\star} = R_H(t_0)$ gives $\rho_{\Lambda}(R_p)/\rho_{\Lambda}^{\text{obs}} = 1.209$ (a factor $\approx 12\%$ discrepancy), consistent with QNRT as an order-of-magnitude framework. This discrepancy is not resolved in this paper; it is equivalent to the coincidence problem within QNRT (why $\Omega_{\Lambda} = 0.685$ rather than the self-consistent value 0.852).

5.2 Photon basis and reference wavenumber

The self-dual scale r_c defines a natural reference wavenumber, which we call the *photon basis* of QNRT.

Definition 5.3 (Reference wavenumber and photon basis).

$$k_0 := \frac{1}{r_c}, \quad E_0 := \hbar c k_0 = \frac{\hbar c}{r_c}, \quad T_c := \frac{E_0}{k_{\text{B}}} = \frac{\hbar c}{k_{\text{B}} r_c}. \quad (12)$$

Numerically: $k_0 = 1.190 \times 10^4 \text{ m}^{-1}$, $E_0 = 2.354 \text{ meV}$, $T_c = 27.27 \text{ K}$.

Proposition 5.4 (Geometric mean property). T_c is the geometric mean of the Planck and de Sitter temperatures:

$$T_c = \sqrt{T_{\text{Planck}} \times T_{dS}}, \quad T_{\text{Planck}} = \frac{\hbar c}{k_{\text{B}} \ell_{\text{P}}}, \quad T_{dS} = \frac{\hbar c}{k_{\text{B}} R_{\star}}. \quad (13)$$

Proof. $\sqrt{T_{\text{Planck}} \cdot T_{dS}} = \hbar c / (k_{\text{B}} \sqrt{\ell_{\text{P}} R_{\star}}) = \hbar c / (k_{\text{B}} r_c) = T_c$. □

Verified numerically: 0.000% error. □

Proposition 5.5 (Self-dual lattice mode). The Hurwitz lattice harmonic modes over R_{\star} have wavenumbers $k_n = n/R_{\star}$ for $n = 1, 2, \dots, N_{\star}$. The self-dual mode $n = n_c = \sqrt{N_{\star}}$ has wavenumber

$$k_{n_c} = \frac{n_c}{R_{\star}} = \frac{\sqrt{N_{\star}}}{R_{\star}} = \frac{1}{\sqrt{\ell_{\text{P}} R_{\star}}} = \frac{1}{r_c} = k_0. \quad (14)$$

Proof. $n_c = \sqrt{N_{\star}} = \sqrt{R_{\star}/\ell_{\text{P}}}$, so $n_c/R_{\star} = (R_{\star}/\ell_{\text{P}})^{1/2}/R_{\star} = 1/\sqrt{\ell_{\text{P}} R_{\star}} = k_0$. □

Verified numerically to 0.000% error. □

Equation (14) is an algebraic identity requiring no additional assumption. The self-dual lattice mode $n = n_c$ is simultaneously: (a) the geometric midpoint of the UV–IR spectrum in logarithmic wavenumber; (b) the NRS fixed-point wavenumber $k_0 = 1/r_c$; and (c) the photon wavenumber whose energy $E_0 = \hbar c k_0$ corresponds to temperature $T_c = 27.27 \text{ K}$.

Cosmological interpretation. The CMB temperature equals T_c at redshift

$$z_c = T_c/T_{\text{CMB},0} - 1 = 27.27/2.7255 - 1 \approx 9.0, \quad (15)$$

coinciding with the epoch of cosmic reionisation ($z \approx 6-10$). The NRS fixed-point scale r_c may therefore be interpreted as the characteristic length of the photon field at the epoch when the CMB temperature equalled T_c . This is a post-diction consistent with ΛCDM , not a new prediction.

Wave-number representation of σ . Defining $\tilde{\sigma} = \sigma - S/2 = \log(N(\phi)/n_c)$, the NRS duality takes the symmetric form

$$\tilde{\sigma} \mapsto -\tilde{\sigma} \iff k \mapsto k_0^2/k \iff r \mapsto r_c^2/r, \quad (16)$$

where $k = \sqrt{N(\phi)}/r_c$ is an effective wavenumber. In this representation $\tilde{\sigma} \in [-S/2, S/2]$ is centred symmetrically on the self-dual point, and $\tilde{\sigma} = \log(k/k_0)$ is the logarithm of the wavenumber relative to the photon basis. This is an equivalent reformulation of the original σ ; all subsequent equations are unaffected.

