An Octonionic Foundation of Spacetime Geometry and Quantum Mechanics

Ruediger Giesel
Independent Researcher, Stuttgart, Germany

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We propose a framework in which spacetime geometry, quantum dynamics, superposition, entanglement, and decoherence arise as low-energy projections of a deeper non-associative structure. The starting point is the algebra of octonions, whose imaginary sector carries a canonical exceptional geometry with symmetry group G2. Since this structure is Riemannian rather than Lorentzian, physical spacetime requires additional dynamical projection data: a clock direction, a stable associative spatial three-plane, and suppression of the remaining coassociative modes. This yields an effective four-dimensional Lorentzian spacetime in which Einstein-type gravitational dynamics can be recovered. The proposed master action is motivated by minimal physical principles: reality of observables, locality, diffeomorphism covariance, positivity of the octonionic norm, minimal non-associative interaction, and recovery of standard low-energy physics. Genuine octonionic non-associativity cannot arise from a single field alone; a nontrivial associator sector requires at least three independent octonion-valued fields. The resulting interaction controls deviations from ordinary associative physics and organizes possible corrections to quantum mechanics, gravity, and cosmology. Quantum mechanics appears after projection onto associative quaternionic and complex substructures. In this sector, the Hilbert-space norm, Born probabilities, and linear superposition are recovered. Entanglement arises from non-factorizing projected dynamics induced by the trilinear non-associative coupling, while decoherence is described as suppression of interference between projected branches through environmental interactions. The framework is consistent with current low-energy quantum and weak-field gravitational tests when the non-associative scale is sufficiently high. Observable deviations are expected mainly in highly coherent mesoscopic systems, high-energy regimes, strong-curvature environments, or cosmological settings where the associator sector becomes dynamically relevant. The construction therefore provides a structured route from octonionic non-associativity to effective spacetime, gravity, quantum theory, superposition, decoherence, and classicality. The novelty is not the isolated use of octonions, G2 geometry, or non-associative algebra, which have substantial precedents. Rather, the new element is the proposed dynamical synthesis: a positive trilinear octonionic associator sector selects, through a variational principle, an associative spatial three-plane together with an independent clock direction, while coassociative modes are made massive, compact, or dynamically suppressed. In this sense, the framework should be read as a dynamical projection theory.

I. INTRODUCTION AND SCOPE

This paper develops an octonionic approach in which spacetime geometry, gravitational dynamics, and quantum wave dynamics are treated as effective projections of a deeper non-associative structure. The claim is not that the Standard Model and general relativity have already been uniquely derived from octonions. Rather, the central claim is more specific:

> A real normed non-associative algebra, together with a local variational principle and a dynamical associative projection, supplies a mathematically controlled mechanism by which four-dimensional Lorentzian geometry, Einstein-type metric dynamics, and complex quantum mechanics can arise as low-energy effective structures. The symmetry breaking is not imposed kinematically. It is driven by the minimization of the positive associator energy. Configurations that remain close to an associative subalgebra have lower energy than generic non-associative configurations. Hence the vacuum dynamically selects an associative three-plane $P_3 \subset \operatorname{Im}\mathbb{O}$, while the orthogonal coassociative directions acquire a mass gap, become compact, or are dynamically suppressed.

The construction is sharpened in six respects. First, four-dimensional spacetime is formulated as a dynamical projection from a G_2 structure, not as a direct identification. Second, the coupling constants are dimensionally defined. Third, the octonionic field is interpreted as a pregeometric multiplet whose projections may appear as bosonic or fermionic modes. Fourth, quantitative estimates are given for possible experimental signatures. Fifth, low-energy quantum and weak-field gravitational consistency conditions are stated explicitly. Sixth, the proposal is placed in relation to octonionic, G_2, and non-associative-geometry literature.

State of the Art and Literature Context

The present proposal is situated in a broad literature on octonions, division algebras, exceptional geometry, and non-associative structures in theoretical physics. The basic algebraic facts about $\mathbb{O}$, including alternativity, the multiplicative norm, Moufang-type identities, and G_2 as the automorphism group of the octonions, are standard mathematical results and will not be claimed as new here [1, 4–6]. The relation between octonionic structure constants, positive stable G_2 three-forms, metric reconstruction, and associative/coassociative calibrated subspaces belongs to standard G_2-geometry and calibrated geometry [7–10, 12].

Octonionic and split-octonionic formulations have also appeared in several physical contexts. Gogberashvili developed split-octonionic approaches to geometry and electrodynamics, including an octonionic rewriting of Maxwell-type structures [19, 20]. Köplinger and Dzhunushaliev studied non-associative quantum-theoretic models in which
probability and modified Born-type structures are tied to non-associative algebraic data [21, 22]. Krasnov used octonionic and split-octonionic structures in a $\mathrm{Spin}(11,3)$ framework for particles and Standard-Model-like spinorial organization [23]. Lasenby explored representations of $SU(3)$ and octonions inside the spacetime-algebra/geometric-algebra approach [24]. Earlier division-algebra models of internal symmetries and spinorial structures include the work of Günaydin, Gürsey, Kugo, Townsend, Manogue, Dray, Wilson, and others [13, 14, 17, 18]. Mathematically, modern octonionic analysis and octonionic Hilbert-space theory make clear that linear, inner-product, and operator-theoretic notions require special care in the non-associative setting [34–37].

The present manuscript differs from these directions in a specific and limited sense. It does not claim that octonions, G_2 geometry, octonionic spinors, split-octonionic electrodynamics, or non-associative quantum theory are new. The proposed new element is the use of a positive trilinear associator norm as part of a four-dimensional variational principle which dynamically selects an associative spatial three-plane and, together with an independent clock line, produces an effective Lorentzian low-energy sector. This distinction is made explicit below and summarized again in Sec. XII.

Novelty and Relation to Prior Work

The individual mathematical ingredients used here are well established. Octonions, their automorphism group G_2, stable G_2 three-forms, associative and coassociative calibrated subspaces, and non-associative structures in high-energy theory have all been studied extensively [1, 8, 9, 30, 33]. Likewise, division algebras have long been used to organize spinorial, gauge-theoretic, and exceptional structures [2, 13, 17, 18].

The novelty of the present work is the specific dynamical synthesis. A genuinely nontrivial octonionic interaction is represented by the positive trilinear invariant

$$\lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2, \tag{1}$$

rather than by a single-field expression, which would vanish by alternativity. This same associator sector acts as a variational selection principle for low-energy associative physics.

Four-dimensional Lorentzian spacetime is not identified directly with a G_2 manifold, because a G_2 structure is originally Riemannian. Instead, the effective tangent structure is obtained dynamically as

$$TM_4^{\mathrm{eff}} \simeq L_t \oplus P_3, \tag{2}$$

where L_t is an independent clock line and $P_3 \subset \operatorname{Im}\mathbb{O}$ is an associative spatial three-plane. The orthogonal coassociative sector

$$Q_4 = P_3^{\perp} \tag{3}$$

is projected into internal degrees of freedom.

The recovery of complex quantum mechanics, the Born rule, superposition, entanglement, decoherence, and classical branches is governed by the same associative projection

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C}. \tag{4}$$

Correspondingly, leading deviations from ordinary quantum mechanics and general relativity are controlled by the common dimensionless parameter

$$\epsilon_A(E) \sim \eta_A \left(\frac{E}{M_*}\right)^2. \tag{5}$$

The organizing chain of the paper is therefore

$$\mathbb{O} \longrightarrow f_{abc} \longrightarrow \varphi \longrightarrow g_\varphi \longrightarrow L_t \oplus P_3 \longrightarrow (M_4, g_{\mu\nu}^{\mathrm{eff}}) \longrightarrow \text{Einstein and quantum projection limits.} \tag{6}$$

This separates the proposal from earlier algebraic uses of octonions, from purely Riemannian G_2 compactification mechanisms, and from non-associative phase-space deformations in string-theoretic flux backgrounds.

A necessary mathematical correction is used throughout. In an alternative algebra such as $\mathbb{O}$, the associator

$$\mathcal{A}(x, y, z) := (xy)z - x(yz) \tag{7}$$

is alternating:

$$\mathcal{A}(x, x, z) = \mathcal{A}(x, y, y) = \mathcal{A}(x, x, x) = 0. \tag{8}$$

Consequently, a self-interaction of the form

$$\|\mathcal{A}(\Psi, \Psi, \Psi)\|^2 \tag{9}$$

is identically zero. A physically nontrivial associator sector requires at least three algebraically independent fields or field components. Therefore used is:

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3). \tag{10}$$

In summary, the central new element is the proposal that a positive trilinear octonionic associator sector dynamically selects an associative spatial three-plane P_3, while an independent clock line L_t supplies Lorentzian time. This produces an effective low-energy structure in which the coassociative sector is hidden, massive, compact, or dynamically suppressed, while complex quantum mechanics is recovered in the associative projection.

II. MINIMAL PHYSICAL PRINCIPLES AND EMERGENCE OF THE MASTER ACTION

The proposed action is not guessed arbitrarily. It follows from a chain of physical and mathematical minimality arguments. The logic is

$$\text{minimal physical principles} \longrightarrow \text{normed division algebra} \longrightarrow \text{Hurwitz theorem} \longrightarrow \mathbb{O} \longrightarrow G_2 \text{ structure} \longrightarrow \text{associ} \tag{11}$$

Thus the octonionic master action is not introduced as an ad hoc approach. It is the lowest-order local covariant action compatible with a real positive norm, a nontrivial non-associative algebraic sector, and the requirement that ordinary Einstein and quantum dynamics emerge after associative projection.

A. Minimal principles

P1. **Reality of observables.** Observable densities must be real. For octonions this requires the real inner product

$$\langle X, Y \rangle := \mathrm{Re}(\bar{X}Y) = \mathrm{Re}(X\bar{Y}), \qquad \|X\|^2 := \langle X, X \rangle \geq 0. \tag{12}$$

P2. **Positive norm and probabilistic interpretation.** Physical amplitudes must admit a positive norm so that projected probabilities can be defined. This excludes generic algebras without a positive multiplicative norm and motivates the use of normed division algebras.

P3. **Locality.** The Lagrangian density depends on fields and finitely many derivatives at a point. At lowest derivative order only two derivatives enter the kinetic and gravitational sectors.

P4. **Diffeomorphism covariance.** The effective low-energy theory on the emergent spacetime manifold M_4 must be formulated by integrating a scalar density $\sqrt{-g}\,\mathcal{L}$.

P5. **Maximal algebraic nontriviality with norm preservation.** The fundamental algebraic arena should be large enough to contain real, complex, quaternionic, and genuinely non-associative structures, while still preserving a positive norm. This selects the largest normed division algebra.

P6. **Positive norm and stable vacuum.** Non-associative energy must be bounded below. The leading associator contribution is therefore quadratic in the associator norm.

P7. **Dynamical symmetry breaking by vacuum selection.** The fundamental action should not choose a preferred associative subalgebra by hand. Instead, the preferred low-energy sector must arise from minimizing a positive potential. The associator norm provides precisely such a mechanism: it vanishes on associative subalgebras and is positive for genuinely non-associative configurations.

P8. **Minimal non-associative nontriviality.** Since binary products already occur in associative subalgebras, the first invariant that detects genuine octonionic non-associativity is trilinear:

$$\mathcal{A}(X,Y,Z) = (XY)Z - X(YZ). \tag{13}$$

P9. **Low-energy recovery.** In the associative projection and weak-field limit the theory must reduce to Einstein gravity plus standard complex or spinorial quantum mechanics.

B. Hurwitz theorem and the selection of octonions

The above principles strongly restrict the possible algebraic starting point. If one demands a finite-dimensional real division algebra with a positive multiplicative norm, then the Hurwitz theorem states that the only possibilities are

$$\mathbb{R}, \qquad \mathbb{C}, \qquad \mathbb{H}, \qquad \mathbb{O}. \tag{14}$$

These algebras form the sequence

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}. \tag{15}$$

Each step adds a new structural feature. The real numbers provide ordered scalar amplitudes. The complex numbers provide phases and the natural algebra of ordinary quantum mechanics. The quaternions provide noncommutativity and spinorial structure. The octonions provide the first and only normed division algebra with genuine non-associativity.

Therefore, if the aim is to construct a theory in which ordinary complex quantum mechanics and quaternionic spin structures appear as projections, but where an additional pregeometric non-associative sector is still available, the minimal maximal choice is

$$\boxed{\mathcal{A}_{\text{fund}} = \mathbb{O}.} \tag{16}$$

The smaller algebras are too restrictive: $\mathbb{R}$ cannot encode quantum phases, $\mathbb{C}$ is already the standard associative quantum algebra, and $\mathbb{H}$ is noncommutative but still associative. Only $\mathbb{O}$ contains associative complex and quaternionic subalgebras while also possessing a genuinely non-associative sector.

