# The Koide Angle as a Conformal Dimension: $G_2$ Geometry, $\mathrm{SU}(3)_3$ WZW Theory, and Fermion Mass Structure

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**Abstract**

The charged lepton masses satisfy the Koide relation $Q = 2/3$ and are parametrized by a single Brannen phase $\delta_{\text{exp}} = 0.22222(5) \approx 2/9$. We prove that the Brannen parametrization is the exact eigenvalue structure of a democratic element of the exceptional Jordan algebra $J_3(\mathbb{O})$, with $\cos(3\delta) = -\varphi(V)$ where $\varphi$ is the $G_2$ 3-form on the generation 3-plane.

The distinguished value 2/9 appears independently in five mathematical constructions: a Hessian ratio on $\mathrm{Gr}(3, \mathbb{R}^6)$, a Casimir quotient $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3)$, the conformal dimension $h_\square$ of $\mathrm{SU}(3)_3$ WZW theory, a crossing phase in conformal blocks, and the Knizhnik–Zamolodchikov singlet exponent. A Bridge Proposition proves these agree if and only if $N = 3$. A Master Identity $C_2(\mathrm{Sym}^N \square) = k + h^\vee$ uniquely at $N = 3$ implies Casimir ratios equal conformal dimensions for all integrable representations at level $k = 3$.

We prove $Q = 1/3 + d_\square/6$, making $Q = 2/3$ equivalent to the quantum dimension $d_\square = 2$. From two hypotheses—the Sumino $\mathrm{SU}(3)_F$ family gauge symmetry (F) and the WZW identification of the Brannen parameters (W)—we derive $\delta = h_\square = 2/9$ with zero free parameters: the democratic Brannen form follows from the weight geometry of the fundamental representation; the amplitude $A = \sqrt{d_\square}$ is proven from (F) alone; the phase identification $\delta = h_\square$ is the central conjecture (W), motivated by five independent characterizations of 2/9, spectral selection, and 0.02% experimental agreement. The $T^c$ identity selects $k = 3$ as the unique level within SU(3), matching the Sumino mechanism independently. Three selection mechanisms confirm uniqueness: spectral positivity, modular self-consistency, and WZW completeness. A Blindness Theorem shows calibrated geometry is exactly insensitive to $\delta$; combined with CP, transcendence, gauge boson blindness, logarithmic sign, and KZ–circulant obstructions, this characterizes the class of mechanisms that the WZW identification bypasses.

Eighteen distinct conditions select $N = 3$ generations. For up-type quarks, $Q_{\text{up}} = 8/9$ at $0.3\sigma$. The neutrino extension is decisively falsified; $Q_\nu = 2/3$ is arithmetically unattainable for neutrino masses in either hierarchy. Eighty falsified approaches are cataloged.

**Keywords:** Koide formula, octonions, $G_2$ holonomy, exceptional Jordan algebra, Casimir invariants, WZW conformal field theory, neutrino masses, quark masses

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# Contents

# 1 Introduction

The Koide formula [1] relates the charged lepton masses through

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = 0.666661 \approx \frac{2}{3}, \tag{1}$$

an agreement at the $10^{-5}$ level using PDG 2024 values [4]. In the Brannen parametrization [2, 3],

$$\sqrt{m_k} = \mu\big(1 + \sqrt{2}\cos(\delta + 2\pi k/3)\big), \quad k = 0, 1, 2, \tag{2}$$

the $\mathbb{Z}_3$-symmetric structure enforces $Q = 2/3$ identically for any $\delta$, while the single phase $\delta$ controls the entire mass hierarchy: $m_\mu/m_e \approx 207$ and $m_\tau/m_\mu \approx 16.8$.

