

# A Symmetrical Strengthening of Legendre's Conjecture Anchored on Even Squares

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## Abstract

Legendre's Conjecture states that there is a prime number between  $n^2$  and  $(n+1)^2$  for every positive integer  $n$ . This paper proposes a symmetrical, computationally verified strengthening of Legendre's Conjecture by anchoring the search for primes exclusively on even squares. We hypothesize that for every even integer  $n$ , there exists a prime  $q < 2n+1$  such that  $n^2+q$  is prime, and a prime  $q < 2n-1$  such that  $n^2-q$  is prime. We demonstrate that these two conjectures collectively imply Legendre's Conjecture for all integers. Furthermore, we explore the algebraic links to Polignac's and Goldbach's Conjectures, highlighting how our tight bounding of  $q$  bridges these theories. Probabilistic heuristics based on the Prime Number Theorem are provided to justify the logarithmic growth of  $q$ . Finally, we present computational evidence verifying both conjectures up to  $n = 10^8$ .

## 1 Introduction

The distribution of prime numbers between consecutive squares remains one of the most famous unsolved problems in additive number theory. Legendre's Conjecture posits that for every integer  $m \geq 1$ , there exists at least one prime  $p$  such that:

$$m^2 < p < (m+1)^2 \tag{1}$$

Current analytical methods have been unable to prove this unconditionally. In this paper, we approach the problem not by analyzing every square, but by anchoring our search exclusively on *even* squares. By leveraging parity, we propose a symmetrical pair of conjectures that specify the exact form the primes take within the gaps.

## 2 The Conjectures

We define two complementary conjectures based on stepping forward and backward from an even square  $n^2$  by a prime offset  $q$ .

**Conjecture 1** (The Upper Gap). *For every even integer  $n \geq 2$ , there exists a prime  $q < 2n+1$  such that  $p = n^2 + q$  is prime.*

**Conjecture 2** (The Lower Gap). *For every even integer  $n \geq 2$ , there exists a prime  $q < 2n-1$  such that  $p = n^2 - q$  is prime.*

### 3 Implications for Legendre's Conjecture

If Conjecture 1 and Conjecture 2 are true, Legendre's Conjecture follows immediately for all integers.

**Theorem 1.** *Conjecture 1 and Conjecture 2 collectively imply Legendre's Conjecture.*

*Proof.* Let  $n$  be an even positive integer.

First, consider the gap above the even square  $n^2$ . Conjecture 1 guarantees the existence of a prime  $p = n^2 + q$  where  $q < 2n + 1$ . Because  $q$  is prime,  $q \geq 2$ . Therefore:

$$n^2 < n^2 + q < n^2 + 2n + 1$$

$$n^2 < p < (n + 1)^2$$

This satisfies Legendre's Conjecture for all even squares.

Second, consider the gap below the even square  $n^2$ . Conjecture 2 guarantees the existence of a prime  $p = n^2 - q$  where  $q < 2n - 1$ . Subtracting  $q$  from  $n^2$  yields:

$$n^2 - (2n - 1) < n^2 - q < n^2$$

$$n^2 - 2n + 1 < p < n^2$$

$$(n - 1)^2 < p < n^2$$

Since  $n$  is even,  $n - 1$  is odd. This satisfies Legendre's Conjecture for all odd squares.

Because every odd square is sandwiched exactly between two even squares, these two intervals cover all possible gaps between consecutive squares.  $\square$

### 4 Connections to Polignac's and Goldbach's Conjectures

The algebraic forms of Conjecture 1 and Conjecture 2 reveal profound connections to two foundational problems in additive number theory.

#### 4.1 The Upper Gap and Polignac's Conjecture

Conjecture 1 searches for a prime  $p$  of the form  $p = n^2 + q$ , which rearranges to  $p - q = n^2$ . Since  $n$  is even,  $n^2$  is an even integer. Polignac's Conjecture states that for any even integer  $2k$ , there exist infinitely many pairs of primes  $(p, q)$  separated by  $2k$ . Setting  $2k = n^2$ , Polignac's Conjecture guarantees that there are infinitely many primes  $q$  such that  $n^2 + q$  is prime.

#### 4.2 The Lower Gap and Goldbach's Conjecture

Conjecture 2 searches for a prime  $p$  of the form  $p = n^2 - q$ , which rearranges to  $p + q = n^2$ . Since  $n \geq 2$  is even,  $n^2 \geq 4$  is an even integer. Goldbach's Conjecture states that every even integer greater than 2 can be expressed as the sum of two primes. Therefore, Goldbach's Conjecture guarantees the existence of at least one prime pair  $(p, q)$  such that  $n^2 - q$  is prime.