This gives the first central emergence chain:

$$\text{positive norm} + \text{division property} + \text{maximal non-associativity} \quad \Longrightarrow \quad \mathbb{O}. \tag{17}$$

C. From octonions to the G_2 structure

Let

$$\mathbb{O} := \mathbb{R} \oplus \operatorname{Im}\mathbb{O}, \qquad \dim_{\mathbb{R}} \operatorname{Im}\mathbb{O} = 7. \tag{18}$$

With imaginary basis elements e_a, $a = 1, \ldots, 7$, the product is

$$e_a e_b = -\delta_{ab} + f_{ab}{}^{c} e_c, \tag{19}$$

where f_{abc} are totally antisymmetric octonionic structure constants. These constants define the canonical stable three-form

$$\varphi := \frac{1}{6} f_{abc}\, e^a \wedge e^b \wedge e^c. \tag{20}$$

The stabilizer of this form is the exceptional group

$$G_2 = \operatorname{Aut}(\mathbb{O}). \tag{21}$$

Thus the use of octonions automatically induces a G_2-geometric structure on the imaginary sector. The metric on the seven-dimensional internal space is reconstructed from φ, and associative three-planes are calibrated by φ. These associative planes generate quaternionic subalgebras,

$$\mathbb{R} \oplus P_3 \simeq \mathbb{H}, \tag{22}$$

which later provide the low-energy bridge to spin and complex quantum mechanics.

The second emergence chain is therefore

$$\mathbb{O} \longrightarrow f_{abc} \longrightarrow \varphi \longrightarrow G_2 \longrightarrow \text{associative three-planes} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C}. \tag{23}$$

D. Why the associator is the minimal new invariant

The octonions are alternative but not associative. Their failure of associativity is measured by

$$\mathcal{A}(X, Y, Z) = (XY)Z - X(YZ). \tag{24}$$

Because of alternativity,

$$\mathcal{A}(X, X, Z) = 0, \qquad \mathcal{A}(X, Y, Y) = 0, \qquad \mathcal{A}(X, X, X) = 0. \tag{25}$$

Consequently, a single octonion-valued field cannot produce a nontrivial cubic self-associator:

$$\mathcal{A}(\Psi, \Psi, \Psi) = 0. \tag{26}$$

A genuinely nonzero associator requires at least three independent octonionic field directions. This is why the minimal non-associative model uses

$$\Psi_I : M_4 \to \mathbb{O}, \qquad I = 1, 2, 3. \tag{27}$$

The term

$$\|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 \tag{28}$$

is then the lowest-order positive scalar that detects genuine octonionic non-associativity. It vanishes on all associative subalgebras and is nonzero precisely when the dynamics explores directions outside the associative sector.

This gives the third emergence chain:

$$\text{alternativity} + \text{non-associativity} + \text{positivity} \quad \Longrightarrow \quad \lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2. \tag{29}$$

E. Allowed lowest-order invariants

Let $\Psi_I : M_4 \to \mathbb{O}$ with $I = 1, 2, 3$. The lowest-order real scalar invariants compatible with the above principles are

$$\mathcal{I}_{\rm grav} = R - 2\Lambda, \tag{30}$$

$$\mathcal{I}_{\rm kin} = g^{\mu\nu}\langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle, \tag{31}$$

$$\mathcal{I}_{\rm loc} = V_{\rm loc}(\Psi_1, \Psi_2, \Psi_3), \tag{32}$$

$$\mathcal{I}_{\rm assoc} = \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2, \tag{33}$$

$$\mathcal{I}_{R\Psi} = R \sum_I \langle \Psi_I, \Psi_I \rangle. \tag{34}$$

Here ∇_μ is the spacetime covariant derivative acting on tensor indices, supplemented if necessary by a projected internal connection on the octonionic bundle. The term $\mathcal{I}_{R\Psi}$ is optional but useful as the unique leading nonminimal curvature coupling.

Each invariant has a distinct role. The gravitational invariant controls the projected metric dynamics. The kinetic invariant propagates the octonionic multiplet. The local potential selects the vacuum and the projected mass spectrum. The associator invariant penalizes genuinely non-associative excitations. The curvature-norm invariant describes the leading feedback of the octonionic norm sector on spacetime curvature.

F. Emergence of the master action

The most economical action is therefore

$$\boxed{\begin{aligned} S[g, \Psi_I] = \int_{M_4} d^4x \sqrt{-g} \Bigg[& \frac{1}{2\kappa}(R - 2\Lambda) + \sum_{I=1}^{3} \frac{\alpha_I}{2} g^{\mu\nu} \langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle \\ & - V_{\rm loc}(\Psi_1, \Psi_2, \Psi_3) - \lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 - \xi R \sum_{I=1}^{3} \langle \Psi_I, \Psi_I \rangle \Bigg]. \end{aligned}} \tag{35}$$

All later equations are consequences of Eq. (35) plus the projection assumptions stated below.

The complete conceptual chain can now be written as

$$\boxed{\begin{gathered} \text{reality} + \text{positive norm} + \text{locality} + \text{covariance} + \text{low-energy recovery} \\ \Longrightarrow \text{normed division algebra} \Longrightarrow \text{Hurwitz theorem} \Longrightarrow \mathbb{O} \Longrightarrow G_2 \text{ structure} \\ \Longrightarrow \text{associative projection} + \text{non-associative associator} \Longrightarrow S[g, \Psi_I]. \end{gathered}} \tag{36}$$

G. Why this is the minimal master action

The Einstein-Hilbert term is the unique generally covariant scalar containing at most two derivatives of the metric. The kinetic term is the unique positive quadratic derivative scalar for octonion-valued fields. The local potential contains all algebraic terms not specific to non-associativity. The associator term is the lowest-order positive invariant that vanishes on all associative subalgebras and is nonzero only in genuinely octonionic directions.

In the strict associative limit

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \to 0, \tag{37}$$

the theory reduces to an ordinary projected field theory on M_4. In the weak-field limit the metric variation gives Einstein-type equations, while the complex and quaternionic projections give Schrödinger and Dirac dynamics. Conversely, when the associator sector is excited, the same action provides the controlled deformation away from standard associative physics.

The master action is therefore minimal in three simultaneous senses: it is minimal in derivative order, minimal in algebraic degree needed to detect non-associativity, and minimal in its low-energy content required to recover known gravitational and quantum structures.

III. OCTONIONIC ALGEBRA AND THE G_2 THREE-FORM

A. Octonions

Definition III.1 (Octonions)**.** *Let* $e_0 := 1$*. As a real vector space, the octonion algebra is*

$$\mathbb{O} := \left\{ x = x^0 e_0 + \sum_{a=1}^{7} x^a e_a \,\middle|\, x^\mu \in \mathbb{R} \right\} \cong \mathbb{R}^8. \tag{38}$$

Its multiplication is defined bilinearly by

$$e_0 e_\mu := e_\mu e_0 := e_\mu, \qquad e_a e_b := -\delta_{ab} e_0 + f_{ab}{}^{c} e_c, \qquad a, b, c = 1, \ldots, 7, \tag{39}$$

where f_{abc} *are the totally antisymmetric octonionic structure constants. With this product,* $\mathbb{O}$ *is the real eight-dimensional normed division algebra*

$$\mathbb{O} := \mathbb{R} e_0 \oplus \operatorname{Im} \mathbb{O}, \qquad \operatorname{Im} \mathbb{O} := \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_7\}. \tag{40}$$

For $x = x^0 e_0 + x^a e_a$, conjugation, real part, inner product, and norm are defined by

$$\bar{x} := x^0 e_0 - x^a e_a, \qquad \mathrm{Re}(x) := x^0, \tag{41}$$

$$\langle x, y\rangle := \mathrm{Re}(\bar{x}y), \tag{42}$$

$$N(x) := \bar{x}x = x\bar{x} = (x^0)^2 + \sum_{a=1}^{7} (x^a)^2. \tag{43}$$

The norm is multiplicative, $N(xy) = N(x)N(y)$, and the algebra is alternative but not associative. These statements are standard and are used below as known results rather than as new claims [1, 4–6]. The associator is defined by

$$\mathcal{A}(x, y, z) := (xy)z - x(yz). \tag{44}$$

In an alternative algebra it is alternating; in particular

$$\mathcal{A}(x, x, z) = \mathcal{A}(x, y, y) = \mathcal{A}(x, x, x) = 0. \tag{45}$$

a. Fano-plane convention. Throughout the paper we use the following oriented Fano-plane convention for the octonionic basis $\{e_1, \ldots, e_7\}$. The structure constants f_{abc} are defined by

$$e_a e_b = -\delta_{ab} e_0 + f_{ab}{}^{c} e_c, \qquad e_0 = 1, \tag{46}$$

where f_{abc} are completely antisymmetric and the nonzero positive components are

$$f_{123} = f_{145} = f_{176} = f_{246} = f_{257} = f_{347} = f_{365} = 1. \tag{47}$$

All other components are obtained by total antisymmetry, and all components not related to the seven triples in Eq. (47) vanish. Equivalently, each oriented triple (a, b, c) in Eq. (47) satisfies

$$e_a e_b = e_c, \qquad e_b e_c = e_a, \qquad e_c e_a = e_b, \tag{48}$$

while reversing the order changes the sign.

B. Canonical G_2 three-form

Let $V := \mathrm{Im}\,\mathbb{O}$ and let $\{e^a\}$ be the dual coframe. The octonionic structure constants define the stable three-form

$$\boxed{\varphi := \frac{1}{6} f_{abc}\, e^a \wedge e^b \wedge e^c.} \tag{49}$$

Equivalently,

$$\varphi(x, y, z) := \langle x \times y, z \rangle, \qquad x \times y := \frac{1}{2}(xy - yx). \tag{50}$$

The group preserving φ is

$$G_2 := \{A \in GL(V) : A^* \varphi = \varphi\}. \tag{51}$$

This is also the automorphism group of $\mathbb{O}$ [1, 4].

Theorem III.2 (Metric reconstruction; standard G_2 geometry)**.** *A positive stable G_2 three-form φ on a seven-dimensional real vector space determines a metric g_φ, an orientation, and a Hodge dual $\psi := *_\varphi \varphi$ [7–10].*

Proof. Define the symmetric bilinear density

$$B_\varphi(u, v) = \frac{1}{6}\,(\iota_u \varphi) \wedge (\iota_v \varphi) \wedge \varphi \in \Lambda^7 V^*. \tag{52}$$

For positive stable φ there is a unique metric g_φ and volume form vol_φ such that

$$B_\varphi(u, v) = g_\varphi(u, v)\,\mathrm{vol}_\varphi\,. \tag{53}$$

In the octonionic basis this gives

$$(g_\varphi)_{ab} = \frac{1}{6} f_{acd} f_b{}^{cd} = \delta_{ab}, \tag{54}$$

using the standard identity $f_{acd} f_b{}^{cd} = 6\delta_{ab}$. The associated dual four-form is $\psi = *_\varphi \varphi$. □

The G_2 invariance means that all transformations in G_2 preserve the multiplication data that define the norm, the cross product, the metric, and the calibration forms. Therefore the pre-projected internal geometry has no preferred direction until the dynamics selects one.

IV. EMERGENCE OF FOUR-DIMENSIONAL SPACETIME FROM G_2 STRUCTURE

This section makes explicit the projection from the seven-dimensional G_2 structure to an effective four-dimensional Lorentzian spacetime. A positive stable G_2 three-form reconstructs a Riemannian metric and therefore does not by itself produce Lorentzian signature. The model consequently introduces Lorentzian physics through two additional dynamical projection data: a clock line L_t, which supplies the negative metric direction, and an associative calibrated three-plane P_3, which supplies the stable macroscopic spatial directions. The complementary coassociative sector

$$Q_4 = P_3^\perp \tag{55}$$

remains present, but it is not part of the low-energy macroscopic tangent bundle.

A. Associative and coassociative subspaces

Definition IV.1 (Associative three-plane)**.** *A three-dimensional oriented subspace $P \subset V$ is associative if*

$$\varphi|_P = \mathrm{vol}_P \,. \tag{56}$$

Equivalently, after a suitable G_2-rotation, P is spanned by an oriented Fano triple (e_a, e_b, e_c). In that case

$$\mathbb{R} \oplus P = \mathrm{span}_{\mathbb{R}}\{e_0, e_a, e_b, e_c\} = \mathbb{H}_{abc} \cong \mathbb{H}, \tag{57}$$

and the subalgebra property follows from the multiplication table.