The experimental value $\delta_{\text{exp}} = 0.22222(5)$ is remarkably close to $2/9 = 0.22222\ldots$, suggesting an algebraic origin. This paper shows that 2/9 arises independently in multiple mathematical constructions and investigates whether the identification $\delta = 2/9$ can be given a structural explanation. The value 2/9 appears simultaneously as:

0. The exact eigenvalue structure of a democratic $J_3(\mathbb{O})$ element, with $\cos(3\delta) = -\varphi(V)$ (Theorem 2.4);
1. A geometric ratio from the Hessian of the $G_2$ 3-form on $\text{Gr}(3, \mathbb{R}^6)$ (Theorem 3.6);
2. The Casimir quotient $C_2(\bar{3})/C_2(\text{Sym}^3 3)$ in $\text{SU}(3) \subset G_2$ (Observation 3.7);
3. The conformal dimension $h_\square = C_2(\square)/(k + h^\vee)$ of $\text{SU}(3)_3$ WZW theory (Theorem 4.1);
4. The crossing phase $\Delta h \times 2/3$ in $\text{SU}(3)_3$ conformal blocks (Theorem 4.10).

A Bridge Proposition (Proposition 5.1) shows that constructions 1 and 3 coincide if and only if $N = 3$, providing a structural explanation for the agreement rather than a numerical coincidence. The identification $\delta_{\text{phys}} = h_\square = 2/9$ remains a conjecture, supported by 0.02% agreement with experiment and the uniqueness results established below.

Crucially, we prove that the Brannen parametrization is not merely empirical: it is the exact eigenvalue structure of a democratic element of the exceptional Jordan algebra $J_3(\mathbb{O})$, with the Brannen phase $\delta$ identically equal to the angle of the $G_2$ 3-form on the generation 3-plane (Theorem 2.4). The connection to $J_3(\mathbb{O})$ provides a natural algebraic home for the democratic mass matrix ansatz [20, 21, 22]. Only in $J_3(\mathbb{O})$ does the non-associativity of the octonions introduce the 3-form $\varphi$ as an additional degree of freedom—and it is this degree of freedom that becomes the Brannen phase $\delta$.

We prove six structural obstructions that sharply constrain the class of viable dynamical mechanisms, collectively characterizing $\delta$ as a quantity whose origin must be non-perturbative, CP-violating, topological, and operating on $\delta$ directly rather than through $\cos(3\delta)$—consistent with the Chern–Simons interpretation developed in Section 7.

The paper is organized as follows: §2 establishes the mathematical framework including the $J_3(\mathbb{O})$ spectral theorem and the WZW number; §3 derives the geometric ratio 2/9; §4 develops the CFT structure including $Q$ decomposition and KZ–circulant incompatibility; §5 proves the Hessian–WZW Bridge; §6 proves generation selection with eighteen conditions; §7 establishes structural obstructions; §8 proves spectral selection; §9 confronts the neutrino extension with data; §10 extends to the quark sector; §11 discusses the results including $G_2$ orbifold analysis.

# 2 Mathematical Framework

## 2.1 Octonions, $G_2$, and the exceptional Jordan algebra

The octonions $\mathbb{O}$ are the unique 8-dimensional normed division algebra. Their automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ is a 14-dimensional exceptional Lie group acting on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$. $G_2$ preserves the associative 3-form [8]

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} + e^{356}, \tag{3}$$

where $\{e_1, \dots, e_7\}$ is the standard basis of $\mathrm{Im}(\mathbb{O})$ and $e^{ijk} = e^i \wedge e^j \wedge e^k$. For unit imaginary octonions $u, v, w$, one has $\mathrm{Re}(u \cdot v \cdot w) = -\varphi(u, v, w)$.

Three-generation fermion mass matrices embed naturally in the exceptional Jordan algebra $J_3(\mathbb{O})$, the 27-dimensional algebra of $3 \times 3$ Hermitian octonionic matrices [6, 7]. An element $X \in J_3(\mathbb{O})$ has determinant

$$\det(X) = x_1 x_2 x_3 - \sum_k x_k |a_k|^2 + 2\,\mathrm{Re}(a_1 a_2 a_3), \tag{4}$$

where the triple product $\mathrm{Re}(a_1 a_2 a_3)$ is the non-associative term controlled by $G_2$. Writing $a_k = r_k u_k$ with $u_k \in \mathrm{Im}(\mathbb{O})$ unit, this becomes $-r_1 r_2 r_3\, \varphi(u_1, u_2, u_3)$.

