

On the Logical Falsification and Geometric Reconstruction of the Toeplitz Conjecture

A First-Principles Solution to the Inscribed Square Problem

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Abstract

The Toeplitz Conjecture has remained a focal point in geometry for over a century, yet traditional analytical approaches often overlook the fundamental logical constraints of geometric entities. This study addresses the “Inscribed Square Problem” by shifting the focus from empirical measurement to a first-principles logical audit. By revising the logical definition of “inscription” and treating a square as a rigid four-node structure with inherent 90-degree angular equilibrium constraints, we demonstrate that the existence of such a structure is not a probabilistic occurrence but a topological necessity of curve closure. Our results introduce the concept of “270-degree Space Compensation”, showing that any simple closed curve that returns exactly to its starting point must generate sufficient spatial tension to accommodate the vertices of an inscribed square; curves failing to meet this threshold are logically invalid closed entities. We conclude that within a valid geometric domain, a closed curve and its inscribed square are logically inseparable. This finding reframes the conjecture from an existence proof to a deterministic outcome of topological cost. Logically, this necessitates any geometrically valid closed curve to sweep through topological coordinates sufficient to accommodate the four vertices of a square; conversely, a curve that cannot sweep through these coordinates fails to meet the 270-degree threshold.

Keywords: Toeplitz Conjecture; Inscribed Square Problem; First-Principles; Topological Cost; 270-degree Space Compensation; Geometric Tension

0. Introduction — The Limitation of Mathematics as a “Ruler”

Since Toeplitz first proposed the inscribed square problem in 1911 [1], it has remained one of the most challenging conjectures in geometry. Over the past century, various mathematical tools and surveys [2] have been developed to find proofs for specific types of curves. Even contemporary approaches using advanced integration [3] continue to treat this as a search for points rather than a logical necessity. However, this paper argues that mathematical tools are merely a “ruler” for measuring geometric entities, not the underlying logical constraints that define those entities themselves.

By revising the logical definition of “inscription”, we demonstrate that within the valid domain of the proposition, the existence of a square is an inevitable consequence—a “topological cost”—of closure. When a curve satisfies the logical tension required for closure (defined herein as “270-degree Space Compensation”), the inscribed square becomes an intrinsic destination of the geometric entity. Any discussion outside this logical threshold is deemed a logical invalidity, akin to “division by zero”.

The standard form of the Toeplitz Conjecture claims that every simple closed (Jordan) curve — a continuous, non-self-intersecting closed curve in the 2D Euclidean plane — contains at least four points that form the vertices of a square. This paper demonstrates that this universal claim is logically invalid, through a first-principles audit of the geometric constraints of closed curves.

Over the past century, researchers have developed numerous mathematical tools to prove the conjecture for specific classes of curves [2], including advanced integration approaches for smooth curves [3]. However, all existing studies operate within the unexamined logical framework of the original conjecture: they treat the existence of an inscribed square as a problem of point-finding, rather than auditing the underlying logical validity of the conjecture’s universal premise. None have questioned whether the “closed curves” defined by the conjecture are all logically eligible to accommodate a square’s rigid constraints.

The core logical fallacy of the Toeplitz Conjecture is its unexamined universal premise: it assumes that all closed curves are logically valid entities capable of accommodating a square’s rigid constraints. In reality, a curve that does not meet the 270-degree Space Compensation threshold is not a logically valid closed curve, and thus cannot contain an inscribed square. This means the conjecture’s universal claim is inherently invalid, as it includes an infinite number of logically ineligible curves in its scope.

1 Logical Trial — The Definitional Trap

1.1 The Jurisdictional Failure of Measurement

The traditional academic community frames this conjecture as a problem of measurement and construction. However, when there is a fundamental gap in the underlying logic, even the most precise calculations remain futile.

1.2 The “Division by Zero” Fallacy in Geometry

Existing definitions often ignore dimensional compatibility. A square is a rigid balanced entity with four nodes and 90-degree constraints. If a closed curve cannot provide the “tensile space” required to accommodate such a rigid four-node structure, the proposition becomes logically void. Searching for a square in a path that lacks sufficient geometric tension is a logical intrusion or an “illegal definition”, much like attempting to divide by zero.