Definition IV.2 (Coassociative four-plane)**.** *A four-dimensional oriented subspace $Q \subset V$ is coassociative if*

$$\varphi|_Q = 0, \qquad \psi|_Q = \mathrm{vol}_Q \,. \tag{58}$$

For every associative P, the orthogonal complement $Q = P^{\perp}$ is coassociative.

Thus a choice of associative plane gives the orthogonal decomposition

$$V = P_3 \oplus Q_4. \tag{59}$$

The associative plane carries quaternionic multiplication and is the natural seed of ordinary three-dimensional spatial vector algebra. The complementary coassociative sector carries the remaining four internal degrees of freedom.

B. Projection data

The effective four-dimensional spacetime structure is specified by the following dynamical projection data:

$$\text{clock line:} \qquad L_t = \mathrm{span}(u), \qquad \tau(u) = 1, \tag{60}$$

$$\text{spatial associative bundle:} \qquad P_3 \subset \mathrm{Im}\,\mathbb{O}, \qquad \varphi|_{P_3} = \mathrm{vol}_{P_3}, \tag{61}$$

$$\text{internal coassociative bundle:} \qquad Q_4 = P_3^{\perp}. \tag{62}$$

The emergent spacetime tangent bundle is

$$\boxed{TM_4^{\mathrm{eff}} \cong L_t \oplus P_3.} \tag{63}$$

The effective Lorentz metric is

$$\boxed{g_{\mu\nu}^{\text{eff}} = -\tau_\mu \tau_\nu + h_{\mu\nu},} \tag{64}$$

where $h_{\mu\nu}$ is the pullback of $g_\varphi|_{P_3}$. This equation states precisely where Lorentz signature enters: the three spatial directions come from the G_2 metric on an associative plane, while the negative sign comes from the clock order parameter.

C. Dynamical reduction $G_2 \to SO(1,3) \times H_{\text{int}}$

Let Π_P and Π_Q be the projectors onto P_3 and Q_4. A vacuum configuration is a pair $(\Psi_I^{(0)}, P_3)$ minimizing the effective potential

$$V_{\text{eff}} = V_{\text{loc}} + \lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 + \frac{M_Q^2}{2} \sum_I \|\Pi_Q \Psi_I\|^2 + \frac{M_{\text{mis}}^2}{2} \|\varphi|_{P_3} - \text{vol}_{P_3}\|^2. \tag{65}$$

The terms have clear roles. The positive associator term suppresses non-associative excitations. The M_Q term gives a mass gap to coassociative modes. The M_{mis} term energetically selects calibrated associative planes.

The stabilizer inside G_2 of a fixed associative plane is isomorphic to $SO(4)$ at the level of rotations of the normal data. After the clock line is included, the effective symmetry decomposes as

$$\boxed{G_2 \quad \longrightarrow \quad SO(1,3) \times H_{\text{int}},} \tag{66}$$

where $SO(1,3)$ acts on $L_t \oplus P_3$ and $H_{\text{int}} \subset SO(Q_4)$ acts on the massive or compact internal coassociative sector. Equation (66) is not a purely group-theoretic identity inside compact G_2; it is the symmetry of the low-energy projected theory after the clock field has produced Lorentzian signature.

The origin of the symmetry breaking is the competition between the G_2-symmetric pregeometric phase and the energetically preferred associative vacuum. The full octonionic structure is invariant under $G_2 = \text{Aut}(\mathbb{O})$, but a specific vacuum configuration $(\Psi_I^{(0)}, P_3)$ is not invariant under all of G_2. It selects a calibrated associative three-plane P_3 and therefore reduces the symmetry to the subgroup that preserves this choice. In this precise sense the symmetry breaking is spontaneous: the action is formulated in G_2-covariant terms, while the vacuum chooses a particular associative sector. The positive associator term favors configurations for which

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \approx 0,$$

because such configurations lie close to an associative subalgebra. The mass term for $\Pi_Q \Psi_I$ suppresses excursions into the coassociative complement $Q_4 = P_3^\perp$, while the misalignment term fixes the calibration condition

$$\varphi|_{P_3} = \mathrm{vol}_{P_3}.$$

Together these terms dynamically select the low-energy splitting

$$\mathrm{Im}\,\mathbb{O} = P_3 \oplus Q_4.$$

D. Why exactly four macroscopic dimensions remain

The model gives a precise stability criterion:

$$E_{\mathrm{macro}} = E[L_t] + E[P_3] + E[Q_4]. \tag{67}$$

The clock line is required for Hamiltonian evolution and causal ordering. The associative calibrated plane is the largest spatial subspace on which the octonionic product closes associatively as a quaternionic algebra. Therefore it is dynamically stable under the associator potential. Adding any additional macroscopic spatial direction forces generic products out of the associative subalgebra and excites Q_4 modes, increasing the energy by approximately

$$\Delta E_Q \sim M_Q^2 \|\Pi_Q \Psi\|^2 + \lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2. \tag{68}$$

Thus the stable low-energy sector is one temporal dimension plus three associative spatial dimensions. The remaining four Q_4 directions are not absent; they are either compact, massive, decohered, or projected into internal quantum numbers.

E. Associative Projection Principle

Theorem IV.3 (Associative Projection Principle)**.** *Let the vacuum potential be bounded below by Eq. (65) with $\lambda > 0$, $M_Q^2 > 0$, and $M_{\mathrm{mis}}^2 > 0$. Then every low-energy configuration with energy $E \ll M_Q$ and $E \ll M_{\mathrm{mis}}$ admits an effective description on $L_t \oplus P_3$, while non-associative and coassociative corrections are suppressed by powers of E/M_Q, E/M_{mis}, and λE^2.*

Proof. At energies below the positive gaps, excitations with nonzero $\Pi_Q \Psi_I$ or misalignment $\varphi|_{P_3} \neq \mathrm{vol}_{P_3}$ are energetically inaccessible. The remaining fields lie, up to suppressed corrections, in $\mathbb{R} \oplus P_3 \cong \mathbb{H}$, which is associative. Therefore the projected low-energy algebra is associative and supports the usual complex and quaternionic structures used in quantum mechanics and spinor theory. Corrections are controlled by the inverse mass gaps and by the associator coupling. □

F. Novel aspects of the projection mechanism

The associative projection principle should not be confused with the standard mathematical statement that associative three-planes exist inside a G_2 vector space. That statement is purely geometric. The additional physical claim made here is dynamical: the low-energy sector of the variational problem is biased toward configurations for which the associator energy, coassociative excitation energy, and calibration-misalignment energy are simultaneously minimized.

This gives the model three distinctive features. First, the emergence of $1 + 3$ dimensions is not postulated kinematically. It follows from the fact that the largest associative imaginary subspace of the octonions is three-dimensional, while one further independent direction is required for time evolution and causal ordering. Second, the remaining four imaginary directions are interpreted as a coassociative internal sector rather than as additional macroscopic spatial dimensions. Third, deviations from ordinary low-energy physics are not arbitrary; they are controlled by the same non-associative scale M_* and the same associator coupling λ that appear in the master action.

Accordingly, the mechanism is neither a conventional Kaluza–Klein compactification nor a direct G_2 holonomy compactification. It is a dynamical associative projection: the octonionic algebra supplies the possible directions, the G_2 three-form supplies the metric and calibrated subspaces, and the associator potential selects the sector in which effective Lorentzian spacetime and standard quantum theory are recovered.

V. PHYSICAL INTERPRETATION AND CONSTRAINTS OF THE FUNDAMENTAL COUPLINGS

A. Natural-unit dimensions

In $c = \hbar = 1$ units the action is dimensionless and $[\mathcal{L}] = 4$. From the canonical kinetic term

$$\frac{1}{2} g^{\mu\nu} \langle \nabla_\mu \Psi, \nabla_\nu \Psi \rangle \tag{69}$$

one obtains

$$[\Psi] = 1, \qquad [\nabla_\mu] = 1. \tag{70}$$

Since the associator is cubic in fields,

$$[\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)] = 3, \qquad [\|\mathcal{A}\|^2] = 6. \tag{71}$$

Therefore

$$\boxed{[\lambda] = -2.} \tag{72}$$

It is useful to write

$$\lambda = \frac{\eta_A}{M_*^2}, \tag{73}$$

where η_A is dimensionless and M_* is the non-associative scale. If $M_* \sim M_{\rm Pl}$, then $\lambda \sim M_{\rm Pl}^{-2}$ for $\eta_A \sim 1$.

B. Associator coupling λ

The associator action is

$$S_A = -\int d^4x\sqrt{-g}\,\lambda\|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2. \tag{74}$$

The corresponding energy density is positive for

$$\boxed{\lambda \geq 0.} \tag{75}$$

If $\lambda < 0$, the potential is unbounded below in non-associative directions unless stabilized by higher-order terms. Therefore a stable minimal model requires $\lambda \geq 0$.

The dimensionless expansion parameter at energy E is

$$\epsilon_A(E) \sim \eta_A \left(\frac{E}{M_*}\right)^2. \tag{76}$$

Low-energy consistency requires $\epsilon_A(E) \ll 1$ for all experimentally tested energies.

C. Gravitational coupling κ

The Einstein-Hilbert term is not an independent phenomenological insertion. Once the octonionic projection has produced an effective Lorentzian metric $g^{\rm eff}_{\mu\nu}$, diffeomorphism covariance, locality, and second-order metric dynamics uniquely select the Einstein-Hilbert scalar R as the leading gravitational invariant. Thus gravity emerges in two steps: the octonionic G_2 sector supplies the projected metric structure, and the variational principle supplies the Einstein dynamics of this metric.

The Einstein-Hilbert sector is

$$S_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda). \tag{77}$$

In SI units

$$\boxed{\kappa = \frac{8\pi G}{c^4}.} \tag{78}$$

Since $[G] = \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and $[c^4] = \mathrm{m}^4\,\mathrm{s}^{-4}$, one has

$$[\kappa] = \mathrm{kg}^{-1}\,\mathrm{m}^{-1}\,\mathrm{s}^2 = \mathrm{m/J}. \tag{79}$$

In natural units $\kappa = 8\pi G = M_{\mathrm{redPl}}^{-2}$, where $M_{\mathrm{redPl}} = 1/\sqrt{8\pi G}$ is the reduced Planck mass.

The associator sector modifies the gravitational equations only through its stress tensor and possible nonminimal coupling $\xi R\langle\Psi, \Psi\rangle$. Therefore an effective gravitational coupling may be written as

$$\frac{1}{2\kappa_{\mathrm{eff}}} = \frac{1}{2\kappa} - \xi \sum_I \langle \Psi_I, \Psi_I \rangle. \tag{80}$$

Weak-field consistency requires

$$\left| 2\kappa\xi \sum_I \langle \Psi_I, \Psi_I \rangle \right| \ll 1 \tag{81}$$

in the solar-system and laboratory environments.

VI. PHYSICAL INTERPRETATION OF THE OCTONIONIC FIELD

The field Ψ_I is best interpreted as a pregeometric octonionic multiplet. It is not initially a standard scalar, spinor, or gauge field. Those interpretations arise only after projection to the effective four-dimensional Lorentzian spacetime and to an associative subalgebra of $\mathbb{O}$.

The index

$$I = 1, 2, 3 \tag{82}$$

labels the minimal number of independent octonionic fields required for a nontrivial associator interaction. A genuinely non-associative invariant at lowest order therefore requires three algebraically independent field directions,

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \neq 0 \tag{83}$$

in general. Thus the triplet Ψ_I should not be interpreted at the outset as three ordinary particle species. Rather, it represents the minimal algebraic content needed to probe octonionic non-associativity dynamically.

A. Origin of Gravity in the Projected Metric Sector

Gravity arises because the associative projection converts the pregeometric octonionic data into an effective Lorentzian metric field. Once this field is dynamical, the leading generally covariant action at lowest derivative order is the Einstein-Hilbert action. The metric variation then yields Einstein-type equations, while the octonionic fields provide the effective stress-energy source.

B. Mass dimension and Lorentz behavior

With canonical second-order kinetic normalization in four dimensions,

$$\boxed{[\Psi_I] = 1.} \tag{84}$$

Before projection, Ψ_I transforms under internal G_2 automorphisms acting on the imaginary octonionic components. At this level the field is more naturally regarded as an algebra-valued order parameter than as a conventional spacetime particle field.

After projection to M_4^{eff}, its components may be decomposed into Lorentz representations. A purely componentwise projection gives eight real scalar fields,

$$\Psi_I = \Psi_I^{(0)} + \sum_{a=1}^{7} \Psi_I^{(a)} e_a. \tag{85}$$

A quaternionic projection permits spinorial packaging through the relationship between quaternions, Pauli matrices, and low-dimensional Clifford modules. Thus the same underlying octonionic multiplet can generate different effective low-energy fields depending on the projection channel.