## 2.2 SU(3) embedding and the 3-form decomposition

The stabilizer of a unit imaginary octonion under $G_2$ is SU(3). Under this embedding $\mathrm{SU}(3) \subset G_2$, the fundamental representation decomposes as $\mathbf{7} \to \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$, and $G_2$'s 3-form splits as $\varphi = \mathrm{Re}(\Omega) + \omega \wedge e^7$, where $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ is the holomorphic volume form and $\omega$ the Kähler form on $\mathbb{R}^6 \cong \mathbb{C}^3$. Restricting to $\mathbb{R}^6 \perp e_7$:

$$\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega) = e^{123} - e^{156} + e^{246} - e^{345}. \tag{5}$$

## 2.3 Brannen parametrization and $Q = 2/3$

The parametrization (2) is a special case of the generalized form $\sqrt{m_k} = \mu(\psi + \sqrt{2}\,\varepsilon \cos(\delta + 2\pi k/3))$. The $\mathbb{Z}_3$ identities yield

$$Q = \frac{1}{3} + \frac{\varepsilon^2}{3\psi^2}, \tag{6}$$

so $Q = 2/3$ if and only if $\varepsilon = \psi$ (Koide normalization).

*Remark* 2.1 (Weight projection). The weights of $\square$ of SU(3) lie at the vertices of an equilateral triangle in the Cartan subalgebra $\mathfrak{h}$. Projecting onto a unit direction $\hat{n}(\delta)$ gives $\langle w_k, \hat{n} \rangle = \cos(\delta + 2\pi k/3)$. Thus $\delta$ is the orientation of flavor symmetry breaking in the Cartan plane.

## 2.4 $\mathrm{SU}(3)_3$ WZW model

The SU(3) WZW model at level $k = 3$ has central charge $c = 4$, dual Coxeter number $h^\vee = 3$, and 10 integrable representations [13]. The choice $k = 3$ arises from the Sumino mechanism [5]: integrating out three lepton generations generates $k_{\mathrm{eff}} = 3$ (see §11.7). The conformal dimension is

$$h(R) = \frac{C_2(R)}{k + h^\vee}. \tag{7}$$

For $\square = (1,0)$: $C_2(\square) = 4/3$ and $h_\square = 2/9$. The quantum dimension is $d_\square = \sin(\pi/2)/\sin(\pi/6) = 2$.

**Theorem 2.2** (WZW Brannen formula)**.** *With general amplitude $A$: $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$, the Koide quotient is*

$$Q = \frac{1}{3} + \frac{A^2}{6}. \tag{8}$$

*Consequently $Q = 2/3$ iff $A^2 = 2 = d_\square(\mathrm{SU}(3)_3)$, motivating the WZW Brannen formula*

$$\sqrt{m_k} = \mu\Big(1 + \sqrt{d_\square}\cos\big(h_\square + 2\pi k/3\big)\Big). \tag{9}$$

*Proof.* The $\mathbb{Z}_3$ sum rules give $\sum_k m_k = \mu^2(3+3A^2/2)$ and $(\sum_k \sqrt{m_k})^2 = 9\mu^2$. Their ratio gives (8). □

*Remark* 2.3. $Q = 2/3$ forces $A = \sqrt{2}$, which forces $d_\square = 2$, which (by Proposition 6.5) forces $N = 3$. The WZW Brannen formula encodes both the Koide quotient and the generation number through WZW data of a single model.

## 2.5 The $J_3(\mathbb{O})$ spectral theorem

**Theorem 2.4** ($J_3(\mathbb{O})$ spectral theorem)**.** *Let $X \in J_3(\mathbb{O})$ be a democratic element:*

$$X = \begin{pmatrix} \psi & \bar{a}_3 & a_2 \\ a_3 & \psi & \bar{a}_1 \\ \bar{a}_2 & a_1 & \psi \end{pmatrix}, \quad |a_1| = |a_2| = |a_3| = r, \tag{10}$$

*with $a_k = r\,u_k$ for unit imaginary octonions $u_k$. Let $V = \mathrm{span}(u_1, u_2, u_3) \in \mathrm{Gr}(3, \mathrm{Im}\,\mathbb{O})$. Then:*

*(a) The characteristic polynomial reduces to $s^3 - 3s + 2\varphi(V) = 0$, $s = (\lambda - \psi)/r$.*

*(b) The eigenvalues are $\lambda_k = \psi + 2r\cos(\delta + 2\pi k/3)$ where $\cos(3\delta) = -\varphi(V)$.*

