Inflation of Universe, Zero Total Energy and Dark Energy as a Consequence of Octonionic Non-Associativity

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Abstract

We formulate a cosmological model in which inflation, the vanishing of the total energy of the Universe, and present-day dark energy arise as structural consequences of octonionic non-associativity. The fundamental action is assumed to contain a positive-definite scalar constructed from the octonionic associator. This term behaves as a vacuum component, enforcing exponential expansion without the introduction of an ad hoc inflaton field. We demonstrate that inflation is dynamically unavoidable (forced inflation), that the zero-total-energy condition holds at all cosmic epochs, and that a minimal residual associator norm naturally yields the observed dark energy density. Numerical estimates from the inflationary epoch to the present Universe are included.

1 Introduction

Standard cosmology successfully describes the Universe within the ΛCDM framework. Nevertheless, several fundamental questions remain open:

1. Why did inflation occur?
2. Total energy of the Universe?
3. Why does dark energy exist and why does it dominate today?

We propose that these features arise from the algebraic structure of the octonions $\mathbb{O}$, the largest normed division algebra. Their essential property — non-associativity — introduces a positive scalar invariant which acts as a geometric vacuum energy.

The central thesis is:

> Non-associativity structurally enforces inflation and implies a zero-total-energy Universe with a residual dark energy term.

2 Octonionic Structure

The octonions $\mathbb{O}$ are an 8-dimensional normed division algebra with multiplication satisfying:

$$N(xy) = N(x)N(y). \tag{1}$$

They are non-associative. The associator is defined as:

$$[x, y, z] := (xy)z - x(yz). \tag{2}$$

In general,

$$[x, y, z] \neq 0. \tag{3}$$

We define the positive-definite scalar invariant

$$\mathcal{A} := \langle [\phi_i, \phi_j, \phi_k], [\phi_i, \phi_j, \phi_k] \rangle, \tag{4}$$

where ϕ_i denote octonionic-valued fields.

Because it is a norm square,

$$\mathcal{A} \geq 0, \tag{5}$$

and vanishes only if the algebra effectively reduces to an associative subalgebra.

3 Fundamental Action and Derivation of the Field Equations

3.1 Motivation for the Associator Term

The octonions $\mathbb{O}$ are non-associative. Their associator

$$[x, y, z] := (xy)z - x(yz) \tag{6}$$

does not vanish in general.

Since the associator measures the failure of associativity, it represents an intrinsic geometric invariant of the algebra. To construct a scalar quantity suitable for inclusion in a gravitational action, we define the norm-square invariant

$$\mathcal{A} = \langle [\phi_i, \phi_j, \phi_k], [\phi_i, \phi_j, \phi_k] \rangle. \tag{7}$$

Because it is a norm square, it satisfies

$$\mathcal{A} \geq 0. \tag{8}$$

This scalar is:

- algebraically invariant,
- positive semi-definite,
- independent of coordinate choice,
- non-vanishing whenever non-associativity is present.

It is therefore the natural candidate for a geometric vacuum contribution.

3.2 Construction of the Action

We postulate that the gravitational dynamics are governed by

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \lambda \mathcal{A} \right), \tag{9}$$

where:

- R is the Ricci scalar curvature,
- G is Newton's constant,
- λ is a coupling parameter,

- $\mathcal{A}$ encodes octonionic non-associativity.

The first term reproduces standard Einstein gravity. The second term introduces the algebraic invariant.

No additional scalar field is introduced. The associator term is purely geometric.

3.3 Variation with Respect to the Metric

To derive the field equations, we vary the action with respect to $g^{\mu\nu}$.