This distinction is important. The full octonionic field is not directly observed as an ordinary field on spacetime. Observable matter fields correspond to projected excitations of Ψ_I after the dynamical selection of

$$L_t \oplus P_3 \tag{86}$$

and after suppression of the coassociative directions. In this sense the octonionic field is pregeometric, while scalars, spinors, gauge-like modes, and dark-sector components are effective projected degrees of freedom.

C. Bosonic and fermionic projections

Let $\Pi_H : \mathbb{O} \to \mathbb{H}$ project to a chosen quaternionic subalgebra generated by an associative plane P_3. Then

$$\psi_H = \Pi_H \Psi \tag{87}$$

can be represented as a two-component complex spinor after choosing a complex unit inside $\mathbb{H}$.

Explicitly, for

$$\mathbb{H} = \mathrm{span}\{1, i, j, k\}, \tag{88}$$

one may choose the complex subalgebra

$$\mathbb{C}_i = \mathrm{span}\{1, i\}. \tag{89}$$

Every quaternion can then be written as

$$q = z_1 + z_2 j, \qquad z_1, z_2 \in \mathbb{C}_i, \tag{90}$$

and represented as

$$q \longleftrightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{91}$$

This is the algebraic reason why quaternionic projection naturally leads to spinorial structures.

Fermionic behavior is not the statement that the original classical octonion field is Grassmann-valued. Rather, fermionic fields emerge when the projected degrees of freedom are quantized in a spinor representation of the Clifford algebra and assigned anticommutation relations. Bosonic scalar modes correspond to the real component, norm fluctuations, and projection-invariant amplitudes.

Thus bosonic and fermionic sectors are not introduced as unrelated fundamental objects. They arise as different representation channels of the same octonionic multiplet. The bosonic sector is associated with scalar, amplitude, and norm degrees of freedom, while the fermionic sector is associated with quaternionic spinorial packaging after Lorentzian projection.

D. Relation to Dirac and Schrodinger structures

Choose $\mathbb{H} = \mathrm{span}\{1, i, j, k\} \subset \mathbb{O}$ and a complex unit i. A projected four-real-component object can be written as a two-component complex spinor. With the usual Clifford representation,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{92}$$

the leading Lorentz-covariant first-order equation is

$$(i\gamma^\mu D_\mu - m)\psi = \delta\mathcal{D}_A\psi, \tag{93}$$

where $\delta\mathcal{D}_A$ denotes suppressed non-associative corrections.

The covariant derivative contains the projected geometric and internal connections,

$$D_\mu = \partial_\mu + \Omega_\mu + \mathcal{A}_\mu. \tag{94}$$

Here Ω_μ is the spin connection of the emergent Lorentzian geometry, while $\mathcal{A}_\mu$ denotes possible projected internal gauge connections. In the exact associative limit,

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \to 0, \qquad \delta\mathcal{D}_A \to 0, \tag{95}$$

one recovers the ordinary Dirac equation

$$(i\gamma^\mu D_\mu - m)\psi = 0. \tag{96}$$

The important point is that the Dirac equation is not imposed as an unrelated low-energy equation. It is the leading first-order Lorentz-covariant equation available after the quaternionic projection has supplied a spinorial module and the clock projection has supplied the effective Lorentzian metric. In this sense, the two structures needed for Dirac theory—spinors and Lorentzian Clifford matrices—arise from different parts of the projection:

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \text{spinor module}, \qquad L_t \oplus P_3 \longrightarrow \mathrm{Cl}(1,3). \tag{97}$$

Taking the nonrelativistic approach

$$\psi = e^{-imc^2t/\hbar}\chi \tag{98}$$

and expanding to order v^2/c^2 gives

$$i\hbar\partial_t\chi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V + \delta H_A + \delta H_R\right]\chi. \tag{99}$$

The Schrödinger equation is then the low-velocity, positive-frequency limit of the projected relativistic dynamics. The rest-energy phase is removed by

$$\psi = e^{-imc^2t/\hbar}\chi, \tag{100}$$

and the remaining slowly varying amplitude χ obeys the ordinary nonrelativistic equation up to the suppressed corrections δH_A and δH_R. Therefore standard quantum mechanics is recovered whenever

$$\frac{\|\delta H_A\|}{\|H_{\mathrm{QM}}\|} \ll 1, \qquad \frac{\|\delta H_R\|}{\|H_{\mathrm{QM}}\|} \ll 1. \tag{101}$$

Thus Dirac and Schrodinger dynamics are projection limits, while δH_A and δH_R encode possible octonionic corrections.

The associator correction scales schematically as

$$\frac{\|\delta H_A\|}{\|H_{\mathrm{QM}}\|} \sim \epsilon_A(E) \sim \eta_A \left(\frac{E}{M_*}\right)^2, \tag{102}$$

whereas the curvature correction is controlled by ξR. Therefore the ordinary Dirac and Schrodinger equations are recovered whenever non-associative excitations are suppressed and curvature-induced mass shifts are negligible.

The hierarchy of emergence can be summarized as

$$\Psi_I \longrightarrow \Pi_H \Psi_I \longrightarrow \text{spinorial projection} \longrightarrow \text{Dirac dynamics} \longrightarrow \text{nonrelativistic Schrodinger dynamics.} \tag{103}$$

E. Particle and cosmological interpretation

The projected P_3 sector supplies ordinary spatial and spinorial low-energy degrees of freedom. The Q_4 coassociative sector may appear as heavy internal states, sterile scalar modes, dark-sector excitations, or vacuum energy contributions. In a low-energy effective description one may decompose

$$\Psi_I = \Pi_P \Psi_I + \Pi_Q \Psi_I, \tag{104}$$

where $\Pi_P \Psi_I$ belongs to the associative sector and $\Pi_Q \Psi_I$ belongs to the coassociative sector.

The associative part gives standard projected matter and quantum degrees of freedom. The coassociative part is naturally hidden at low energies if it has a large mass gap, weak coupling to the visible sector, or rapid decoherence into internal degrees of freedom. It is therefore a natural candidate for sterile, dark, or cosmological modes.

Its cosmological relevance follows from the vacuum expectation value

$$\rho_A = \lambda \left\langle \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 \right\rangle + \langle V_{\mathrm{loc}} \rangle, \tag{105}$$

which can behave as a dark-energy-like component if it is slowly varying, or as dark matter if massive projected modes are long-lived and nonrelativistic. The pressure associated with this sector depends on the time dependence of the expectation value. A nearly constant associator energy has an equation of state close to vacuum energy, whereas oscillating massive modes behave approximately as cold matter after averaging over many oscillations.

Thus the same octonionic multiplet has a dual interpretation. Microscopically, it generates the projected fields of quantum theory after associative reduction. Cosmologically, its non-associative and coassociative remnants may contribute to dark-sector phenomenology if they remain dynamically active on large scales.

VII. FIELD EQUATIONS FROM THE MASTER ACTION

A. Metric variation

Varying Eq. (35) with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T^{\text{eff}}_{\mu\nu}, \tag{106}$$

where

$$T^{\text{eff}}_{\mu\nu} = T^{\text{kin}}_{\mu\nu} + T^{\text{pot}}_{\mu\nu} + T^{\text{assoc}}_{\mu\nu} + T^{\xi}_{\mu\nu}, \tag{107}$$

$$T^{\text{kin}}_{\mu\nu} = \sum_I \alpha_I \left(\langle \nabla_\mu \Psi_I, \nabla_\nu \Psi_I \rangle - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \langle \nabla_\rho \Psi_I, \nabla_\sigma \Psi_I \rangle \right), \tag{108}$$

$$T^{\text{pot}}_{\mu\nu} = -g_{\mu\nu} V_{\text{loc}}, \tag{109}$$

$$T^{\text{assoc}}_{\mu\nu} = -g_{\mu\nu} \lambda \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2. \tag{110}$$

For $F = \sum_I \langle \Psi_I, \Psi_I \rangle$, the nonminimal term contributes

$$T^{\xi}_{\mu\nu} = 2\xi \left[F G_{\mu\nu} + g_{\mu\nu} \Box F - \nabla_\mu \nabla_\nu F \right]. \tag{111}$$

Equivalently, the $FG_{\mu\nu}$ part may be moved to the left-hand side, producing the effective coupling in Eq. (80).

B. Field variation and associator adjoints

Let

$$A := \mathcal{A}(\Psi_1, \Psi_2, \Psi_3). \tag{112}$$

The variation of the associator is

$$\delta A = \mathcal{A}(\delta\Psi_1, \Psi_2, \Psi_3) + \mathcal{A}(\Psi_1, \delta\Psi_2, \Psi_3) + \mathcal{A}(\Psi_1, \Psi_2, \delta\Psi_3). \tag{113}$$

Define adjoint maps $\mathcal{D}_I^\dagger$ by

$$\langle A, \mathcal{A}(\Psi_1, \dots, \delta\Psi_I, \dots, \Psi_3)\rangle = \left\langle \mathcal{D}_I^\dagger A, \delta\Psi_I \right\rangle . \tag{114}$$

Then the field equations are

$$\boxed{\alpha_I \Box_g \Psi_I - \frac{\partial V_{\mathrm{loc}}}{\partial \Psi_I} - 2\lambda \mathcal{D}_I^\dagger \mathcal{A}(\Psi_1, \Psi_2, \Psi_3) - 2\xi R \Psi_I = 0, \qquad I = 1, 2, 3.} \tag{115}$$

These are the mathematically precise replacements for a schematic single-field equation.

VIII. QUANTUM PROJECTION AND BORN RULE

The associative projection gives

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C}. \tag{116}$$

This projection has two distinct meanings. First, the map $\mathbb{O} \to \mathbb{H}$ restricts the octonionic degrees of freedom to the dynamically selected associative quaternionic subalgebra determined by the associative three-plane P_3. Second, the map $\mathbb{H} \to \mathbb{C}$ corresponds to the choice of a complex unit inside this quaternionic subalgebra. Once this choice has been made, the projected state space becomes an ordinary complex linear space.

Let

$$\Pi_{\mathbb{H}} : \mathbb{O} \to \mathbb{H}, \qquad \Pi_{\mathbb{C}} : \mathbb{H} \to \mathbb{C} \tag{117}$$

denote the corresponding projections. The effective quantum wavefunction is then

$$\psi = \Pi_{\mathbb{C}} \Pi_{\mathbb{H}} \Psi. \tag{118}$$

The essential point is that the complex Hilbert-space norm is not introduced as an external postulate. It descends from the positive octonionic norm. For octonions one has the real inner product

$$\langle X, Y\rangle_{\mathbb{O}} = \mathrm{Re}(\bar{X}Y), \tag{119}$$

and therefore

$$\|X\|_{\mathbb{O}}^2 = \langle X, X\rangle_{\mathbb{O}} = \bar{X}X \geq 0. \tag{120}$$

After projection to $\mathbb{C}$, this becomes the usual complex inner product

$$\langle \psi, \chi\rangle_{\mathbb{C}} = \int d^3x \, \bar{\psi}(x)\chi(x), \tag{121}$$

and in particular

$$\langle \psi, \psi \rangle_{\mathbb{C}} = \int d^3x \, \bar{\psi}(x)\psi(x) = \int d^3x \, |\psi(x)|^2. \tag{122}$$

Thus the positivity of the probability density is inherited from the normed-division-algebra structure.

Let $\{\phi_n\}$ be an orthonormal basis of the projected complex Hilbert space and expand

$$\psi = \sum_n c_n \phi_n. \tag{123}$$

Using orthonormality,

$$\langle \phi_m, \phi_n \rangle_{\mathbb{C}} = \delta_{mn}, \tag{124}$$

one obtains

$$\|\psi\|^2 = \left\langle \sum_m c_m \phi_m, \sum_n c_n \phi_n \right\rangle_{\mathbb{C}} = \sum_{m,n} \bar{c}_m c_n \langle \phi_m, \phi_n \rangle_{\mathbb{C}} = \sum_n |c_n|^2. \tag{125}$$

For a normalized state,

$$\|\psi\|^2 = 1, \tag{126}$$

this gives

$$\sum_n |c_n|^2 = 1. \tag{127}$$

The natural probability assigned to the outcome associated with the basis state ϕ_n is therefore

$$P_n = |c_n|^2, \qquad \sum_n P_n = 1. \tag{128}$$

In the continuous position representation the same argument yields

$$P(\Omega) = \int_\Omega d^3x \, |\psi(x)|^2 \tag{129}$$

for the probability of detecting the projected state in a spatial region Ω. Hence the usual position-space Born rule is recovered as the local norm density of the complex projection.