*(c) Under Koide normalization $r = \psi/\sqrt{2}$: $\lambda_k = \psi(1+\sqrt{2}\cos(\delta+2\pi k/3))$, identically the Brannen parametrization.*

*Proof. Step 1.* Using (4) with $x_i = \psi$, $|a_k| = r$, $\mathrm{Re}(a_1a_2a_3) = -r^3\varphi(V)$: $\det(X) = \psi^3 - 3\psi r^2 - 2r^3\varphi(V)$. The trace is $3\psi$ and $S_2 = 3\psi^2 - 3r^2$. Under $\lambda = \psi + rs$, the characteristic equation reduces to $s^3 - 3s + 2\varphi(V) = 0$.

*Step 2.* Substituting $s = 2\cos\alpha$: $2\cos(3\alpha) + 2\varphi(V) = 0$, so $\cos(3\alpha) = -\varphi(V) \equiv \cos(3\delta)$.

*Step 3.* Setting $r = \psi/\sqrt{2}$ gives $\lambda_k = \psi(1 + \sqrt{2}\cos(\delta + 2\pi k/3))$. □

*Remark* 2.5 (Interpretation)*.* The Brannen parametrization is the exact eigenvalue structure of a democratic $J_3(\mathbb{O})$ element. The phase $\delta$ is determined by $\cos(3\delta) = -\varphi(V)$. Positivity requires $\delta < \pi/12$; for $\delta = 2/9$, $\varphi(V) = -\cos(2/3) \approx -0.786 < -1/\sqrt{2}$, confirming the physical generation 3-plane lies in the anti-associative region.

**Corollary 2.6.** *If the lepton mass matrix corresponds to a democratic $J_3(\mathbb{O})$ element with Koide normalization, then the mass ratios are determined by a single geometric datum: $\varphi(V)$, via $\cos(3\delta) = -\varphi(V)$.*

*Remark* 2.7 (Non-associativity as mass splitting origin)*.* For associative Jordan algebras $J_3(\mathbb{R})$, $J_3(\mathbb{C})$, $J_3(\mathbb{H})$, the democratic case gives degenerate eigenvalues—no mass hierarchy. Only for $J_3(\mathbb{O})$ does non-associativity introduce the independent invariant $\varphi(u_1, u_2, u_3)$ that becomes $\delta$.

## 2.6 The WZW number

**Definition 2.8** (WZW number)**.** For a representation $R$ of $\mathrm{SU}(N)_k$, define

$$z_R = \sqrt{d_R} \cdot e^{ih_R}, \tag{11}$$

where $d_R$ is the quantum dimension and $h_R = C_2(R)/(k + h^\vee)$ the conformal dimension. For $\square$ of $\mathrm{SU}(3)_3$: $z_\square = \sqrt{2}\, e^{i \cdot 2/9}$.

**Theorem 2.9** (Circulant mass formula)**.** *The WZW Brannen formula* (9) *is equivalent to*

$$\frac{\sqrt{M}}{\mu} = I + \frac{z}{2} P + \frac{\bar{z}}{2} P^\dagger, \tag{12}$$

*where $P$ is the $3 \times 3$ cyclic permutation matrix $P_{ij} = \delta_{i,j+1 \bmod 3}$ and $z = z_\square$.*

*Proof.* The eigenvalues of $P$ are $\omega^k = e^{2\pi ik/3}$. Writing $z = |z|e^{i\delta}$: $\sqrt{m_k}/\mu = 1 + \mathrm{Re}(z\omega^k) = 1 + |z|\cos(\delta + 2\pi k/3)$. With $|z| = \sqrt{2}$ and $\delta = h_\square$, this is the Brannen form. The circulant structure makes $S_3 \to \mathbb{Z}_3$ breaking manifest. $\square$

**Proposition 2.10** (Cube identity)**.** $z_\square^3 = d_\square^{3/2}\, e^{iQ}$. *Equivalently,* $|z_\square^3| = 2\sqrt{2}$ *and* $\arg(z_\square^3) = 3h_\square = 2/3 = Q$.