The variation of the Einstein–Hilbert term gives

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu}\right) + \text{boundary terms.} \tag{10}$$

We now consider the associator term:

$$S_A = \int d^4x\sqrt{-g}\,\lambda\mathcal{A}. \tag{11}$$

Its variation is

$$\delta S_A = \int d^4x\left(\lambda\delta(\sqrt{-g})\mathcal{A} + \lambda\sqrt{-g}\delta\mathcal{A}\right). \tag{12}$$

Using the identity

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{13}$$

and assuming $\mathcal{A}$ does not depend explicitly on curvature (but only algebraically on internal fields), the dominant metric dependence enters through $\sqrt{-g}$.

Therefore

$$\delta S_A = -\frac{1}{2}\int d^4x\sqrt{-g}\lambda\mathcal{A}g_{\mu\nu}\delta g^{\mu\nu}. \tag{14}$$

3.4 Effective Energy–Momentum Tensor

By definition,

$$T^{(A)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_A}{\delta g^{\mu\nu}}. \tag{15}$$

Substituting the variation yields

$$T^{(A)}_{\mu\nu} = \lambda\mathcal{A}g_{\mu\nu}. \tag{16}$$

We identify the effective energy density

$$\rho_A = \lambda\mathcal{A}. \tag{17}$$

Thus

$$T^{(A)}_{\mu\nu} = \rho_A g_{\mu\nu}. \tag{18}$$

3.5 Equation of State

For a perfect fluid in FRW spacetime,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{19}$$

Comparing with

$$T^{(A)}_{\mu\nu} = \rho_A g_{\mu\nu}, \tag{20}$$

we obtain

$$p_A = -\rho_A. \tag{21}$$

Hence the equation-of-state parameter is

$$w = \frac{p_A}{\rho_A} = -1. \tag{22}$$

This is identical to a cosmological constant.

3.6 Resulting Field Equations

Combining Einstein–Hilbert variation and the associator contribution, we obtain

$$G_{\mu\nu} = 8\pi G \left(T^{(matter)}_{\mu\nu} + \rho_A g_{\mu\nu} \right). \tag{23}$$

The associator therefore enters as an effective vacuum term

$$\Lambda_{eff} = 8\pi G \rho_A. \tag{24}$$

3.7 Physical Interpretation

The logical structure is:

- Non-associativity produces a positive scalar invariant.
- The invariant contributes to the action.
- Variation yields a vacuum-type stress tensor.
- The equation of state is $w = -1$.
- A cosmological constant term emerges dynamically.

Thus the cosmological constant is not inserted ad hoc, but arises as a projection of octonionic non-associativity.

4 Forced Inflation as a Consequence of the Associator Term

4.1 Field Equations Including the Associator Contribution

From Chapter 3 we obtained the modified Einstein equations

$$G_{\mu\nu} = 8\pi G \left(T^{(matter)}_{\mu\nu} + \rho_A g_{\mu\nu} \right), \tag{25}$$

where

$$\rho_A = \lambda \mathcal{A}, \qquad p_A = -\rho_A. \tag{26}$$

The associator therefore behaves exactly like a cosmological constant. We now analyze the cosmological consequences in a homogeneous and isotropic Universe.

4.2 FRW Geometry

For a spatially flat Friedmann–Robertson–Walker metric,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \tag{27}$$

the dynamics are governed by the Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho_{tot}, \tag{28}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_{tot} + 3p_{tot}\right), \tag{29}$$

where

$$H = \frac{\dot{a}}{a}. \tag{30}$$

4.3 Inflation from the Associator Density

Assume that at early times the associator term dominates:

$$\rho_{tot} \approx \rho_A. \tag{31}$$

Using

$$p_A = -\rho_A, \tag{32}$$

we compute

$$\rho_{tot} + 3p_{tot} = \rho_A - 3\rho_A = -2\rho_A. \tag{33}$$

Substituting into the second Friedmann equation yields

$$\frac{\ddot{a}}{a} = +\frac{8\pi G}{3}\rho_A. \tag{34}$$

Since $\rho_A > 0$, we obtain

$$\ddot{a} > 0. \tag{35}$$

Thus the Universe undergoes accelerated expansion.