This construction also clarifies the role of superposition. Since the final projected algebra $\mathbb{C}$ is associative and linear, two projected states ψ_1 and ψ_2 may be combined as

$$\psi = c_1\psi_1 + c_2\psi_2, \qquad c_1, c_2 \in \mathbb{C}. \tag{130}$$

The probability weights of the resulting state are again determined by the same projected norm. In this sense, linear superposition and Born probabilities arise together from the associative complex projection.

The argument does not prove the full measurement postulates from algebra alone. In particular, it does not by itself explain why a specific outcome is realized in an individual measurement. Rather, it establishes the more precise statement that, once the octonionic system is dynamically projected onto an associative complex Hilbert space, the positive probability measure required by quantum mechanics is naturally inherited from the octonionic norm. The Born rule is therefore not an additional arbitrary structure in the projected theory, but the probabilistic image of the positive normed-division-algebra geometry.

IX. SUPERPOSITION, ENTANGLEMENT, AND DECOHERENCE FROM ASSOCIATIVE PROJECTION

The preceding section showed that the positive Hilbert-space norm and the Born probabilities arise after the projection

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C}. \tag{131}$$

We now clarify how ordinary quantum superposition, entanglement, and decoherence appear in the same projected framework.

A. Superposition as linearity inside an associative projection

Let $P_3 \subset \operatorname{Im}\mathbb{O}$ be the dynamically selected associative spatial three-plane and let

$$\Pi_{\mathbb{H}} : \mathbb{O} \to \mathbb{H}_{P_3} \tag{132}$$

denote the projection onto the quaternionic subalgebra generated by

$$\mathbb{H}_{P_3} = \mathbb{R} \oplus P_3. \tag{133}$$

After choosing a complex unit $i \in P_3$, one obtains the further complex projection

$$\Pi_{\mathbb{C}} : \mathbb{H}_{P_3} \to \mathbb{C}_i. \tag{134}$$

The projected quantum state is therefore

$$\psi = \Pi_{\mathbb{C}}\Pi_{\mathbb{H}}\Psi. \tag{135}$$

Since $\mathbb{C}_i$ is associative and linear, the projected state space is a complex vector space. Hence, for two projected solutions ψ_1, ψ_2 and complex coefficients $c_1, c_2 \in \mathbb{C}_i$, the combination

$$\psi = c_1\psi_1 + c_2\psi_2 \tag{136}$$

is again a valid projected state.

This gives the precise meaning of superposition in the present framework: superposition is not assumed at the fundamental octonionic level as an independent postulate, but appears as the linear structure of the associative complex projection. The non-associative sector controls deviations from exact linearity.

At leading order the projected Hamiltonian has the form

$$H_{\text{eff}} = H_{\text{QM}} + \delta H_A + \delta H_R, \tag{137}$$

where H_{QM} is the standard complex quantum Hamiltonian, δH_A is the suppressed associator-induced correction, and δH_R is the curvature-induced correction. Therefore the projected Schrödinger equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = (H_{\text{QM}} + \delta H_A + \delta H_R)\, \psi. \tag{138}$$

Exact linear superposition is recovered whenever

$$\|\delta H_A\| \ll \|H_{\text{QM}}\|, \qquad \|\delta H_R\| \ll \|H_{\text{QM}}\|. \tag{139}$$

B. Associator corrections to the superposition principle

Let

$$\psi_a := \Pi_{\mathbb{C}} \Pi_{\mathbb{H}} \Psi_a, \qquad a = 1, 2, \tag{140}$$

be two projected states. At the octonionic level, the product of field configurations is not generically associative. Therefore the deviation from exact projected linearity may be measured by the associator functional

$$\Delta_A(\Psi_1, \Psi_2, \Psi_3) = \Pi_{\mathbb{C}} \Pi_{\mathbb{H}} A(\Psi_1, \Psi_2, \Psi_3). \tag{141}$$

In the low-energy associative sector one has

$$A(\Psi_1, \Psi_2, \Psi_3) \approx 0, \tag{142}$$

and therefore

$$\Delta_A \approx 0. \tag{143}$$

The ordinary superposition principle is thus recovered as the low-energy limit

$$E \ll M_*, \qquad \epsilon_A(E) = \eta_A \left(\frac{E}{M_*}\right)^2 \ll 1. \tag{144}$$

Corrections to superposition are controlled by the same small parameter that controls the interference corrections discussed later:

$$\frac{\|\delta H_A\|}{\|H_{\mathrm{QM}}\|} \sim \epsilon_A(E). \tag{145}$$

Thus the model is automatically consistent with standard interference experiments if M_* is sufficiently large or if η_A is sufficiently small.

C. Entanglement from non-factorizing projected dynamics

Consider two projected subsystems A and B with states

$$\psi_A = \Pi_{\mathbb{C}}\Pi_{\mathbb{H}}\Psi_A, \qquad \psi_B = \Pi_{\mathbb{C}}\Pi_{\mathbb{H}}\Psi_B. \tag{146}$$

If the full octonionic dynamics contains only separable terms, the effective Hamiltonian takes the factorized form

$$H_{\mathrm{eff}} = H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B. \tag{147}$$

Such a Hamiltonian cannot create entanglement from an initially product state.

In contrast, the octonionic interaction naturally contains a trilinear non-associative coupling. For a mediating field Ξ one obtains the interaction density

$$V_{\mathrm{ent}} = \lambda \left\| A(\Psi_A, \Xi, \Psi_B) \right\|^2 . \tag{148}$$

After projection this induces an effective interaction

$$H_{\mathrm{ent}} = \Pi_{\mathbb{C}}\Pi_{\mathbb{H}} \left[\lambda \left\| A(\Psi_A, \Xi, \Psi_B) \right\|^2 \right] . \tag{149}$$

Generically this term cannot be written as

$$H_{\mathrm{ent}} \neq H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B. \tag{150}$$

Therefore an initially separable state

$$|\psi(0)\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{151}$$

evolves into

$$|\psi(t)\rangle = \sum_{mn} c_{mn}(t)|m\rangle_A \otimes |n\rangle_B, \tag{152}$$

with, in general,

$$c_{mn}(t) \neq a_m(t)b_n(t). \tag{153}$$

The non-associative associator sector therefore supplies a natural microscopic source of non-factorizing quantum correlations. Standard Bell and CHSH correlations are recovered in the projected Hilbert space, while possible non-associative deviations are suppressed by $\epsilon_A(E)$.

D. Reduced density matrix and decoherence

To describe decoherence, split the projected degrees of freedom into a measured subsystem S and an environment E:

$$\mathcal{H}_{\text{tot}} := \mathcal{H}_S \otimes \mathcal{H}_E. \tag{154}$$

The total state may be written as

$$|\Psi_{\text{tot}}\rangle = \sum_n c_n|n\rangle_S \otimes |E_n\rangle. \tag{155}$$

The reduced density matrix of the subsystem is

$$\rho_S = \text{Tr}_E\,|\Psi_{\text{tot}}\rangle\langle\Psi_{\text{tot}}| = \sum_{mn} c_m c_n^* \langle E_n|E_m\rangle |m\rangle\langle n|. \tag{156}$$

The off-diagonal coherence terms are controlled by the overlaps

$$\Gamma_{mn}(t) = \langle E_n(t)|E_m(t)\rangle. \tag{157}$$

Decoherence occurs when

$$|\Gamma_{mn}(t)| \longrightarrow 0 \qquad (m \neq n), \tag{158}$$

so that

$$\rho_S \longrightarrow \sum_n |c_n|^2 |n\rangle\langle n|. \tag{159}$$

Thus the Born weights obtained from the projected norm become the diagonal probabilities of the decohered reduced state.

E. Octonionic contribution to the decoherence rate

In the present model the environment is not only an ordinary external bath. It also includes suppressed coassociative and non-associative degrees of freedom. The total environment may be schematically decomposed as

$$E = E_{\text{ordinary}} \oplus Q_4 \oplus E_A, \tag{160}$$

where Q_4 denotes the coassociative sector and E_A denotes associator excitations.

The effective decoherence rate may therefore be written as

$$\Gamma_{\text{dec}} = \Gamma_{\text{env}} + \Gamma_A + \Gamma_R. \tag{161}$$

Here Γ_{env} is the standard environmental decoherence rate, Γ_A is the associator-induced contribution, and Γ_R is the curvature-induced contribution. At leading order one may parameterize

$$\Gamma_A \sim \frac{1}{\hbar^2} S_A(\omega) \left(\Delta O_A\right)^2, \tag{162}$$

where $S_A(\omega)$ is the spectral density of associator fluctuations and ΔO_A is the difference of the associator-sensitive observable between two branches. Since the associator sector is suppressed at low energy,

$$S_A(\omega) \propto \epsilon_A(E)^2, \tag{163}$$

one obtains the estimate

$$\Gamma_A \sim \Gamma_0 \eta_A^2 \left(\frac{E}{M_*}\right)^4, \tag{164}$$

with Γ_0 a system-dependent microscopic rate.

This result has two important consequences. First, for M_* near the Planck scale, associator-induced decoherence is negligible in ordinary laboratory quantum mechanics. Second, mesoscopic interference experiments are especially relevant, because they maximize the product of coherence time, mass, spatial separation, and environmental isolation.

F. Pointer basis from the associative projection

The preferred decoherence basis is selected by the interaction Hamiltonian. In the present framework the dynamically stable low-energy states are those that minimize the excitation of coassociative and non-associative modes. Therefore pointer states are approximately eigenstates of the associative projection operator:

$$\Pi_{\mathbb{H}}|\pi_n\rangle \approx |\pi_n\rangle, \qquad \Pi_Q|\pi_n\rangle \approx 0. \tag{165}$$

Equivalently, stable classical branches satisfy

$$\|A(\Psi_1, \Psi_2, \Psi_3)\|^2 \approx 0 \tag{166}$$

within each branch. This gives a physical interpretation of classicality: classical branches are those projected histories that remain close to associative subalgebras and therefore do not strongly excite the non-associative sector.

G. Classical limit

The classical limit is reached when three conditions hold simultaneously:

$$\epsilon_A(E) \ll 1, \tag{167}$$

$$|\Gamma_{mn}(t)| \ll 1 \quad (m \neq n), \tag{168}$$

$$\frac{S_{\text{cl}}}{\hbar} \gg 1. \tag{169}$$

The first condition suppresses fundamental non-associative corrections. The second suppresses interference between macroscopically distinct branches. The third is the usual WKB condition for classical trajectories. Under these assumptions the reduced density matrix becomes diagonal in the pointer basis,

$$\rho_S \simeq \sum_n p_n|\pi_n\rangle\langle\pi_n|, \qquad p_n = |c_n|^2, \tag{170}$$

and the dynamics of the dominant branches follows the classical projected equations of motion.

Thus the chain is

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C} \longrightarrow \text{linear quantum superposition} \longrightarrow \text{entanglement} \longrightarrow \text{decohered classical branches.} \tag{171}$$

The same associative projection that produces the complex Hilbert space also identifies the low-energy stable branches, while the associator sector controls possible deviations from exact quantum linearity and standard decoherence.

X. QUANTITATIVE PREDICTIONS AND EXPERIMENTAL SIGNATURES

The estimates in this section are not intended as parameter-free predictions. They are effective-field-theory estimates that translate the proposed non-associative sector into observable quantities. Their novelty lies in the fact that interference corrections, curvature-induced quantum shifts, decoherence contributions, and high-energy dispersion deviations are all traced back to the same trilinear associator interaction. This makes the framework falsifiable in principle: a future bound on one class of deviations constrains the same scale M_* and dimensionless strength η_A that control the others. The model is effective. Quantitative predictions are therefore expressed in terms of the non-associative scale M_*, the dimensionless strength η_A, and curvature coupling ξ.

A. Quantum-interference corrections

For a coherent process with duration T and characteristic energy E, the associator-induced Hamiltonian correction can be parameterized as

$$\delta H_A \sim E\,\epsilon_A(E), \qquad \epsilon_A(E) = \eta_A \left(\frac{E}{M_*}\right)^2. \tag{172}$$

The phase shift is

$$\boxed{\Delta\phi_A \sim \frac{\delta H_A T}{\hbar} \sim \frac{ET}{\hbar}\eta_A \left(\frac{E}{M_* c^2}\right)^2.} \tag{173}$$

For $M_* = M_{\rm Pl} = 1.22 \times 10^{19}\,\text{GeV}$ and $\eta_A = 1$:

$$E = 1\,\text{eV}, \quad T = 1\,\text{s} \Rightarrow \Delta\phi_A \sim 1.5 \times 10^{15} \times (10^{-28})^2 \sim 10^{-41}, \tag{174}$$

$$E = 1\,\text{TeV}, \quad T = 10^{-24}\,\text{s} \Rightarrow \Delta\phi_A \sim 1.5 \times 10^{3} \times (10^{-16})^2 \sim 10^{-29}. \tag{175}$$

Thus Planck-scale non-associativity is invisible in ordinary low-energy interferometry. Observable laboratory effects require either a much lower M_*, coherent enhancement, or a special near-degenerate system.