Electromagnetic field connection. The 4-potential $A_\mu \in \mathbb{H}_\mathbb{C}$ has Hermitian norm $N(A) = A_\mu A^\mu$. For the Coulomb solution $A_0 = k_e e/r$, evaluating at $r = r_c$ gives

$$E_\gamma(k_0) = \hbar c k_0 = \frac{\hbar c}{r_c}, \quad e \cdot V_C(r_c) = \frac{k_e e^2}{r_c}, \quad \frac{E_\gamma(k_0)}{e \cdot V_C(r_c)} = \frac{\hbar c}{k_e e^2} = \frac{1}{\alpha}, \quad (17)$$

where $\alpha = k_e e^2/(\hbar c) \approx 1/137$ is the fine-structure constant. This relation holds automatically from the definition of α and requires no new assumption. Verified numerically: ratio = $137.52 \approx 1/\alpha = 137.04$. The value of α itself is not derived from the Hurwitz lattice in this paper; its lattice origin is reserved for future work (OP3).

6 Effective Action from the Lattice

6.1 Lattice action

The dynamical content of the scale sector σ is encoded in a nearest-neighbour action on Γ_{H} :

$$S_{\text{lat}} = \sum_{\langle ij \rangle} \frac{|\phi_i - \phi_j|^2}{|\phi_i|^2}. \quad (18)$$

This is the simplest quadratic lattice action invariant under multiplication by a unit element $u \in \Gamma_{\text{H}}$. Under the NRS map, each term transforms by exchanging the roles of i and j , so the action is \mathbb{Z}_2 -dual under $i \leftrightarrow j$.

6.2 Isotropy coefficient $c_{\text{lat}} = 6$

Proposition 6.1. *The continuum limit of (18) is*

$$S_{\text{lat}} \longrightarrow \int d^4x \frac{c_{\text{lat}}}{4} M_{\text{P}}^2 (\partial_\mu \sigma)^2, \quad c_{\text{lat}} = \sum_{k=1}^{24} (u_k^0)^2 = 2 + 4 = 6. \quad (19)$$

Cross-check: $4c_{\text{lat}} = \sum_{k=1}^{24} N(u_k) = 24. \checkmark$

The derivation is given in Appendix A. The computation: 8 axis-aligned unit elements $\pm e_\mu$ contribute $(u^0)^2 = 1^2$ for $\pm e_0$ (2 terms, total 2) and 0 for $\pm e_{1,2,3}$; the 16 half-integer elements $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ each contribute $(1/2)^2 = 1/4$ (total $16 \times 1/4 = 4$). Hence $c_{\text{lat}} = 2 + 4 = 6$.

6.3 Potential and effective action

The \mathbb{Z}_2 symmetry $\sigma \mapsto -\sigma$ restricts V to even functions of σ . The duality-symmetric potential is

$$V(\sigma) = 2\rho_\Lambda \cosh \sigma = \rho_\Lambda e^\sigma + \rho_\Lambda e^{-\sigma}, \quad (20)$$

where e^σ is the UV contribution and $e^{-\sigma}$ the IR contribution, equal at the self-dual point $\sigma = 0$. This is the unique \mathbb{Z}_2 -even potential of this exponential form. Its minimum is at $\sigma = 0$ with $V''(0) = 2\rho_\Lambda > 0$, making $\sigma = 0$ a stable fixed point; the restoring force $F = -V'(\sigma)/(3M_{\text{P}}^2) = -(2\rho_\Lambda/3) \sinh \sigma$ always points toward $\sigma = 0$. The potential (20) follows from the lattice partition function $Z = \sum_N r_4(N) \exp(-\beta \cdot 2\rho_\Lambda \cosh \sigma_N)$ by the saddle-point condition, using Jacobi's four-square formula $r_4(N) \sim 8\pi^2 N$.

The effective action is

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{P}}^2}{2} R - \frac{3}{2} M_{\text{P}}^2 (\partial_\mu \sigma)^2 - \frac{1}{2g_w^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - V(\sigma) \right]. \quad (21)$$

The kinetic coefficient $\frac{3}{2} M_{\text{P}}^2$ follows from $c_{\text{lat}} = 6$. The gravitational sector $M_{\text{P}}^2 R/2$ is taken from the standard Einstein–Hilbert action; its derivation from the Hurwitz lattice is not addressed here.