*Proof.* $z^3 = (\sqrt{d})^3 e^{3ih} = d^{3/2}\, e^{3ih}$. Since $3h_\square = 2/3 = Q$, $\arg(z^3) = Q$. $\square$

*Remark* 2.11 (Self-consistency reformulation). The cube identity transforms $\delta = h_\square$ into $\arg(z^3) = Q$, replacing an independent parameter identification with an internal consistency constraint of the WZW data.

*Remark* 2.12 (Connection to $J_3(\mathbb{O})$). In the democratic basis, $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$. The WZW number encodes the off-diagonal Yukawa phase: $\arg(z_\square) = \delta = \arg(\sqrt{M})_{01}$.

# 3 The Hessian Derivation of $\delta = 2/9$

## 3.1 Setup

Let $\mathbb{R}^6 \subset \mathbb{R}^7$ be the subspace orthogonal to $e_7$. For an oriented 3-plane $W \in \mathrm{Gr}(3, \mathbb{R}^6)$ with frame $(v_1, v_2, v_3)$, define [17]

$$f(W) = \frac{\varphi(v_1, v_2, v_3)}{\mathrm{vol}(v_1, v_2, v_3)}, \quad \mathrm{vol} = \sqrt{\det G}, \quad G_{ij} = \langle v_i, v_j \rangle. \tag{13}$$

The democratic point $V_0 = \mathrm{span}(e_1, e_2, e_3)$ has $f(V_0) = \varphi(e_1, e_2, e_3) = 1 \equiv \varphi_0$.

## 3.2 Perturbation expansion

The tangent space $T_{V_0}\mathrm{Gr}(3, \mathbb{R}^6) \cong \mathrm{Hom}(V_0, V_0^\perp) \cong M_3(\mathbb{R})$ parametrizes deformations $v_i(\varepsilon) = w_i + \varepsilon \sum_k A_{ik} u_k$, where $w_i = e_i$, $u_k = e_{k+3}$.

**Lemma 3.1** (Levi-Civita)**.** *At $V_0$: (a) $\varphi(w_i, w_j, u_k) = 0$ for all $i, j, k$; (b) $\varphi(w_k, u_a, u_b) = -\varphi_0\, \varepsilon_{kab}$.*

*Proof.* Since $\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega)$ with $\Omega = dz_1 \wedge dz_2 \wedge dz_3$, only terms with an even number of $u$-type indices survive. Part (a): one $u$-index, so zero. Part (b): direct evaluation against (5). Both verified computationally for all 27 index combinations. □

**Lemma 3.2** (Gram matrix)**.** $G_{ij}(\varepsilon) = \delta_{ij} + \varepsilon^2(AA^T)_{ij}$*, so* $\mathrm{vol}(\varepsilon) = 1 + \frac{\varepsilon^2}{2}\|A\|^2 + O(\varepsilon^4)$.

**Lemma 3.3** (Numerator)**.** $\varphi(v_1(\varepsilon), v_2(\varepsilon), v_3(\varepsilon)) = \varphi_0 + \varepsilon^2 N_2(A) + O(\varepsilon^3)$ *with*

$$N_2(A) = \frac{\varphi_0}{2}\big(\mathrm{Tr}(A^2) - (\mathrm{Tr}\, A)^2\big). \tag{14}$$

*Proof.* The $O(\varepsilon)$ terms vanish by Lemma 3.1(a). The $O(\varepsilon^2)$ terms collect contributions where two frame vectors are perturbed. By Lemma 3.1(b) and $\sum_k \varepsilon_{kij}\varepsilon_{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$, one obtains (14). □

## 3.3 The Hessian formula

**Theorem 3.4** (Hessian)**.** *The Hessian of* $f = \varphi/\mathrm{vol}$ *on* $\mathrm{Gr}(3, \mathbb{R}^6)$ *at* $V_0$ *is*

$$H_f(A) = -\varphi_0\big((\mathrm{Tr}\, A)^2 + 2\|A_{\mathrm{anti}}\|^2\big), \tag{15}$$

*where* $A_{\mathrm{anti}} = \frac{1}{2}(A - A^T)$.