A third-order-interference observable may be written as

$$\kappa_3 = \frac{I_{ABC} - I_{AB} - I_{AC} - I_{BC} + I_A + I_B + I_C - I_0}{I_{ABC}}. \tag{176}$$

Standard quantum mechanics gives $\kappa_3 = 0$. The octonionic estimate is

$$|\kappa_3| \lesssim C_A \epsilon_A(E), \tag{177}$$

with C_A a geometry-dependent coefficient. Existing null tests therefore bound $C_A \eta_A (E/M_*)^2$.

B. Curvature-modified quantum dynamics

The nonminimal term $-\xi R\langle\Psi,\Psi\rangle$ induces an effective curvature-dependent mass shift. For a projected scalar or nonrelativistic matter mode this may be written as

$$m_{\text{eff}}^2 = m^2 + \frac{\xi\hbar^2}{c^2}R. \tag{178}$$

In the nonrelativistic limit this gives the Hamiltonian correction

$$\delta H_R \simeq \frac{\hbar^2 \xi R}{2m}. \tag{179}$$

The corresponding frequency shift is

$$\Delta\nu_R = \frac{\delta H_R}{h} = \frac{\hbar\xi R}{4\pi m}. \tag{180}$$

In exterior vacuum Schwarzschild spacetime $R = 0$, so this specific Ricci-scalar correction vanishes outside a spherical body. This does not mean that spacetime curvature is absent; the Riemann tensor is nonzero. It only means that a coupling proportional to R gives no exterior vacuum correction. Other curvature couplings, for example contractions involving $R_{\mu\nu\rho\sigma}$, would have to be introduced separately.

Inside nonrelativistic matter, Einstein's equation gives approximately

$$R \simeq \frac{8\pi G}{c^2}\rho, \tag{181}$$

when pressure and Λ are negligible. For Earth mean density

$$\rho_\oplus \simeq 5.5 \times 10^3\,\mathrm{kg\,m^{-3}}, \tag{182}$$

one obtains

$$R_\oplus \sim 1.0 \times 10^{-22}\,\mathrm{m^{-2}}. \tag{183}$$

For a neutron with

$$m_n = 1.675 \times 10^{-27}\,\mathrm{kg}, \tag{184}$$

this gives

$$\Delta\nu_R \sim 5 \times 10^{-31}\xi\,\mathrm{Hz}. \tag{185}$$

Near nuclear-density matter,

$$\rho \sim 10^{17}\,\mathrm{kg\,m^{-3}}, \tag{186}$$

one obtains

$$R \sim 2 \times 10^{-9}\,\mathrm{m}^{-2}, \tag{187}$$

and therefore

$$\Delta\nu_R \sim 10^{-17}\xi\,\mathrm{Hz} \tag{188}$$

for neutron-mass probes. Thus Ricci-scalar-induced quantum shifts are far below laboratory sensitivity for $\xi \sim 1$, and even at nuclear densities they remain extremely small unless ξ is very large or additional curvature couplings are present.

C. High-energy deviations

At energies approaching the non-associative scale, the associative projection need not remain exact. The leading effect can be described by an effective modified dispersion relation. Let

$$E_0^2 = p^2c^2 + m^2c^4 \tag{189}$$

be the standard relativistic dispersion relation. A leading non-associative correction may be parameterized as

$$E^2 = E_0^2 + \zeta_A \frac{E_0^4}{M_*^2c^4} + \mathcal{O}\left(\frac{E_0^6}{M_*^4c^8}\right), \tag{190}$$

where M_* is the non-associative scale and ζ_A is a dimensionless coefficient depending on the projection channel and the normalization of the effective associator operator.

For

$$|\zeta_A|\left(\frac{E_0}{M_*c^2}\right)^2 \ll 1, \tag{191}$$

the fractional energy shift is

$$\frac{\delta E}{E_0} \simeq \frac{\zeta_A}{2}\left(\frac{E_0}{M_*c^2}\right)^2. \tag{192}$$

This has the same suppression structure as the low-energy associator parameter

$$\epsilon_A(E_0) \sim \eta_A\left(\frac{E_0}{M_*c^2}\right)^2. \tag{193}$$

Thus high-energy deviations, interference corrections, and associator-induced quantum effects are controlled by the same basic hierarchy.

For a Planck-scale non-associative scale,

$$M_*c^2 \simeq 1.22 \times 10^{19}\,\mathrm{GeV}, \tag{194}$$

one finds at collider energies

$$E_0 = 10\,\mathrm{TeV} = 10^4\,\mathrm{GeV}, \tag{195}$$

and hence

$$\frac{\delta E}{E_0} \simeq \frac{\zeta_A}{2}\left(\frac{10^4}{1.22 \times 10^{19}}\right)^2 \sim 3.4 \times 10^{-31}\zeta_A. \tag{196}$$

For ultra-high-energy cosmic rays with

$$E_0 = 10^{11}\,\mathrm{GeV}, \tag{197}$$

one obtains

$$\frac{\delta E}{E_0} \simeq \frac{\zeta_A}{2}\left(\frac{10^{11}}{1.22 \times 10^{19}}\right)^2 \sim 3.4 \times 10^{-17}\zeta_A. \tag{198}$$

Planck-scale non-associative corrections are therefore completely negligible at present collider energies and remain small even for ultra-high-energy cosmic rays unless ζ_A is large, M_* lies below the Planck scale, or the relevant process contains a coherent enhancement mechanism.

This parametrization should be understood as an effective high-energy expansion, not as a unique prediction of the minimal action alone. A complete calculation would require deriving ζ_A from the projected associator operator in a specified particle or field channel.

TABLE I. Indicative parameter scales. Values are order-of-magnitude estimates for $\eta_A, \zeta_A, C_A \sim 1$, with $M_*c^2 \simeq 1.22 \times 10^{19}$ GeV.

Observable	Scaling	Planck-scale estimate	Comment
Interference phase	$\Delta\phi_A \sim (ET/\hbar)(E/M_*c^2)^2$	10^{-41} for $E = 1\,\mathrm{eV}$, $T = 1\,\mathrm{s}$	negligible unless enhanced
Third-order interference	$\lvert\kappa_3\rvert \lesssim C_A(E/M_*c^2)^2$	10^{-56} at eV scale	strong null consistency
Curvature shift, Earth	$\Delta\nu_R = \hbar\xi R/(4\pi m)$	$5 \times 10^{-31}\xi\,\mathrm{Hz}$	negligible for $\xi \sim 1$
Curvature shift, nuclear density	same	$10^{-17}\xi\,\mathrm{Hz}$	tiny unless $\xi \gg 1$
Dispersion deviation, 10 TeV	$\delta E/E_0 \simeq (\zeta_A/2)(E_0/M_*c^2)^2$	$3.4 \times 10^{-31}\zeta_A$	collider-safe
Dispersion deviation, 10^{11} GeV	same	$3.4 \times 10^{-17}\zeta_A$	cosmic-ray regime

XI. LOW-ENERGY AND WEAK-FIELD CONSISTENCY

A. Quantum mechanics

Low-energy quantum mechanics is recovered when

$$\epsilon_A(E) = \eta_A(E/M_*)^2 \ll 1. \tag{199}$$

Then

$$H_{\text{eff}} = H_{\text{QM}} + \delta H_A + \delta H_R, \qquad \|\delta H_A\| \ll \|H_{\text{QM}}\|, \qquad \|\delta H_R\| \ll \|H_{\text{QM}}\|. \tag{200}$$

Consequently double-slit interference, Bell tests, CHSH correlations, atom interferometry, precision QED, and quantum optics remain standard to current accuracy. In Bell and CHSH experiments the projected complex Hilbert space and Born rule are unchanged at leading order. Non-associative corrections enter only as suppressed higher-order phase or state-evolution perturbations.

B. General relativity

Let

$$g_{\mu\nu} := g^{\text{GR}}_{\mu\nu} + \delta g^{(A)}_{\mu\nu}. \tag{201}$$

Linearizing the Einstein equation gives schematically

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma}\delta g^{(A)}_{\rho\sigma} = \kappa T^{(A)}_{\mu\nu} + \mathcal{O}(\delta g^2), \tag{202}$$

where $\mathcal{E}$ is the usual Lichnerowicz operator and

$$T^{(A)}_{\mu\nu} = -g_{\mu\nu}\lambda\|\mathcal{A}\|^2 + T^{\xi}_{\mu\nu} + \cdots . \tag{203}$$

Weak-field consistency requires

$$\boxed{|\delta g^{(A)}_{\mu\nu}| \ll 1} \tag{204}$$

in regimes where light deflection, Shapiro delay, perihelion precession, gravitational redshift, binary-pulsar timing, and gravitational waves have been tested. In the associative low-energy vacuum $\mathcal{A} \approx 0$ and F nearly constant, this condition is naturally satisfied. The Schwarzschild exterior is recovered when $R_{\mu\nu} = 0$, $\mathcal{A} = 0$, and $\nabla_\mu \Psi_I = 0$ outside the source.

C. PPN constraint form

The parametrized post-Newtonian metric may be written as

$$g_{00} = -1 + 2U - 2\beta_{\rm PPN}U^2 + \delta g_{00}^{(A)}, \tag{205}$$

$$g_{ij} = (1 + 2\gamma_{\rm PPN}U)\delta_{ij} + \delta g_{ij}^{(A)}. \tag{206}$$

The model is compatible with solar-system tests if

$$|\delta\gamma_A| \ll 10^{-5}, \qquad |\delta\beta_A| \ll 10^{-4}, \tag{207}$$

which translates into bounds on $\kappa\lambda\|\mathcal{A}\|^2$ and $\kappa\xi F$ in the solar system.

XII. RELATION TO EXISTING WORK AND NOVELTY OF THE PRESENT APPROACH

Octonions have a long history in mathematical physics. Their normed division algebra structure, their relation to triality, and their exceptional automorphism group G_2 are standard results [1, 4–6]. The use of octonions in attempts to organize spinors, internal symmetries, exceptional groups, and Standard-Model-like structures is also well established [2, 13, 14, 17, 18, 23, 24]. Similarly, G_2 geometry, stable three-forms, metric reconstruction, and associative/coassociative calibrated submanifolds are classical subjects in differential geometry [7–10, 12].

The present work does not claim novelty for these ingredients individually. Instead, its contribution is the way in which they are combined into a dynamical low-energy projection mechanism. Table eftab:prior-comparison summarizes the most relevant distinctions.

In particular, the paper differs from earlier algebraic octonionic model building in four respects.

First, the associator is not used only as an algebraic curiosity or as a kinematic obstruction. It is promoted to a positive variational energy density,

$$\lambda\|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2, \tag{208}$$

which vanishes on associative subalgebras and penalizes genuinely octonionic directions. This makes non-associativity both physically active and dynamically controlled.

Second, the model corrects a common single-field temptation. Since the octonions are alternative,

$$\mathcal{A}(\Psi, \Psi, \Psi) = 0, \tag{209}$$

Reference direction	Main role of octonions/non-associativity	Relation to spacetime/quantum theory	Difference of the present work
Gogberashvili [19, 20]	Split-octonionic geometry and electrodynamics; Maxwell/Dirac-type structures in octonionic notation	Field equations are represented in split-octonionic form	Here the associator norm is a positive variational sector selecting an associative low-energy projection rather than mainly a reformulation of electrodynamics
Köplinger–Dzhunushaliev [21, 22]	Non-associative quantum-theoretic models; emergent probability and modified Born-type ideas	Non-associativity modifies quantum-theoretic structure directly	Here ordinary complex Hilbert space and Born weights are recovered only after the dynamical chain $\mathbb{O} \to \mathbb{H} \to \mathbb{C}$, with deviations controlled by ϵ_A
Krasnov [23]	Octonionic and split-octonionic realization of spinorial particle structure in a $\mathrm{Spin}(11,3)$ setting	Organizes Standard-Model-like fermionic degrees of freedom and symmetry breaking	The present paper does not construct a $\mathrm{Spin}(11,3)$ particle model; it focuses on a trilinear associator potential and emergent $1+3$-dimensional projection
Lasenby [24]	Representation of $SU(3)$ and octonions inside spacetime/geometric algebra	Links octonions to spacetime algebra and internal symmetry representations	The present paper uses an octonionic G_2 starting point and a dynamical associative projection rather than embedding octonions into spacetime algebra
Modern octonionic analysis [34–37]	Functional analysis, kernels, Hilbert spaces, weak associativity and non-associative operator subtleties	Establishes mathematical caution needed for octonionic linear structures	The present paper uses projected complex Hilbert spaces to avoid treating the full non-associative algebra as an ordinary Hilbert scalar field
Non-geometric string flux models [30–33]	Non-associative products arise effectively from flux backgrounds or phase-space deformations	Non-associativity is usually induced by a background deformation	Here non-associativity is primitive at the algebraic level and enters through $\lambda\|\mathcal{A}\|^2$

TABLE II. Relation to representative prior work. The table is intended to clarify what is adopted from the literature and what is specific to the present proposal.

and therefore a nontrivial associator sector requires at least three independent octonion-valued fields or field components. The trilinear structure of $\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)$ is therefore not an optional complication but the minimal nontrivial possibility.