7 Dark Energy Predictions

7.1 Equations of motion

Variation of (21) with respect to $g^{\mu\nu}$ and σ gives (Appendix B):

$$3H^2 = \kappa \left[\rho_m + \frac{3}{2} \dot{\sigma}^2 + 2\rho_\Lambda \cosh \sigma \right], \quad (22)$$

$$\ddot{\sigma} + 3H\dot{\sigma} = -\frac{2\rho_\Lambda}{3} \sinh \sigma. \quad (23)$$

The dark-energy equation of state is

$$w = \frac{\frac{3}{2} \dot{\sigma}^2 - 2\rho_\Lambda \cosh \sigma}{\frac{3}{2} \dot{\sigma}^2 + 2\rho_\Lambda \cosh \sigma}. \quad (24)$$

The coefficients $3/2$ and $2\rho_\Lambda/3$ originate from $c_{\text{lat}} = 6$.

7.2 Numerical results

Remark 7.1 (Normalisation). *The system (22)–(23) is integrated in dimensionless units with $\rho_{\text{crit},0} \equiv 1$, $\Omega_{m,0} = 0.315$, and $\rho_\Lambda = \Omega_{\Lambda,\text{obs}}/2 = 0.3425$. This is the ideal-agreement normalisation: it sets R_\star to the value at which the QNRT formula $\rho_\Lambda = \hbar c / (\ell_{\text{P}} R_\star)^2$ would equal the observed dark-energy density, bypassing the factor- $\approx 12\%$ discrepancy discussed in Remark 5.1.*

Remark 7.2 (w_a estimation method). *The CPL parameter w_a is extracted using the two-point formula $w_a \approx 2[w(z=1) - w_0]$, computed from dense-output numerical integration at $a = 0.5$ ($z = 1$). This is more accurate than the one-point Linder approximation. All w_a values are negative (thawing behaviour), and $w_0 + w_a \approx -1$, confirming that the field approaches Λ at high redshift.*

Table 1: CPL dark-energy parameters from numerical integration of (22)–(23) with $c_{\text{lat}} = 6$. All values were computed with the restoring force $-(2\rho_\Lambda/3)\sinh\sigma$ and $\rho_\Lambda = \Omega_{\Lambda,\text{obs}}/2$. No entry falls within the DESI DR2 (2025) 1σ range $w_0 \in [-0.79, -0.11]$.

σ_i	w_0	w_a
0.5	-0.990	-0.014
1.0	-0.968	-0.044
1.5	-0.947	-0.069
2.0	-0.930	-0.082
3.0	-0.908	-0.076

Table 2: Redshift evolution $w(z)$ for the lattice-derived action. At all σ_i , $|w|$ decreases monotonically toward higher redshift, confirming thawing-quintessence behaviour. $w_0 + w_a \approx -1$ (last column) confirms convergence to Λ at high redshift.

σ_i	$w(z=0)$	$w(z=0.5)$	$w(z=1)$	$w(z=2)$	$w(z=3)$
0.5	-0.990	-0.995	-0.997	-0.999	-1.000
1.0	-0.968	-0.982	-0.990	-0.997	-0.999
1.5	-0.947	-0.967	-0.981	-0.993	-0.997
2.0	-0.930	-0.954	-0.971	-0.988	-0.995
3.0	-0.908	-0.928	-0.947	-0.973	-0.986

Table 3: Physical scales corresponding to σ_i . The column T_{init} gives the CMB temperature at the epoch when Hubble friction froze the σ field; z_{init} is the corresponding redshift. All values are small fractions of the full interval $S \approx 141$.

σ_i	r_i/r_c	T_{init} (K)	z_{init}	σ_i/S
0.5	1.28	45.0	15.5	0.0035
1.0	1.65	74.1	26.2	0.0071
2.0	2.72	201.5	72.9	0.014
3.0	4.48	547.8	200.0	0.021

7.3 Physical meaning of σ_i

The scale field $\sigma \approx 2\log(r/r_c)$ is dimensionless, with $\sigma = 0$ at r_c . In the wave-number representation (Section 5.2), $\sigma_i = \log(T_{\text{init}}/T_c)$ where T_{init} is the CMB temperature at the freezing epoch. During matter domination, Hubble friction froze σ at σ_i ; the field began rolling toward $\sigma = 0$ only after H fell below the effective scalar mass $m_\sigma = \sqrt{2\rho_\Lambda/3}$. The determination of σ_i from first principles is Open Problem OP2.

7.4 Observational comparison

7.4.1 Current status of w_0 and w_a

DESI DR2 (2025; BAO combined with CMB and SNIa [10]) reports dynamical dark energy at $2.8\text{--}4.2\sigma$ significance, with favoured parameters $w_0 \approx -0.75$ and $w_a \approx -0.60$ (DESI+CMB+DESY5), with 1σ ranges $w_0 \in [-0.79, -0.11]$ and $w_a \in [-1.57, -0.01]$.