4.4 Constant Hubble Parameter

From the first Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho_A. \tag{36}$$

Because ρ_A is constant, it follows that

$$H = \text{const.} \tag{37}$$

The scale factor therefore satisfies

$$a(t) = a_0 e^{Ht}. \tag{38}$$

This is exponential expansion.

4.5 Algebraic Necessity of Inflation

In conventional inflationary models, accelerated expansion requires:

- a scalar inflaton field,
- a specially chosen potential,
- finely tuned initial conditions.

In contrast, within the octonionic framework:

$$\text{Non-associativity} \Rightarrow \mathcal{A} > 0 \Rightarrow \rho_A > 0 \Rightarrow \ddot{a} > 0. \tag{39}$$

As long as the algebra is not fully associative, $\mathcal{A}$ cannot vanish. Inflation is therefore structurally enforced.

4.6 Dynamical Attractor Behavior

Now include additional matter or radiation components:

$$\rho_{tot} = \rho_A + \rho_m + \rho_r, \tag{40}$$

with

$$\rho_m \propto a^{-3}, \qquad \rho_r \propto a^{-4}. \tag{41}$$

For exponential expansion,

$$a(t) \sim e^{Ht}, \tag{42}$$

we obtain

$$\rho_m \propto e^{-3Ht}, \qquad \rho_r \propto e^{-4Ht}. \tag{43}$$

Both contributions decay exponentially.
Hence

$$\rho_{tot} \to \rho_A \quad \text{as} \quad t \to \infty. \tag{44}$$

The inflationary solution is therefore a dynamical attractor.

4.7 Termination of Inflation

Inflation ends when the associator energy is partially converted into standard matter and radiation degrees of freedom. In the octonionic picture, this corresponds to a partial relaxation of the associator configuration.

However, a minimal residual value

$$\mathcal{A}_{min} > 0 \tag{45}$$

remains, which later appears as dark energy.

4.8 Structural Chain of Implications

The logical structure of forced inflation is:

$$\text{Non-associativity} \Rightarrow \mathcal{A} > 0 \Rightarrow \rho_A = \text{const} \Rightarrow a(t) \sim e^{Ht} \Rightarrow \text{Inflation.} \tag{46}$$

Inflation is therefore not a special initial condition, but a structural consequence of octonionic algebra.

5 Zero Total Energy of the Universe: Detailed Energy Composition

5.1 Decomposition of Positive Energy

The total positive energy inside the Hubble sphere is composed of

$$E_{pos} = E_{matter} + E_{radiation} + E_{vacuum}. \tag{47}$$

Each contribution is given by

$$E_i = \rho_i V_H c^2, \tag{48}$$

where the Hubble volume is

$$V_H = \frac{4\pi}{3} R_H^3, \qquad R_H = \frac{c}{H}. \tag{49}$$

Thus

$$E_{pos} = (\rho_m + \rho_r + \rho_A) \frac{4\pi}{3} \frac{c^3}{H^3} c^2. \tag{50}$$

Using Friedmann:

$$H^2 = \frac{8\pi G}{3} \rho_{tot}, \tag{51}$$

we obtain

$$R_H \propto \rho_{tot}^{-1/2}, \quad V_H \propto \rho_{tot}^{-3/2}. \tag{52}$$

Hence

$$E_{pos} \propto \rho_{tot} \rho_{tot}^{-3/2} = \rho_{tot}^{-1/2}. \tag{53}$$

5.2 Gravitational Energy

For a homogeneous sphere:

$$E_{grav} = -\frac{3}{5} \frac{GM^2}{R_H}. \tag{54}$$

Since

$$M = \rho_{tot} V_H \propto \rho_{tot}^{-1/2}, \tag{55}$$

we find

$$E_{grav} \propto -\frac{\rho_{tot}^{-1}}{\rho_{tot}^{-1/2}} = -\rho_{tot}^{-1/2}. \tag{56}$$

Thus

$$E_{total} = E_{pos} + E_{grav} = 0. \tag{57}$$

The cancellation is scale independent and holds for any epoch.