Third, four-dimensional Lorentzian spacetime is not identified directly with the seven-dimensional G_2 structure. A positive G_2 three-form reconstructs a Riemannian metric. The Lorentzian theory emerges only after a clock line L_t and an associative spatial three-plane P_3 are dynamically selected:

$$TM_4^{\text{eff}} \simeq L_t \oplus P_3, \qquad g_{\mu\nu}^{\text{eff}} = -\tau_\mu \tau_\nu + h_{\mu\nu}. \tag{210}$$

This distinguishes the present construction from direct identifications of octonionic imaginary directions with spacetime coordinates.

Fourth, the coassociative complement

$$Q_4 = P_3^\perp \tag{211}$$

is retained as a physical sector. It may appear as massive internal modes, compact degrees of freedom, sterile excitations, dark-sector components, or suppressed environmental degrees of freedom. Thus the projection is not a discarding of four directions, but a dynamical separation between macroscopic associative spacetime directions and non-macroscopic coassociative modes.

The proposal also differs from non-associative geometries appearing in string-theoretic flux backgrounds. In those settings, non-associativity often arises as a deformation of phase-space products or as an effective structure induced by non-geometric fluxes. Here, by contrast, non-associativity is taken as primitive at the algebraic level and enters the action through a positive octonionic associator norm.

Finally, the quantum-mechanical part of the construction is not an alternative postulate of Hilbert space. The Hilbert structure is recovered through the projection chain

$$\mathbb{O} \longrightarrow \mathbb{H} \longrightarrow \mathbb{C}. \tag{212}$$

The complex norm gives the Born weights, the complex vector space gives linear superposition, the projected trilinear interaction gives non-factorizing dynamics and hence entanglement, and the suppression of coassociative and associator modes contributes to decoherence and classical branch stability.

The distinctive claim of the paper is therefore the following: a single positive trilinear associator sector, together with a dynamical associative projection, can organize the simultaneous emergence of effective Lorentzian spacetime, Einstein-type metric dynamics, complex quantum mechanics, entanglement, decoherence, and controlled deviations from standard low-energy physics. This combined mechanism appears to be the genuinely new element of the framework.

XIII. PHENOMENOLOGICAL OUTLOOK: ASSOCIATOR-MEDIATED INTERACTIONS

The framework developed in this paper does not require an observable fifth force. However, it contains a natural phenomenological channel through which such an interaction may arise. If the associator sector contains a sufficiently light projected fluctuation and if this fluctuation couples weakly to ordinary projected matter, then it can mediate an additional finite-range interaction. This possibility should be understood as an outlook, not as a necessary consequence of the minimal model.

A. Light associator mode

The non-associative part of the master action is governed by

$$V_A = \lambda \left\| \mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \right\|^2 . \tag{213}$$

In the associative low-energy vacuum,

$$\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \approx 0. \tag{214}$$

A small projected fluctuation away from this vacuum may be described by an effective scalar mode

$$a(x) \equiv \delta \left[\mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \right]_{\mathrm{proj}} . \tag{215}$$

Expanding the associator potential around the vacuum gives

$$V_A = V_A^{(0)} + \frac{1}{2} m_A^2 a^2 + \mathcal{O}(a^3), \tag{216}$$

where m_A is the effective associator-mode mass. Its Compton wavelength defines the force range,

$$\ell_A = \frac{\hbar}{m_A c}. \tag{217}$$

If m_A is large, the interaction is short-ranged and practically invisible. If m_A is small, the mode may generate an observable macroscopic correction to gravity.

B. Coupling to projected matter

At low energies, the leading coupling of the associator mode to ordinary matter can be written as

$$\mathcal{L}_{\text{int}} = -g_A a\,\mathcal{O}_m, \tag{218}$$

where g_A is a dimensionless coupling and $\mathcal{O}_m$ is a projected matter operator. For nonrelativistic macroscopic matter, the dominant scalar source is approximately the rest-mass energy density,

$$\mathcal{O}_m \simeq \rho_m c^2. \tag{219}$$

For a point particle of mass m, this becomes

$$\mathcal{L}_{\text{int}} \simeq -g_A a\, mc^2. \tag{220}$$

The coupling is universal if $a(x)$ couples only to total projected mass-energy. It becomes composition-dependent if the coupling depends on the detailed octonionic projection channel of different types of matter.

C. Yukawa-type correction to Newtonian gravity

For two nonrelativistic bodies with masses m_1 and m_2, exchange of a scalar associator mode produces the effective potential

$$V_A(r) = -\alpha_A \frac{Gm_1m_2}{r} e^{-r/\ell_A}. \tag{221}$$

Here ℓ_A is the force range and α_A is the strength relative to Newtonian gravity. The total potential may be written as

$$V_{\text{tot}}(r) = -\frac{Gm_1m_2}{r}\left[1 + \alpha_A e^{-r/\ell_A}\right]. \tag{222}$$

The corresponding force is

$$F_{\text{tot}}(r) = -\frac{Gm_1m_2}{r^2}\left[1 + \alpha_A\left(1 + \frac{r}{\ell_A}\right) e^{-r/\ell_A}\right]. \tag{223}$$

Equivalently, the fractional deviation from Newtonian gravity is

$$\frac{\Delta F}{F_N} = \alpha_A\left(1 + \frac{r}{\ell_A}\right) e^{-r/\ell_A}, \qquad F_N = \frac{Gm_1m_2}{r^2}. \tag{224}$$

Thus the phenomenology takes the standard Yukawa form used in searches for fifth forces. In the present interpretation, the microscopic origin of the mediator is the projected associator mode.

D. Relation between microscopic and phenomenological parameters

The observable parameters are

$$\alpha_A, \qquad \ell_A. \tag{225}$$

The microscopic parameters are the associator stiffness, the non-associative scale, the matter coupling, and the associator-mode mass:

$$\lambda, \qquad M_*, \qquad g_A, \qquad m_A. \tag{226}$$

The range is directly fixed by m_A,

$$\ell_A = \frac{\hbar}{m_A c}. \tag{227}$$

The strength depends on the normalization of $a(x)$ and on its coupling to matter. Schematically, one may write

$$\alpha_A \sim \frac{g_A^2}{4\pi G m_{\rm ref}^2}, \tag{228}$$

where $m_{\rm ref}$ is the mass scale that normalizes the effective matter coupling. A complete phenomenological analysis must derive $m_{\rm ref}$ from the chosen projection channel.

The associator-mode mass is controlled by the curvature of the associator potential:

$$m_A^2 \sim \frac{\partial^2}{\partial a^2} \lambda \left\| \mathcal{A}(\Psi_1, \Psi_2, \Psi_3) \right\|^2 \Big|_{\rm vac}. \tag{229}$$

Large λ typically makes non-associative modes heavy and short-ranged. However, an approximately flat direction in the associator sector could keep one projected mode light while the remaining non-associative excitations stay suppressed.

E. Consistency with existing tests

Existing fifth-force and inverse-square-law tests already constrain the pair

$$(\alpha_A, \ell_A). \tag{230}$$

The present framework remains consistent with current null results if the associator mode is weakly coupled, sufficiently massive, short-ranged, screened, or absent from the low-energy spectrum:

$$\alpha_A \ll 1, \tag{231}$$

$$\ell_A \ll r_{\rm exp}, \tag{232}$$

$$g_A \ll 1, \tag{233}$$

$$m_A \gg \frac{\hbar}{c\, r_{\rm exp}}. \tag{234}$$

Here $r_{\rm exp}$ denotes the characteristic length scale of a given experiment.

Relevant laboratory probes include torsion-balance experiments, inverse-square-law tests, Casimir-force measurements, atom interferometry, and equivalence-principle tests. Astrophysical and cosmological constraints may arise from binary systems, stellar dynamics, large-scale structure, and the expansion history.

F. Composition dependence and equivalence principle tests

If the associator mode couples universally to total projected mass-energy, then it modifies gravity without violating composition independence at leading order. If, however, the coupling depends on internal octonionic projection data, different materials may source or respond to $a(x)$ with slightly different strengths. This would lead to an apparent equivalence-principle violation, usually quantified by an Eötvös parameter

$$\eta_{12} = 2\frac{|a_1 - a_2|}{a_1 + a_2}. \tag{235}$$

In the universal-coupling limit,

$$\eta_{12} \approx 0, \tag{236}$$

whereas a composition-dependent associator coupling would give

$$\eta_{12} \neq 0. \tag{237}$$

This makes equivalence-principle measurements a particularly sensitive probe of the projection structure of the model.

G. Connection to dark-sector phenomenology

A very light and weakly coupled associator mode can affect cosmology. The same degree of freedom that mediates a possible finite-range fifth interaction may contribute to the cosmic dark sector if it survives as a slowly varying background, a coherently oscillating condensate, or a population of long-lived projected excitations. Thus fifth-force phenomenology and dark-sector cosmology are two possible infrared manifestations of the same non-associative sector.

The relevant effective energy density is

$$\rho_A = \lambda \left\langle \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 \right\rangle + \langle V_{\rm loc} \rangle + \rho_{\rm kin}^{(A)}, \tag{238}$$

where $\rho_{\rm kin}^{(A)}$ denotes possible kinetic contributions from time-dependent associator modes. If the projected associator fluctuation is described by an effective scalar mode $a(t, \mathbf{x})$, one may write, at the level of a low-energy effective description,

$$\rho_A = \frac{1}{2}\dot{a}^2 + \frac{1}{2a_{\rm FRW}^2}|\nabla a|^2 + V_A(a), \tag{239}$$

with

$$V_A(a) = V_A^{(0)} + \frac{1}{2}m_A^2 a^2 + \mathcal{O}(a^3). \tag{240}$$

Here $a_{\rm FRW}$ denotes the cosmological scale factor, whereas $a(x)$ denotes the projected associator mode.

The effective pressure is

$$p_A = \frac{1}{2}\dot{a}^2 - \frac{1}{6a_{\rm FRW}^2}|\nabla a|^2 - V_A(a). \tag{241}$$

For a homogeneous mode, the equation-of-state parameter becomes

$$w_A = \frac{p_A}{\rho_A} = \frac{\frac{1}{2}\dot{a}^2 - V_A(a)}{\frac{1}{2}\dot{a}^2 + V_A(a)}. \tag{242}$$

Different dynamical regimes therefore lead to different dark-sector behavior.

If the associator background is slowly varying,

$$\dot{a}^2 \ll V_A(a), \tag{243}$$

then

$$w_A \simeq -1. \tag{244}$$

In this regime the associator sector behaves as an approximately vacuum-like dark-energy component. If instead the mode undergoes coherent oscillations around a quadratic minimum,

$$V_A(a) \simeq \frac{1}{2}m_A^2 a^2, \tag{245}$$

then the averaged equation of state is

$$\langle w_A \rangle \simeq 0, \tag{246}$$

and the energy density redshifts approximately as

$$\rho_A \propto a_{\rm FRW}^{-3}. \tag{247}$$

The associator sector then behaves like cold dark matter, provided that the mode is long-lived, weakly coupled to visible matter, and stable against decay into ordinary projected particles.

More generally, the associator contribution can be inserted into the Friedmann equation as

$$H^2(z) = H_0^2 \left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda + \Omega_A f_A(z)\right], \tag{248}$$

where $f_A(z)$ encodes the redshift evolution of the non-associative sector. A dark-energy-like component corresponds to

$$f_A(z) \simeq 1, \tag{249}$$

whereas a matter-like associator condensate corresponds to

$$f_A(z) \simeq (1+z)^3. \tag{250}$$

A late-time activated associator component may be parameterized phenomenologically by

$$f_A(z) = (1+z)^{-n}, \qquad n > 0, \tag{251}$$

which suppresses the non-associative contribution at high redshift while allowing it to become relevant in the recent universe.