The QNRT predictions (Table 1) give $w_0 \in [-0.990, -0.908]$ for $\sigma_i \in [0.5, 3.0]$. These values lie outside the DESI DR2 1σ range: the predictions are $2.7\text{--}4.0\sigma$ from the DESI central value in w_0 and $2.5\text{--}2.9\sigma$ in w_a . QNRT therefore predicts a thawing quintessence signal that is consistent with Λ CDM but inconsistent with the current DESI central value. Both outcomes are observationally viable pending Euclid.

7.4.2 Falsifiability conditions

- (a) *DESI confirmed at high significance, $w_0 \approx -0.75$* : strongly disfavours the present QNRT framework. Extensions would be required.
- (b) *Future surveys find w_0 consistent with -1* : Λ CDM remains viable; QNRT with any $\sigma_i \in [0.5, 3]$ is consistent.
- (c) *$w_a > 0$ measured at high significance*: freezing-quintessence behaviour, incompatible with the thawing dynamics of QNRT; this would falsify the model.
- (d) *Euclid joint (w_0, w_a) measurement*: QNRT predicts a one-parameter curve in the (w_0, w_a) plane. Euclid’s forecasted precision $\sigma(w_0) \sim 0.02$ [11] would distinguish $w_0 = -0.95$ from $w_0 = -1$ at $\sim 2.5\sigma$.

7.4.3 Thawing quintessence

Table 2 shows that $|w(z)|$ decreases monotonically toward higher redshift, confirming thawing quintessence [9]: the field was more Λ -like in the past. QNRT always produces $w_a < 0$, consistent with the DESI-preferred quadrant $w_a < 0$. The thawing signal is small ($|w_a| \lesssim 0.08$) and currently below detection threshold ($\sigma(w_a) \approx 0.58$).

7.4.4 Sensitivity to the kinetic coefficient

Table 4: Dark-energy predictions for the lattice-derived ($3M_{\text{P}}^2/2$) and Brans-Dicke ($M_{\text{P}}^2/16\pi$) kinetic coefficients. Both choices produce thawing behaviour ($w_a < 0$); the lattice-derived coefficient gives larger dynamical deviation from Λ CDM.

σ_i	Lattice ($3M_{\text{P}}^2/2$)		Brans-Dicke ($M_{\text{P}}^2/16\pi$)	
	w_0	w_a	w_0	w_a
0.5	-0.990	-0.014	-1.000	-0.000
1.0	-0.968	-0.044	-1.000	-0.000
2.0	-0.930	-0.082	-1.000	-0.001
3.0	-0.908	-0.076	-0.999	-0.000

The kinetic coefficient $\alpha_{\text{lat}} = c_{\text{lat}}M_{\text{P}}^2/4 = (3/2)M_{\text{P}}^2$ is derived from the F_4 stencil (Proposition 6.1) and confirmed by the Voronoi–Regge correspondence (OP4). The ratio

$\alpha_{\text{lat}}/\alpha_{\text{BD}} = 24\pi$ decomposes as 8π (Dicke normalisation convention) $\times 3$ (F_4 geometry, $c_{\text{lat}}(F_4)/c_{\text{lat}}(\text{cubic}) = 6/2$). The coefficient is understood geometrically and is not a free parameter.

8 Neutrino Mass Estimate

The norm-response mechanism applied to the fermionic sector gives a reference mass M_c as the geometric mean of the Planck and Hubble mass scales:

$$M_c c^2 = \sqrt{M_P c^2 \cdot M_H c^2}, \quad M_P = \sqrt{\hbar c/G}, \quad M_H = \frac{\hbar}{R_* c}, \quad (25)$$

giving

$$M_c c^2 \approx 4.2 \text{ meV} \quad (R_* = R_H(t_0), H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}). \quad (26)$$

This lies within the active-neutrino mass window ($\lesssim 50 \text{ meV}$ [12]) and below the KATRIN upper bound ($< 450 \text{ meV}$ [13]). The same geometric-mean structure that determines $r_c = \sqrt{\ell_P R_*}$ gives $M_c = \sqrt{M_P M_H}$, unifying cosmological and fermionic scales within a single framework.

Remark 8.1. *Equation (25) is a dimensional estimate, not a precision prediction. It does not yield individual mass eigenvalues or the mass ordering; precision predictions require the gauge embedding of $OP3$ and the generation structure of Part II.*

9 Connection to Matter Fields

9.1 The eight degrees of freedom of $\mathbb{H}_{\mathbb{C}}$

The algebra $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$ has real dimension 8. To make the physical content of the theory transparent, we exhibit all eight degrees of freedom via the polar decomposition of ϕ .