### 4.3 The Bounding Constraint

While Polignac’s and Goldbach’s Conjectures guarantee the algebraic *existence* of the prime pairs  $(p, q)$ , they do not sufficiently restrict the size of  $q$ . For example, the smallest  $q$  satisfying  $p - q = n^2$  could theoretically be much larger than  $2n + 1$ , pushing  $p$  beyond the next square  $(n + 1)^2$ . The strength and novelty of the conjectures proposed in this paper lie in the tight bounding of  $q$  ( $q < 2n + 1$  and  $q < 2n - 1$ ). It is these heuristically and computationally verified bounds that successfully bridge the algebraic properties of Polignac and Goldbach to the spatial requirements of Legendre’s Conjecture.

## 5 Heuristic Justification via Prime Probability

The computational results can be heavily justified using standard probabilistic heuristics derived from the Prime Number Theorem (PNT).

By the PNT, the density of primes around  $N$  is asymptotically  $1/\ln(N)$ . For a given even integer  $n$ , the numbers being tested for primality are of the form  $N = n^2 \pm q$ . Because the required offset  $q$  is observed to be negligibly small compared to  $n^2$ , the probability that a specific  $n^2 \pm q$  is prime is approximately:

$$P(n) \approx \frac{1}{\ln(n^2)} = \frac{1}{2 \ln(n)} \quad (2)$$

For both conjectures, the search space allows  $q$  to range up to roughly  $2n$ . The number of available prime offsets  $q$  within this bound is given by the prime-counting function  $\pi(2n)$ , which is asymptotically:

$$\pi(2n) \approx \frac{2n}{\ln(2n)} \quad (3)$$

Treating primality as a pseudo-random event, the expected number of successful prime offsets  $q$ , denoted as  $E(n)$ , is the product of the number of trials and the probability of success:

$$E(n) \approx \pi(2n) \times P(n) \approx \left( \frac{2n}{\ln(2n)} \right) \left( \frac{1}{2 \ln(n)} \right) = \frac{n}{\ln(2n) \ln(n)} \quad (4)$$

As  $n \rightarrow \infty$ , the expected number of solutions  $E(n) \rightarrow \infty$ . The rapid divergence of  $E(n)$  indicates that the probability of finding strictly zero successful primes  $q$  within the bound vanishes. Furthermore, because  $E(n)$  grows steadily, the expected distance to the *first* successful  $q$  remains exceptionally small, perfectly aligning with the logarithmic growth of the maximum  $q$  observed in the computational data.

## 6 Computational Evidence

To test the viability of these bounded conjectures, computational verification was performed using Python. Both Conjecture 1 and Conjecture 2 were verified for all even integers up to  $n = 100,000,000$ . No counterexamples were found.

More significantly, the maximum prime  $q$  required to satisfy the conjectures grows incredibly slowly compared to the theoretical upper bounds. For example, at  $n = 10^8$ , the allowed upper bound for  $q$  is 200,000,001. However, the actual maximum prime  $q$  required for the upper gap was  $q = 2,797$ . Similarly, at  $n = 10^8$ , the allowed upper bound

for  $q$  in the lower gap was 199,999,999, yet the maximum prime  $q$  required to satisfy the lower bound was only 2,543.