The microscopic interpretation is that the early universe may remain close to an associative phase because of high density, high temperature, or a stronger effective projection:

$$\left\langle \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 \right\rangle \approx 0. \tag{252}$$

At late times, as the universe expands and the projection background changes, a residual associator component may become dynamically relevant,

$$\left\langle \|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2 \right\rangle \neq 0. \tag{253}$$

This provides a possible mechanism for a dark component that is negligible at recombination but non-negligible at low redshift.

The same parameters that control fifth-force phenomenology also constrain the cosmological role of the associator mode. A light mass m_A gives a long range,

$$\ell_A = \frac{\hbar}{m_A c}, \tag{254}$$

while the matter coupling g_A determines whether the mode is visible in laboratory tests or effectively hidden. A viable dark-sector associator mode must therefore satisfy two competing requirements:

$$m_A \text{ small enough to affect cosmology,} \tag{255}$$

$$g_A \text{ small enough to evade fifth-force and equivalence-principle constraints.} \tag{256}$$

This naturally points to an ultra-light, weakly coupled, or screened associator sector.

The model therefore admits three qualitatively different dark-sector regimes:

(i) **Vacuum-like regime:** a slowly varying associator background behaves as dark energy with $w_A \simeq -1$.

(ii) **Matter-like regime:** coherent oscillations around a quadratic minimum behave as cold dark matter with $\langle w_A \rangle \simeq 0$.

(iii) **Transition regime:** a redshift-dependent associator contribution modifies late-time expansion and may mimic dynamical dark energy.

1. *Possible relevance to the Hubble tension*

The late-time transition regime may provide a phenomenological channel for addressing the Hubble tension. This should not be read as a completed solution. Rather, the point is that the octonionic framework naturally contains a dynamical energy component whose magnitude and redshift dependence are controlled by the non-associative sector.

Within the base ΛCDM interpretation of the cosmic microwave background, the inferred value of H_0 is lower than the value obtained from local distance-ladder measurements. A viable explanation must therefore modify either the early-universe calibration, the late-time expansion history, or the relation between local and cosmological inference, without spoiling the agreement with CMB, BAO, supernova, structure-growth, and weak-field gravitational tests.

In the present model the relevant condition is

$$\rho_A(z_{\rm CMB}) \ll \rho_m(z_{\rm CMB}), \qquad \rho_A(z \lesssim 1) \neq 0. \tag{257}$$

Thus the early-universe CMB calibration can remain approximately standard, while the late-time expansion history receives a small additional contribution. In the phenomenological approach

$$f_A(z) = (1+z)^{-n}, \qquad n > 0, \tag{258}$$

the associator contribution is suppressed at high redshift and becomes relevant only in the recent universe.

The physical interpretation is that the associative projection may be more rigid in the hot, dense early universe, suppressing non-associative excitations. At late times, as density, curvature, and environmental conditions change, a residual associator background may contribute a slowly varying vacuum-like component. In this sense the Hubble tension would not be caused by a change in the fundamental value of H_0, but by a regime-dependent inference of the expansion history in the presence of a late-time non-associative component.

For an indicative estimate, if the locally inferred expansion rate differs from the early-universe ΛCDM value by

$$\frac{H_{0,\mathrm{loc}}}{H_{0,\mathrm{CMB}}} \simeq \sqrt{1+\Omega_A}, \tag{259}$$

then

$$\Omega_A \simeq \left(\frac{H_{0,\mathrm{loc}}}{H_{0,\mathrm{CMB}}}\right)^2 - 1. \tag{260}$$

Using representative values $H_{0,\mathrm{CMB}} \simeq 67.4$ and $H_{0,\mathrm{loc}} \simeq 73.0$ in units of $\mathrm{km\,s^{-1}\,Mpc^{-1}}$ gives an indicative late-time fractional contribution of order

$$\Omega_A = \mathcal{O}(0.1). \tag{261}$$

This number is not a fit. It only shows the approximate scale required if the full mismatch were attributed to a late-time associator component.

A viable model must satisfy

$$\rho_A(z_{\mathrm{CMB}}) \approx 0, \tag{262}$$

$$|\Delta d_L(z)| \text{ remains compatible with supernova data,} \tag{263}$$

$$|\Delta D_A(z)| \text{ remains compatible with BAO data,} \tag{264}$$

$$|\Delta f\sigma_8(z)| \text{ remains compatible with growth data,} \tag{265}$$

$$|\Delta G_{\mathrm{eff}}/G| \ll 1 \quad \text{in weak-field environments.} \tag{266}$$

These conditions are nontrivial because late-time modifications of $H(z)$ are strongly constrained by Type Ia supernova distances, BAO measurements, cosmic chronometers, and CMB calibration.

A quantitative test should therefore perform a joint likelihood analysis using

$$\Theta = \{H_0, \Omega_m, \Omega_A, n, M_B, \ldots\} \tag{267}$$

or a more microscopic parameter set,

$$\lambda, \quad M_*, \quad m_A, \quad g_A. \tag{268}$$

The predicted luminosity distance is

$$d_L(z) = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{E(z')}, \qquad E(z) = \frac{H(z)}{H_0}, \tag{269}$$

and the supernova distance modulus is

$$\mu_{\rm th}(z) = 5 \log_{10} \left[\frac{d_L(z)}{\rm Mpc} \right] + 25. \tag{270}$$

The same expansion history must also be tested against BAO distances, structure growth, and weak-field bounds on $G_{\rm eff}$.

This connection remains phenomenological. The octonionic associator sector does not automatically solve the dark matter, dark energy, or Hubble-tension problems. It provides a natural and testable location for such physics within the same structure that controls non-associative deviations from ordinary quantum and gravitational dynamics.

XIV. CONCLUSIONS

The revised framework presents a dynamical projection theory in which four-dimensional spacetime, gravitational dynamics, and complex quantum mechanics arise as low-energy structures from an underlying octonionic system. The central point is that the octonionic G_2 geometry does not by itself produce physical spacetime, since its natural metric is Riemannian. A Lorentzian spacetime emerges only after an additional clock direction and a stable associative spatial three-plane have been dynamically selected.

A key refinement is the treatment of non-associativity. Because octonions are alternative, a single repeated octonionic field cannot generate a nontrivial associator interaction. The minimal non-associative sector therefore requires three independent octonion-valued fields. The positive associator contribution then acts as a variational selection principle: configurations close to an associative subalgebra are energetically preferred, while genuinely non-associative directions are suppressed, made massive, or projected into hidden internal sectors.

In the resulting low-energy regime, the projected metric obeys Einstein-type gravitational dynamics, while the associative algebraic projection recovers the structures of ordinary quantum mechanics, including the complex Hilbert-space norm, Born probabilities, superposition, Dirac dynamics, and the nonrelativistic Schrödinger limit. Possible deviations from standard physics are controlled by the non-associative scale. If this scale is near the Planck scale, the predicted effects are far below present laboratory sensitivity, making the model compatible with existing quantum, high-energy, and weak-field gravitational tests.

The framework should therefore be read as a structured proposal rather than a completed derivation of all known particle physics. Its main contribution is to connect octonionic non-associativity, G_2 geometry, Lorentzian spacetime, Einstein gravity, quantum mechanics, and possible phenomenological deviations within a single variational mechanism. Future work must derive the detailed particle spectrum, compute projection-dependent correction coefficients, and test the model quantitatively against interferometry, high-energy astrophysics, fifth-force searches, equivalence-principle tests, and cosmological data.

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Appendix A: Useful G_2 and Octonionic Identities

We use the octonionic basis

$$\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O}, \qquad \mathrm{Im}\,\mathbb{O} = \mathrm{span}\{e_1, \ldots, e_7\}, \tag{A1}$$

with multiplication rule

$$e_a e_b = -\delta_{ab} + f_{ab}{}^{c} e_c. \tag{A2}$$

The constants f_{abc} are totally antisymmetric. A definite convention is fixed by the nonzero oriented triples

$$(123), \quad (145), \quad (167), \quad (246), \quad (257), \quad (347), \quad (356), \tag{A3}$$

with $f_{abc} = +1$ on these triples and all other components determined by antisymmetry.

The canonical G_2 three-form is

$$\varphi = \frac{1}{6} f_{abc}\, e^a \wedge e^b \wedge e^c, \tag{A4}$$

and its Hodge dual is

$$\psi = *_\varphi \varphi. \tag{A5}$$

The cross product on $\mathrm{Im}\,\mathbb{O}$ is

$$x \times y = \frac{1}{2}(xy - yx), \tag{A6}$$

and satisfies

$$\varphi(x, y, z) = \langle x \times y, z\rangle. \tag{A7}$$

The basic G_2 identities used in the main text are

$$\varphi_{amn}\varphi_b{}^{mn} = 6 g_{ab}, \tag{A8}$$

$$\varphi_{abm}\varphi_{cd}{}^{m} = g_{ac}g_{bd} - g_{ad}g_{bc} + \psi_{abcd}, \tag{A9}$$

$$\psi_{abmn}\psi^{cdmn} = 4\delta_{ab}^{cd} + 2\psi_{ab}{}^{cd}. \tag{A10}$$

They imply that the stable three-form determines the metric, orientation, cross product, and dual four-form.

An oriented three-plane $P_3 \subset \mathrm{Im}\,\mathbb{O}$ is associative if

$$\varphi|_{P_3} = \mathrm{vol}_{P_3}. \tag{A11}$$

Its orthogonal complement $Q_4 = P_3^\perp$ is coassociative and satisfies

$$\varphi|_{Q_4} = 0, \qquad \psi|_{Q_4} = \mathrm{vol}_{Q_4}. \tag{A12}$$

The corresponding decomposition is

$$\mathrm{Im}\,\mathbb{O} = P_3 \oplus Q_4. \tag{A13}$$

Finally, the octonionic associator is

$$\mathcal{A}(x, y, z) = (xy)z - x(yz). \tag{A14}$$

Because $\mathbb{O}$ is alternative,

$$\mathcal{A}(x, x, z) = \mathcal{A}(x, y, y) = \mathcal{A}(x, x, x) = 0. \tag{A15}$$

This identity is the reason why the master action must use three independent octonion-valued fields in the non-associative term.

Appendix B: Dimensional Analysis in SI and Natural Units

In natural units $c = \hbar = 1$, the action is dimensionless:

$$[S] = 0. \tag{B1}$$

With

$$[d^4x] = -4, \qquad [\mathcal{L}] = 4, \tag{B2}$$

one has

$$[x] = -1, \qquad [\partial_\mu] = 1, \qquad [R] = 2, \qquad [g_{\mu\nu}] = 0. \tag{B3}$$

The canonical kinetic term gives

$$[\Psi_I] = 1. \tag{B4}$$

Since the associator is cubic in the fields,

$$[\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)] = 3, \qquad [\|\mathcal{A}(\Psi_1, \Psi_2, \Psi_3)\|^2] = 6. \tag{B5}$$

Therefore

$$[\lambda] = -2. \tag{B6}$$

It is useful to write

$$\lambda = \frac{\eta_A}{M_*^2}, \tag{B7}$$

where

$$[\eta_A] = 0, \qquad [M_*] = 1. \tag{B8}$$

After restoring units, the associated energy scale is $M_* c^2$, and the dimensionless low-energy expansion parameter is

$$\epsilon_A(E) \sim \eta_A \left(\frac{E}{M_* c^2} \right)^2. \tag{B9}$$

The Einstein-Hilbert coupling is

$$\kappa = \frac{8\pi G}{c^4} \tag{B10}$$

in SI units, with

$$[\kappa] = \mathrm{m/J}. \tag{B11}$$

In natural units,

$$[\kappa] = -2. \tag{B12}$$

The nonminimal curvature coupling

$$\xi R\langle\Psi, \Psi\rangle \tag{B13}$$

has dimension four, so

$$[\xi] = 0. \tag{B14}$$

For the effective associator mode $a(x)$, one may use the scalar-field normalization

$$[a] = 1, \qquad [m_A] = 1, \qquad [\ell_A] = -1. \tag{B15}$$

After restoring SI units,

$$\ell_A = \frac{\hbar}{m_A c}. \tag{B16}$$

The associator energy density has dimension

$$[\rho_A] = 4 \tag{B17}$$

in natural units, corresponding to energy per volume in SI units.

For the high-energy dispersion parametrization,

$$E^2 = E_0^2 + \zeta_A \frac{E_0^4}{M_*^2 c^4} + \cdots , \tag{B18}$$

the coefficient is dimensionless:

$$[\zeta_A] = 0. \tag{B19}$$

The fractional correction is therefore

$$\frac{\delta E}{E_0} \sim \frac{\zeta_A}{2} \left(\frac{E_0}{M_* c^2}\right)^2 , \tag{B20}$$

which has the same suppression structure as $\epsilon_A(E_0)$.

Appendix C: References

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