Proposition 9.1 (Polar decomposition of $\mathbb{H}_{\mathbb{C}}$). *Every invertible $\phi \in M_2(\mathbb{C})$ admits a unique decomposition*

$$\phi = P \cdot U, \quad (27)$$

where $P = \sqrt{\phi\phi^\dagger}$ is positive-definite Hermitian and $U = P^{-1}\phi$ is unitary. This gives the splitting

$$\underbrace{P}_{4 \text{ real DOF}} = \exp\left(\frac{\sigma}{2} I_2 + \frac{\xi_a \tau^a}{2}\right), \quad \underbrace{U}_{4 \text{ real DOF}} \in \text{U}(2) = \text{U}(1) \times \text{SU}(2), \quad (28)$$

where τ^a ($a = 1, 2, 3$) are the Pauli matrices and $\xi_a \in \mathbb{R}$.

The eight degrees of freedom and their physical roles in the present paper are:

DOF	Field	Type	Role in QNRT
σ	scale field	boson	dark energy (this paper, established)
ξ_1, ξ_2, ξ_3	traceless Hermitian	boson	unidentified; physical role deferred
$\theta \in \text{U}(1)$	complex phase	boson	$\text{U}(1)_Y$ candidate (§9.3)
$u \in \text{SU}(2)$	directional field	boson	$\text{SU}(2)$ gauge field (established)

The identity $\sigma = \log \det P$ connects the scale field to the polar factor: $\det(e^{\sigma/2} I_2) = e^\sigma = N(\phi)$, consistent with equation (5). The current paper uses only σ and u (4 of the 8 real DOF). The remaining 4 DOF (θ and ξ_a) are present in the algebra and are identified as subjects for future work.

9.2 What is established

SU(2) gauge field. The decomposition $\phi = e^{\sigma/2} u$, $u \in \text{SU}(2)$, separates the scale sector σ from the directional sector u . The kinetic term for u in the lattice action (18) gives the SU(2) field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_w [A_\mu, A_\nu]$ in the continuum limit.

Weyl spinors and chirality. Since $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ acts on \mathbb{C}^2 by left matrix multiplication, any $\Psi \in \mathbb{C}^2$ is a Weyl spinor on which $u \in \text{SU}(2)$ acts as $\Psi \mapsto u\Psi$. This action automatically projects onto a definite chirality, providing a geometric basis for weak-interaction handedness. A Dirac spinor is accommodated by $\mathbb{C}^2 \oplus \mathbb{C}^2 \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. The Hurwitz spinor $\Psi : \Gamma_\mathbb{H} \rightarrow \mathbb{H}_\mathbb{C} \otimes S$ is defined on the lattice, with fermionic action

$$S_{\text{fermion}} = \int d^4x \sqrt{-g} \bar{\Psi} (i\gamma^\mu D_\mu - V_m(\sigma_f)) \Psi, \quad V_m(\sigma_f) = M_c \cosh \sigma_f, \quad (29)$$

where $\sigma_f = \log(N(\Psi)/M_c^2)$ is the fermionic scale field and the reference mass $M_c c^2 \approx 4.2 \text{ meV}$ is determined by the NRS fixed-point condition (Section 8).

Summary of established connections. The algebra $\mathbb{H}_\mathbb{C}$ already contains SU(2) gauge bosons and spin-1/2 fermions with left-handed chirality as geometric structures. These emerge from the polar decomposition (27) without additional postulates. The σ field (dark energy), the SU(2) gauge field (weak-boson candidate), and the Weyl/Dirac spinors are all housed within the same algebraic object $\phi \in \mathbb{H}_\mathbb{C}$.

9.3 What remains to be established

The following connections are not claimed in this paper.

- (i) *Identification of lattice SU(2) with SU(2)_L.* Representation content of quarks and leptons under u must be specified; the gauge coupling g_w must match the measured weak coupling.
- (ii) *Physical role of the traceless Hermitian fields ξ_a .* The three DOF ξ_1, ξ_2, ξ_3 in equation (28) are present in $\mathbb{H}_\mathbb{C}$ but their physical identification is open. They are candidates for scalar degrees of freedom (possibly related to electroweak symmetry breaking), but the Standard Model Higgs doublet has four real DOF whereas ξ_a has three; a direct identification is not possible without further structure.
- (iii) *Hypercharge U(1)_Y.* The complex phase $\theta \in \text{U}(1)$ of $U \in \text{U}(2)$ is a candidate for hypercharge. The gauge transformation $\phi \mapsto e^{i\theta} \phi$ leaves $N(\phi)$ invariant and generates a U(1) symmetry. Whether the correct hypercharge assignments for quarks and leptons follow from the geometry of $\Gamma_\mathbb{H}$ is an open question; the gauge coupling g' is not derived here.
- (iv) *Colour gauge group SU(3)_c.* Not contained in $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$; likely requires an octonionic extension of $\Gamma_\mathbb{H}$, where $G_2 \supset \text{SU}(3)$ acts as the automorphism group.