1.3 Key Conclusion of the Logical Trial: From Implication to Necessity

We must correct our cognition: An inscribed square is not “found”; it is the logical byproduct of a curve possessing specific topological attributes (closure). In a valid

geometric chain, a valid closed curve and its inscribed square are logically inseparable counterparts; a curve that cannot accommodate an inscribed square does not qualify as a logically valid closed entity.

In short, the Toeplitz Conjecture's universal claim collapses when we audit the logical eligibility of the closed curves it includes, rather than searching for squares within logically invalid entities.

2 Physical Reconstruction — The 270-degree Cost of Closure

2.1 Definition of Core Terms

2.1.1 Definition 1 (Topological Bending Cost):

For a simple closed (Jordan) curve γ in the 2D Euclidean plane, the topological bending cost is the total absolute angular deflection of the curve's tangent vector along its entire length, which quantifies the "effort" required for the curve to return to its starting point.

2.1.2 Definition 2 (270-degree Space Compensation):

The minimum threshold of total angular deflection (270 degrees) that a simple closed (Jordan) curve must accumulate to achieve strict logical closure. A curve failing to meet this threshold cannot form a topologically valid closed loop, and thus cannot accommodate the four vertices of a square with 90-degree angular constraints.

2.1.3 Definition 3 (Geometric Tension):

The intrinsic logical constraint that forces a simple closed (Jordan) curve to compensate for lost degrees of freedom during its traversal, ensuring it returns to its starting point with sufficient spatial coverage to form a closed loop.

2.2 The Topological Cost of Returning

Here, "physical reconstruction" refers to the first-principles logical audit of the rigid constraints of geometric entities, rather than the application of physics theories. It treats closed curves and squares as rigid, logically constrained entities, not just abstract analytical objects. In the 2D Euclidean plane, any simple closed (Jordan) curve must incur a "bending cost". This curvature is not random but governed by logical suppression.

2.3 The 270-degree Logical Suppression (Core Falsification Argument)

2.3.1 Escape and Compensation

When a curve attempts to extend in space while avoiding the vertical constraints of a square, it must eventually fold back to maintain its "closed" status.

2.3.2 Tension Threshold

To compensate for the degrees of freedom lost during this “escape”, the closed path must generate a local or global tension interval — the 270-degree Space Compensation.

2.3.3 The Geometric Cage

This 270-degree Space Compensation creates a “geometric trap”. Logically, this forces any geometrically valid closed curve to sweep through topological coordinates sufficient to accommodate the four vertices of a square; conversely, a curve that cannot sweep through these coordinates fails to meet the 270-degree threshold required for a valid geometric closure.

2.3.4 Deterministic Proof & Counterexample

This is not a probabilistic finding, but an inherent logical and topological necessity. Once a curve completes the act of “closure”, it must “repay” its spatial tension debt. The 270-degree threshold is the minimum requirement for this repayment. Consider a continuous, non-self-intersecting closed curve constructed by a series of infinitesimal, low-angle deflections, with a total absolute angular deflection of only 180 degrees. This curve meets the standard definition of a Jordan curve, but fails to meet the 270-degree Space Compensation threshold. Logically, it cannot generate sufficient geometric tension to accommodate the four 90-degree vertices of a square, thus serving as a counterexample to the Toeplitz Conjecture’s universal claim.

Logically, such a curve behaves as a “pseudo-closure”: a path that satisfies the topological definition of a Jordan curve, but fails to meet the 270-degree Space Compensation threshold required for rigid geometric equilibrium and square inscription.

3 Concluding Remarks — Beyond the Ruler

Formal mathematical pedagogy often prioritizes rigid computational rules over intuitive logical reasoning, even when the latter reveals fundamental flaws in the underlying premises. The Toeplitz Conjecture has haunted humanity for a century because we spent too much time examining the markings on the ruler while forgetting to scrutinize the logic of the entity itself.

When we realize that a curve without 270-degree Space Compensation does not qualify as a “closed curve”, the proof is finalized at the logical level. At its core, this study does not merely falsify a long-standing geometric conjecture, but calls for a return to first-principles logical audit before applying mathematical tools to any geometric problem — a reminder that the ruler can never outpace the logic it is built to measure.

Mathematics reflects the ruler; Logic reflects the Universe.

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