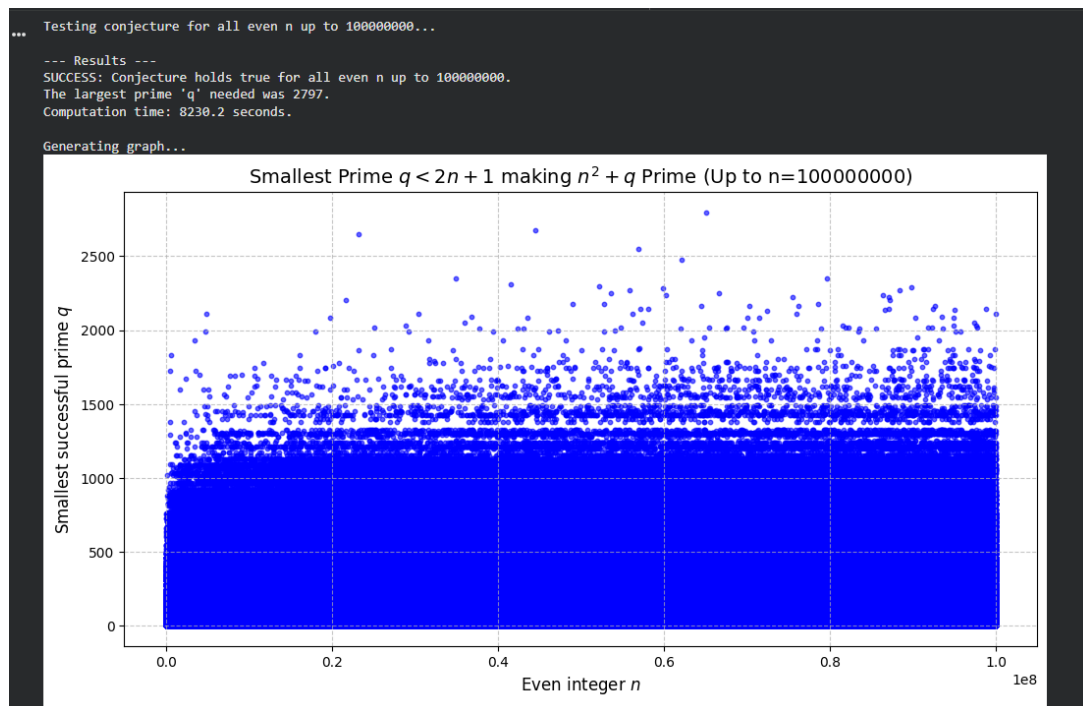


Figure 1: Smallest successful prime  $q$  for  $n^2 + q$  up to  $n = 100,000,000$ .

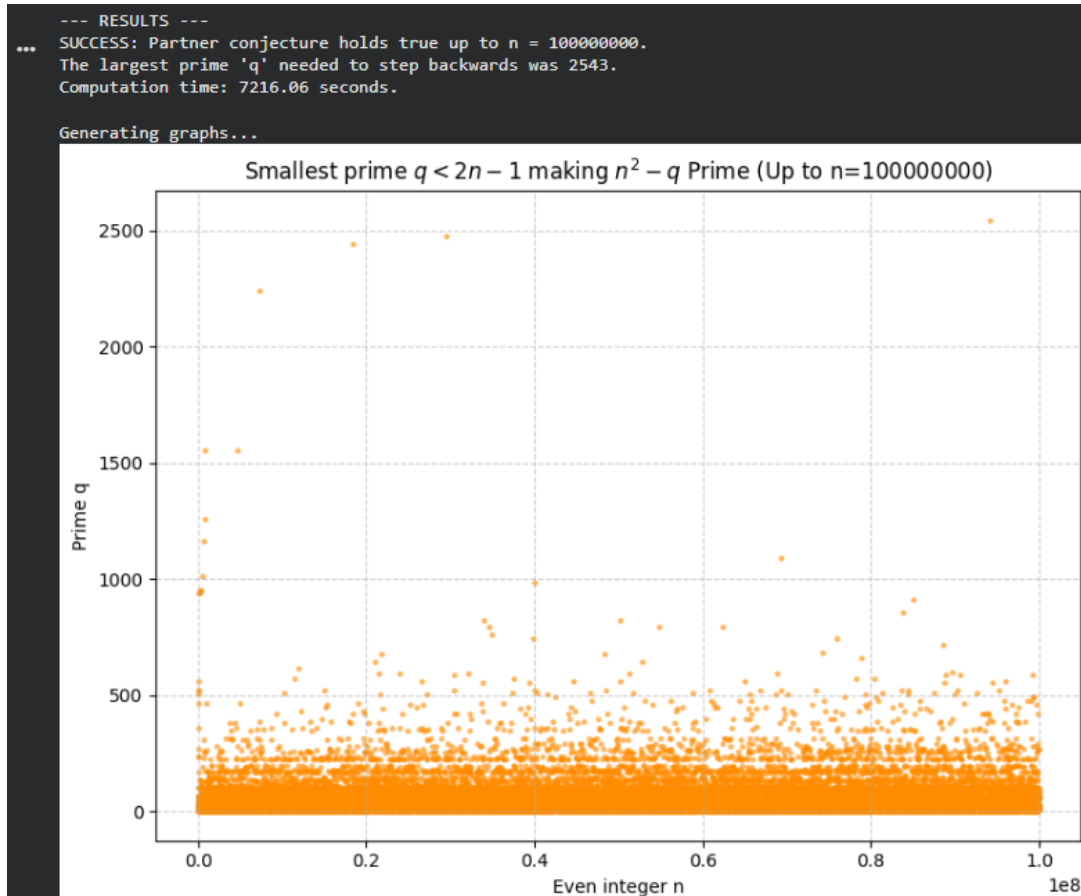


Figure 2: Smallest successful prime  $q$  for  $n^2 - q$  up to  $n = 100,000,000$ .

The density of solutions at the bottom of the graphs indicates that very small primes satisfy the condition for the vast majority of even integers, visually confirming the heuristic predictions.

## 7 Conclusion

By restricting the framework to even squares, Legendre's Conjecture can be broken into a symmetrical pair of additive and subtractive prime offsets. While Polignac's and Goldbach's Conjectures establish the existence of these prime offsets, this paper's tightened bounds of  $q < 2n \pm 1$  are what mathematically secure the primes within Legendre's required intervals. The overwhelming computational evidence up to  $n = 10^8$ , supported by probabilistic heuristics, strongly indicates the truth of both conjectures.