- (v) *Anomaly cancellation.* The correct algebraic context is the E_8 Lie algebra ($E_8 \cong \Gamma_H \oplus \Gamma_H$ as root lattices), which contains $SU(3) \times SU(2) \times U(1)$ among its maximal subgroups [1].
- (vi) *Three fermion generations.* The D_4 sub-root system of F_4 has a triality automorphism of order 3, a natural candidate for generating three generations. The concrete field identification is deferred to Part II.
- (vii) *Fine-structure constant α .* The value $\alpha = 1/137$ is not derived from the Hurwitz lattice in this paper; equation (17) uses α as an input. Its lattice origin is part of OP3.

10 Open Problems

OP1. Physical identification and determination of R_\star . This is the central unsolved problem of QNRT. All physical outputs are conditional on $R_\star \approx R_H(t_0)$, set by hand. Three approaches have been investigated: (i) de Sitter attractor gives $R_\star \approx 0.24$ m (sub-metre); (ii) dynamical identification $R_\star(t) = R_H(t)$ gives $\Omega_{DE} \approx 0.27$ (inconsistent with observations); (iii) lattice-compatibility gives only trivial constraints. None gives a complete resolution.

In the wave-number representation, OP1 becomes: why is the current IR boundary wavenumber $k_{IR} = 1/R_\star \approx 2.3 \times 10^{-27} \text{ m}^{-1}$? Via the classical identity $\rho_\Lambda = c^4/(GR_\star^2)$, this is equivalent to: why does the Universe have $R_\star \approx 4.4 \times 10^{26} \text{ m}$ today? No quantum input (\hbar) is required; OP1 is a purely classical-gravitational question.

OP1 and OP2 are connected: both R_\star (IR cutoff) and σ_i (initial field displacement) are frozen by Hubble friction at the same epoch. A unified treatment of QNRT during inflation and reheating would simultaneously determine R_\star and σ_i . This is the central goal of Part II.

OP2. Physical determination of σ_i . The initial displacement σ_i is a free parameter. In the photon-basis interpretation, $\sigma_i = \log(T_{\text{init}}/T_c)$ corresponds to the CMB temperature at the freezing epoch. Determination requires analysing QNRT dynamics during inflation or reheating, or constraining it via CMB+BAO+SNIa data.

OP3. Standard Model gauge embedding. Identify the lattice $SU(2)$ with $SU(2)_L$; provide $SU(3)_c$ (likely requiring an octonionic extension); verify anomaly cancellation; derive $\alpha = 1/137$ from lattice structure.

OP4. Note on the kinetic coefficient. The Voronoi–Regge correspondence on the F_4 lattice confirms $\alpha_{\text{lat}} = c_{\text{lat}} M_P^2/4 = (3/2)M_P^2$. The ratio $\alpha_{\text{lat}}/\alpha_{\text{BD}} = 24\pi$ decomposes as 8π (normalisation convention) $\times 3$ (F_4 geometry). The coefficient is understood geometrically; it is not an undetermined free parameter.

OP5. Connection to CMB anisotropies. The present paper treats both the σ field and the $SU(2)$ directional field u as spatially homogeneous. Computing their connection to CMB observables requires: (i) a perturbation theory for spatial fluctuations $\delta\sigma(\mathbf{x}, t)$ and $\delta u(\mathbf{x}, t)$, derived from the Hurwitz lattice action; and (ii) the incorporation of the resulting source terms into the Boltzmann hierarchy. The σ -field effective mass $m_\sigma c^2 \approx 10^{-33} \text{ eV}$

places it in the ultra-light scalar regime, with a Compton wavelength comparable to the Hubble radius; its spatial fluctuation spectrum and any imprint on the CMB temperature or polarisation anisotropies are reserved for Part II.

11 Conclusion

QNRT derives observable consequences through a single algebraic operation: $\phi \mapsto \phi^{-1} = \phi^\dagger/N(\phi)$ on the Hurwitz lattice. This revised paper adds three improvements that do not alter the theory's predictions.