6 Initial Planck Energy and Early Universe Limit

6.1 Planck Scale Quantities

The Planck units are

$$M_P = \sqrt{\frac{\hbar c}{G}} \approx 2.18 \times 10^{-8}\,\mathrm{kg}, \tag{58}$$

$$E_P = M_P c^2 \approx 1.96 \times 10^{9}\,\mathrm{J}, \tag{59}$$

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\mathrm{m}, \tag{60}$$

$$\rho_P = \frac{c^5}{\hbar G^2} \approx 5.1 \times 10^{96}\,\mathrm{kg/m^3}. \tag{61}$$

6.2 Initial Hubble Volume at Planck Density

At $\rho_{tot} \sim \rho_P$:

$$H_P = \sqrt{\frac{8\pi G}{3}\rho_P} \sim \frac{1}{t_P}, \tag{62}$$

where

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44}\,\mathrm{s}. \tag{63}$$

Hence

$$R_H \sim \ell_P. \tag{64}$$

The Hubble volume becomes

$$V_H \sim \ell_P^3. \tag{65}$$

6.3 Initial Positive Energy

$$E_{pos}^{(P)} = \rho_P \ell_P^3 c^2. \tag{66}$$

Using the Planck definitions one finds

$$E_{pos}^{(P)} \sim E_P. \tag{67}$$

Thus the initial energy contained in the Planck Hubble volume is of order one Planck energy.

6.4 Gravitational Cancellation at Planck Scale

The corresponding gravitational energy is

$$E_{grav}^{(P)} = -\frac{3}{5}\frac{GM_P^2}{\ell_P} \sim -E_P. \tag{68}$$

Therefore

$$E_{total}^{(P)} = 0. \tag{69}$$

6.5 Interpretation

At the Planck epoch:

- Positive energy is dominated by Planck density.
- The Hubble radius equals the Planck length.
- The energy inside the causal domain is of order E_P.
- Gravitational binding energy exactly compensates it.

Hence the Universe can originate from a Planck-scale configuration with zero total energy.

As the Universe expands:

$$E_{pos} \propto \rho^{-1/2} \to 10^{69}\,\mathrm{J} \tag{70}$$

today, while the negative gravitational contribution grows with equal magnitude.

The zero-energy condition is therefore preserved dynamically from the Planck epoch through inflation to the present Universe.

7 Dark Energy as Associator-Induced Positive Vacuum Energy

7.1 Origin of Vacuum Energy from the Associator

In the octonionic framework the additional fundamental contribution to the action is

$$S_A \propto \int d^4x\sqrt{-g}\,\mathcal{A}, \tag{71}$$

where the associator norm is defined as

$$\mathcal{A} = \langle [\phi_i, \phi_j, \phi_k], [\phi_i, \phi_j, \phi_k] \rangle \geq 0. \tag{72}$$

Since $\mathcal{A}$ is a norm square, it is strictly non-negative and vanishes only if the fields lie in an associative subalgebra.

The corresponding energy density is

$$\rho_A = \lambda \mathcal{A}. \tag{73}$$

Variation with respect to the metric yields

$$T_{\mu\nu}^{(A)} = \rho_A g_{\mu\nu}. \tag{74}$$

Thus the equation of state becomes

$$p_A = -\rho_A, \qquad w = -1. \tag{75}$$

The associator term therefore behaves exactly like a cosmological constant.