(1) $\rho_\Lambda = c^4/(GR_\star^2)$: the classical identity. Substituting $\ell_P^2 = \hbar G/c^3$ shows that \hbar cancels exactly. The cosmological constant problem reduces to why $R_\star/\ell_P \approx 10^{62}$. The NRS fixed-point condition provides the algebraic mechanism selecting $r_c = \sqrt{\ell_P R_\star}$, but the determination of R_\star itself (OP1) remains the central open problem.

(2) Photon basis: $k_0 = 1/r_c$ as the self-dual lattice mode. Proposition 5.5 (algebraically exact, 0.000% error) shows that the Hurwitz lattice harmonic mode at $n = n_c = \sqrt{N_\star}$ has wavenumber $k_0 = 1/r_c$. This is a direct consequence of $r_c = \sqrt{\ell_P R_\star}$, requiring no additional assumption. The reference energy $E_0 = \hbar c k_0 = 2.354$ meV and temperature $T_c = 27.27$ K (geometric mean of Planck and de Sitter temperatures) provide a natural photon-basis interpretation: the σ field, the photon field, and the electromagnetic field share a description in wave-number units with k_0 as the reference. The scale T_c corresponds to redshift $z_c \approx 9$ (cosmic reionisation), giving r_c a cosmologically meaningful interpretation.

(3) Eight degrees of freedom of $\mathbb{H}_\mathbb{C}$ and matter fields. The polar decomposition $\phi = P \cdot U$ (Proposition 9.1) shows that $\mathbb{H}_\mathbb{C} \cong M_2(\mathbb{C})$ already contains, as geometric structures and without additional postulates: the σ scale field (1 DOF), three traceless Hermitian fields ξ_a (3 DOF), a U(1) phase θ (1 DOF), and the SU(2) directional field u (3 DOF). The SU(2) gauge field strength $F_{\mu\nu}$ and the Weyl spinor structure with automatic left-handed chirality are present in the algebra. The current paper uses σ and u (4 of the 8 DOF); the physical identification of the remaining 4 DOF (θ , ξ_a) and their connection to U(1)_Y and electroweak symmetry breaking are reserved for Part II.

(4) Corrected dark-energy table. The w_a values in Tables 1–4 are extracted by the two-point CPL formula $w_a \approx 2[w(z=1) - w_0]$, more accurate than the one-point Linder approximation used in some earlier analyses. The physically natural range $\sigma_i \in [0.5, 1.0]$ gives $w_0 \in [-0.990, -0.968]$ and $w_a \in [-0.044, -0.014]$.

Observational status. The QNRT predictions $w_0 \in [-0.990, -0.908]$ are inconsistent with DESI DR2 (2025) at 2.7–4.0 σ , while remaining consistent with Λ CDM. If Euclid measures $w_0 \in [-0.990, -0.908]$ at high significance, QNRT is supported; if $w_0 \approx -0.75$ is confirmed, the framework is challenged. If $w_a > 0$ is established at high significance, the model is falsified. Five open problems are identified: OP1 (physical determination of R_\star , the reformulated coincidence problem), OP2 (determination of σ_i), OP3 (Standard Model gauge embedding including the derivation of α), OP4 (understood; the lattice kinetic

coefficient is fixed by F_4 geometry), and OP5 (connection of spatial field fluctuations to CMB anisotropies, reserved for Part II).

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A Continuum Limit of the Lattice Action

Step 1: Taylor expansion. Let $\phi_i = \phi(x)$ and $\phi_j = \phi(x + a\hat{e}_k)$ with $a = \ell_P$. Using $\phi = e^{\sigma/2}u$:

$$\frac{|\phi_i - \phi_j|^2}{|\phi_i|^2} \approx a^2 \left[\frac{1}{4} (\hat{e}_k^\mu \partial_\mu \sigma)^2 + N(u^{-1} \hat{e}_k^\mu \partial_\mu u) \right] + O(a^4). \quad (30)$$

Step 2: Sum over 24 neighbours. By F_4 symmetry, $\sum_k u_k^\mu u_k^\nu = c_{\text{lat}} \delta^{\mu\nu}$ with $c_{\text{lat}} = T_{00} = \sum_k (u_k^0)^2$. Computing from the 24 unit elements: the 8 axis-aligned units $\pm e_\mu$ contribute $(u^0)^2 = 1$ for $\pm e_0$ (2 terms, total 2) and 0 for $\pm e_{1,2,3}$; the 16 half-integer units $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ each contribute $(1/2)^2 = 1/4$ (total $16 \times 1/4 = 4$). Hence $c_{\text{lat}} = 2 + 4 = 6$; cross-check: $4 \times 6 = 24 = \sum N(u_k)$. ✓

Step 3: Continuum integral. Converting $\sum_{\text{sites}} \rightarrow (1/a^4) \int d^4x$ with $a = \ell_P = 1/M_P$:

$$S_{\text{lat}} \longrightarrow \int d^4x \frac{c_{\text{lat}}}{4} M_P^2 (\partial_\mu \sigma)^2 = \int d^4x \frac{3}{2} M_P^2 (\partial_\mu \sigma)^2. \quad (31)$$

B Derivation of the FLRW Equations

Starting from the effective action (21) with the SU(2) sector taken constant on cosmological scales:

Einstein equations. $T_{\mu\nu}^\sigma = 3M_P^2 \partial_\mu \sigma \partial_\nu \sigma - g_{\mu\nu} [\frac{3}{2} M_P^2 (\partial\sigma)^2 + V(\sigma)]$. In flat FLRW with $\sigma = \sigma(t)$: $\rho_\sigma = \frac{3}{2} M_P^2 \dot{\sigma}^2 + V(\sigma)$, $p_\sigma = \frac{3}{2} M_P^2 \dot{\sigma}^2 - V(\sigma)$. The (00) component gives (22) with $\kappa = 1/M_P^2$.

Klein-Gordon equation. Varying with respect to σ :

$$-V'(\sigma) + 3M_P^2 \square \sigma = 0, \quad \square \sigma = -\ddot{\sigma} - 3H\dot{\sigma}, \quad V'(\sigma) = 2\rho_\Lambda \sinh \sigma, \quad (32)$$

which yields (23).

C Supplementary Analysis of the R_\star Problem

C.1 Entropy maximisation and the lattice partition function

The Hurwitz lattice partition function

$$Z = \sum_{N=1}^{N_\star} r_4(N) \exp(-\beta \cdot 2\rho_\Lambda \cosh \sigma_N), \quad (33)$$

with $r_4(N) = 8 \sum_{d|N, 4 \nmid d} d$ (Jacobi's four-square formula), converges for any $\beta\rho_\Lambda > 0$: the potential $V(\sigma) = 2\rho_\Lambda \cosh \sigma$ suppresses both UV ($N \ll n_c$) and IR ($N \gg n_c$) contributions simultaneously, providing a natural self-regularisation of the scale integral. The saddle point occurs near $N = n_c$ when $\beta\rho_\Lambda \sim 1$. Setting $T_{\text{lattice}} = T_{dS} = \hbar H / (2\pi k_B)$ and substituting $\rho_\Lambda = \hbar c / (\ell_P R_\star)^2$ gives $R_\star \sim 2\pi \ell_P$ (Planck-scale), not R_H . Simple entropy maximisation does not select $R_\star \sim R_H$.

C.2 Lattice growth law

A growth law $dN_\star/dt = \gamma N_\star^\alpha$ gives $R_\star \propto t^\beta$ with $\beta = 1/(1 - \alpha)$. For $\alpha = 0$ (fixed R_\star , standard QNRT): $\Omega_{\text{DE}} \approx 0.685$ and the correct equation-of-state range. For $\alpha > 2$, Hubble friction naturally freezes the lattice size, structurally identical to the mechanism that freezes σ at σ_i .

C.3 Summary: current understanding of the R_\star problem

- (i) No simple argument selects $R_\star \sim R_H$. De Sitter self-consistency gives $R_\star \approx 0.24 \text{ m}$; entropy maximisation gives $R_\star \rightarrow \infty$; the growth-law analysis with $\beta > 0$ gives Ω_{DE} inconsistent with observations.
- (ii) The partition function Z in (33) converges—a non-trivial algebraic property. The self-dual scale n_c is dynamically preferred when $T_{\text{lattice}} \sim \rho_\Lambda$.
- (iii) R_\star is frozen by Hubble friction during matter domination, exactly as σ is frozen at σ_i .
- (iv) OP1 and OP2 are connected. Both R_\star and σ_i are frozen at the same epoch. A unified treatment of QNRT during inflation and reheating would simultaneously determine both. This is the central goal of Part II.
- (v) The identification $R_\star = R_H(t_0)$ remains the working hypothesis. In the wave-number representation, OP1 asks why $k_{\text{IR}} = 1/R_\star$ has its current value, or equivalently, via $\rho_\Lambda = c^4 / (GR_\star^2)$, why the Universe has the observed dark-energy density today. This is a classical gravitational question; no quantum input (\hbar) is required.