7.2 Contribution to the Positive Energy Budget

The total positive energy inside the Hubble sphere is

$$E_{pos} = (\rho_m + \rho_r + \rho_A)\, V_H c^2, \tag{76}$$

with

$$V_H = \frac{4\pi}{3} R_H^3, \qquad R_H = \frac{c}{H}. \tag{77}$$

The associator contribution is

$$E_A = \rho_A V_H c^2. \tag{78}$$

Unlike matter and radiation,

$$\rho_m \propto a^{-3}, \qquad \rho_r \propto a^{-4}, \tag{79}$$

the associator density satisfies

$$\rho_A = \text{const}. \tag{80}$$

As the Universe expands, the Hubble volume grows and therefore

$$E_A \propto V_H. \tag{81}$$

At late times, when $H \to \text{const}$, the associator contribution dominates the positive energy content.

7.3 Minimal Non-Associativity and Residual Energy

The octonionic structure implies that complete associativity is not globally realizable. Even in the vacuum configuration a minimal residual associator norm persists:

$$\mathcal{A}_{min} > 0. \tag{82}$$

This generates a minimal vacuum energy density

$$\rho_\Lambda = \lambda \mathcal{A}_{min}. \tag{83}$$

Observationally,

$$\rho_\Lambda \approx 5.9 \times 10^{-27}\, \text{kg/m}^3. \tag{84}$$

Dark energy is therefore interpreted as the energetic manifestation of irreducible octonionic non-associativity.

7.4 Dominance in the Present Universe

Today the total energy density satisfies

$$\rho_{tot} = \rho_m + \rho_A, \tag{85}$$

with approximately

$$\rho_A \approx 0.69\, \rho_{tot}. \tag{86}$$

Hence the dominant positive energy contribution is

$$E_A \approx 0.69\, E_{pos}. \quad (87)$$

The gravitational energy compensates the total positive contribution:

$$E_{grav} = -E_{pos}. \quad (88)$$

Therefore

$$E_{total} = 0 \quad (89)$$

remains valid even in the dark-energy dominated epoch.

7.5 Geometric Interpretation

The associator corresponds geometrically to torsion in the internal G_2 structure of the octonionic algebra. Persistent non-associativity induces a constant positive scalar invariant.

Thus the chain of implications is:

$$\text{Non-associativity} \;\Rightarrow\; \mathcal{A} > 0 \;\Rightarrow\; \rho_A > 0 \;\Rightarrow\; \ddot{a} > 0. \quad (90)$$

Dark energy is therefore not an additional field, but the macroscopic manifestation of the algebraic structure of the underlying octonionic geometry.

8 Expansion Behaviour of the Universe Across Cosmological Phases

8.1 Overall Structure of Cosmic Expansion

The cosmic expansion can therefore be separated into four dynamical regimes:

- Inflationary phase: exponential expansion driven by a large associator contribution.
- Radiation-dominated phase: expansion with $a(t) \propto t^{1/2}$.
- Matter-dominated phase: expansion with $a(t) \propto t^{2/3}$.
- Dark-energy dominated phase: renewed exponential expansion caused by the residual associator term.

8.2 Interpretation in the Octonionic Framework

Within the octonionic model both inflation and present-day dark energy originate from the same algebraic structure: the non-associativity of the octonions.

The difference between the early and late accelerated expansion phases lies only in the magnitude of the associator term:

- Early Universe: large associator contribution leading to strong inflation.
- Late Universe: minimal residual associator producing weak accelerated expansion.

Thus two major cosmological phenomena arise from a single underlying algebraic mechanism.

9 Conclusions

We have shown:

1. Non-associativity introduces a positive-definite scalar invariant.
2. This invariant behaves as vacuum energy.
3. Inflation is algebraically enforced.
4. The inflationary solution is a dynamical attractor.
5. The total energy of the Universe vanishes at all epochs.
6. Dark energy corresponds to the minimal residual associator norm.

A Appendix A: Algebraic Structure of the Octonionic Associator

A.1 Basic Identities

The octonions $\mathbb{O}$ form a non-associative but alternative normed division algebra. For $x, y, z \in \mathbb{O}$ the associator is defined as

$$[x, y, z] := (xy)z - x(yz). \tag{91}$$

Alternativity implies:

$$[x, x, y] = [y, x, x] = 0. \tag{92}$$

The associator is totally antisymmetric:

$$[x, y, z] = -[y, x, z] = -[x, z, y]. \tag{93}$$

In a basis $\{e_0 = 1, e_a\}$ with $a = 1, \dots, 7$ one has

$$e_a e_b = -\delta_{ab} + C_{abc} e_c, \tag{94}$$

where C_{abc} are the totally antisymmetric structure constants of the imaginary octonions. The associator can be written as

$$[e_a, e_b, e_c] = 2\,\Phi_{abcd} e_d, \tag{95}$$

where Φ_{abcd} is the G_2-invariant 4-form dual to C_{abc}.

A.2 Positive Definiteness

Define the scalar invariant

$$\mathcal{A} = \langle [\phi_i, \phi_j, \phi_k], [\phi_i, \phi_j, \phi_k] \rangle. \tag{96}$$

Using norm multiplicativity,

$$N(xy) = N(x)N(y), \tag{97}$$

one finds

$$\mathcal{A} \geq 0, \tag{98}$$

with equality if and only if the fields lie in an associative subalgebra.

Thus the associator norm defines a positive semi-definite scalar functional.

B Appendix B: Derivation of the Effective Energy-Momentum Tensor

B.1 Variation of the Associator Term

Consider the action contribution

$$S_A = \int d^4x \sqrt{-g}\, \lambda \mathcal{A}. \tag{99}$$

The variation with respect to the metric yields

$$\delta S_A = \int d^4x \left(\lambda \delta(\sqrt{-g}) \mathcal{A} + \lambda \sqrt{-g} \delta \mathcal{A} \right). \tag{100}$$

Using

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}, \tag{101}$$

and assuming $\mathcal{A}$ depends only algebraically on the fields (not explicitly on curvature), we obtain

$$T^{(A)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_A}{\delta g^{\mu\nu}} = \lambda \mathcal{A} g_{\mu\nu}. \tag{102}$$

Hence,

$$\rho_A = \lambda \mathcal{A}, \qquad p_A = -\rho_A. \tag{103}$$

This confirms the equation of state

$$w = -1. \tag{104}$$

C Appendix C: Proof of Zero Total Energy in FRW Geometry

C.1 Scaling Relations

For a spatially flat FRW universe:

$$H^2 = \frac{8\pi G}{3}\rho. \tag{105}$$

Thus

$$R_H = \frac{c}{H} \propto \rho^{-1/2}. \tag{106}$$

The mass inside the Hubble sphere is

$$M = \rho\frac{4\pi}{3}R_H^3 \propto \rho\rho^{-3/2} = \rho^{-1/2}. \tag{107}$$

C.2 Energy Cancellation

Positive energy:

$$E_{pos} = Mc^2 \propto \rho^{-1/2}. \tag{108}$$

Gravitational energy:

$$E_{grav} = -\frac{3}{5}\frac{GM^2}{R_H} \propto -\frac{\rho^{-1}}{\rho^{-1/2}} = -\rho^{-1/2}. \tag{109}$$

Therefore:

$$E_{total} = E_{pos} + E_{grav} = 0. \tag{110}$$

The cancellation holds independently of the epoch since it depends only on the Friedmann relation.

D Appendix D: Stability of the Inflationary Attractor

Consider perturbations

$$\rho = \rho_A + \delta\rho. \tag{111}$$

Linearizing the Friedmann equation:

$$\dot{\delta\rho} + 3H(1+w)\delta\rho = 0. \tag{112}$$

For matter ($w = 0$):

$$\delta\rho \propto a^{-3}. \tag{113}$$

For radiation ($w = \frac{1}{3}$):

$$\delta\rho \propto a^{-4}. \tag{114}$$

Since

$$a(t) \sim e^{Ht}, \tag{115}$$

all perturbations decay exponentially:

$$\delta\rho \to 0. \tag{116}$$

Thus the inflationary solution is dynamically stable.

Inflation, zero total energy, and dark energy are thus direct consequences of octonionic algebra.

E Appendix E: G_2 Geometry and Octonionic Structure

E.1 The G_2 Invariant 3-Form

The automorphism group of the octonions $\mathbb{O}$ is the exceptional Lie group G_2. It preserves the octonionic multiplication structure.

On $\mathbb{R}^7$ with basis $\{e_a\}$ ($a = 1, \ldots, 7$), define the G_2-invariant 3-form

$$\varphi = \frac{1}{3!} C_{abc}\, dx^a \wedge dx^b \wedge dx^c, \tag{117}$$

where C_{abc} are the totally antisymmetric octonionic structure constants.

The dual 4-form is

$$\Phi = \star\varphi, \tag{118}$$

with components

$$\Phi_{abcd} = \frac{1}{3!} \epsilon_{abcdefg} C_{efg}. \tag{119}$$

These satisfy the fundamental identity

$$C_{abe} C_{cde} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + \Phi_{abcd}. \tag{120}$$

E.2 Associator in Terms of the G_2 4-Form

The octonionic associator can be written as

$$[e_a, e_b, e_c] = 2\, \Phi_{abcd} e_d. \tag{121}$$

Hence the scalar invariant becomes

$$\mathcal{A} = 4\, \Phi_{abcd} \Phi_{abce}\, \phi_d \phi_e. \tag{122}$$

Using the contraction identity

$$\Phi_{abcd} \Phi_{abce} = 6\, \delta_{de}, \tag{123}$$

one obtains

$$\mathcal{A} = 24\,\phi_d\phi_d. \tag{124}$$

Thus the associator norm is directly controlled by the G_2-invariant structure, establishing geometric positivity.

E.3 Nearly Parallel G_2 and Cosmological Constant

If the internal 7-manifold X_7 admits a nearly parallel G_2 structure:

$$d\varphi = \tau_0\,\Phi, \tag{125}$$

then the scalar curvature of X_7 satisfies

$$R_7 = \frac{21}{8}\tau_0^2. \tag{126}$$

Compactification on such a manifold produces an effective 4D vacuum energy

$$\rho_\Lambda \propto \tau_0^2. \tag{127}$$

Thus dark energy corresponds geometrically to torsion in G_2 geometry, which is equivalent to persistent octonionic non-associativity.

F Appendix F: Projective Derivation of the Einstein Equations

F.1 Octonionic Projector

Consider the decomposition

$$\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{O}). \tag{128}$$

Define a projection operator onto spacetime directions:

$$P : \mathbb{O} \to T_pM_4. \tag{129}$$

The effective spacetime metric arises from the projected norm:

$$g_{\mu\nu} = \langle P(e_\mu), P(e_\nu) \rangle. \tag{130}$$

Because the octonionic norm is multiplicative,

$$N(xy) = N(x)N(y), \tag{131}$$

curvature emerges from variations of projected multiplication.

F.2 Emergence of Curvature

Define a projected covariant derivative

$$D_\mu = P \circ \partial_\mu. \tag{132}$$

Non-associativity induces a curvature tensor via

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma. \tag{133}$$

The curvature scalar is obtained by contraction:

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{134}$$

F.3 Effective Gravitational Action

Projecting the full octonionic invariant action onto spacetime yields

$$S_{eff} = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \lambda \mathcal{A} \right). \tag{135}$$

Variation with respect to $g_{\mu\nu}$ gives

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu}^{(matter)} + \lambda \mathcal{A} g_{\mu\nu} \right). \tag{136}$$

Thus Einstein's equations arise as the projected dynamics of octonionic multiplication.

F.4 Interpretation

- The metric arises from projected octonionic norm.
- Curvature originates from non-associative multiplication.
- Vacuum energy corresponds to the invariant associator norm.
- Einstein gravity is the low-energy projection of octonionic geometry.

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