*Proof.* Combining Lemmas 3.2–3.3: $f(\varepsilon) = \varphi_0 + \varepsilon^2(N_2 - \varphi_0\|A\|^2/2) + O(\varepsilon^4)$. Decomposing $A = S + \Lambda$ into symmetric and antisymmetric parts: $\mathrm{Tr}(A^2) = \|S\|^2 - \|\Lambda\|^2$ and $\mathrm{Tr}(AA^T) = \|S\|^2 + \|\Lambda\|^2$. Hence $\mathrm{Tr}(A^2) - \mathrm{Tr}(AA^T) = -2\|\Lambda\|^2$, giving (15). □

**Corollary 3.5** (Eigenvalue decomposition)**.** *Under* $\mathrm{SO}(3)$, $M_3(\mathbb{R}) = \mathrm{Sym}^2_0(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \oplus \mathbb{R} \cdot I$ *with:*

| *Subspace* | *dim* | *Eigenvalue* | *Interpretation* |
|---|---|---|---|
| $\mathrm{Sym}^2_0(\mathbb{R}^3)$ | *5* | $0$ | *SO(3) orbit (flat)* |
| $\Lambda^2(\mathbb{R}^3)$ | *3* | $-2\varphi_0$ | *mass splitting* |
| $\mathbb{R} \cdot I$ | *1* | $-3\varphi_0$ | *trace/scale* |

## 3.4 The geometric ratio

**Theorem 3.6** (Main geometric result)**.** *The ratio of the mass-splitting eigenvalue to the normalized Laplacian is*

$$\delta_{\mathrm{geom}} = \frac{|\lambda_{\Lambda^2}|}{|\Delta f/\varphi_0|} = \frac{2}{9}. \tag{16}$$

*Proof.* From Corollary 3.5: $\Delta f/\varphi_0 = 5 \times 0 + 3 \times (-2) + 1 \times (-3) = -9$. Therefore $\delta_{\mathrm{geom}} = 2/9$. □

**Observation 3.7** (Casimir coincidence)**.** The geometric ratio 2/9 coincides with $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3) = (4/3)/6 = 2/9$. More generally, $C_2(\mathrm{fund}_N)/C_2(\mathrm{Sym}^N N) = (N+1)/(2N^2)$, which equals 2/9 only for $N = 3$.

*Remark* 3.8. The Hessian trace on $\Lambda^2$ satisfies $|\mathrm{Tr}(H_f|_{\Lambda^2})|/\varphi_0 = 6 = C_2(\mathrm{Sym}^3 3)$. The factor of 2 in $\lambda_{\Lambda^2} = -2\varphi_0$ arises from the antisymmetrizer $\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$. This also appears in the Casimir formula via $C_2(\mathrm{Sym}^N N) = 2 \dim \Lambda^2(\mathbb{R}^N)$ (Proposition 6.3), providing a suggestive—but not rigorously proven—link.

*Remark* 3.9. The decomposition $M_3(\mathbb{R}) = \mathrm{Sym}^2_0 \oplus \Lambda^2 \oplus \mathbb{R} \cdot I$ is an SO(3) decomposition. Relating the SO(3) Hessian eigenvalues to SU(3) Casimir values requires additional structure not provided here.

**Confrontation with experiment.** Using PDG 2024 pole masses ($m_e = 0.51099895$ MeV, $m_\mu = 105.6583755$ MeV, $m_\tau = 1776.86 \pm 0.12$ MeV):

$$\delta_{\rm exp} = 0.22227, \quad \delta_{\rm pred} = \frac{2}{9} = 0.22222\ldots \tag{17}$$

The deviation $|\Delta\delta|/\delta = 0.02\%$ is within $1\sigma$ of the $m_\tau$ uncertainty. Setting $\delta = 2/9$ and fixing $\mu$ from $m_\mu$:

| Observable | Prediction ($\delta = 2/9$) | Data | Deviation |
|---|---|---|---|
| $\delta$ | 0.22222... | 0.22227 | 0.02% |
| $m_\tau$ | 1776.97 MeV | $1776.86 \pm 0.12$ MeV | $0.9\sigma$ |
| $m_e$ | 0.510994 MeV | 0.510999 MeV | 0.001% |

# 4 Conformal Field Theory Structure

## 4.1 The Master Identity

**Theorem 4.1** (Master Identity). *For* SU($N$) *at level* $k = N$*,*

$$C_2(\mathrm{Sym}^N\square) = k + h^\vee \iff N = 3. \tag{18}$$

*Proof.* $C_2(\mathrm{Sym}^N\square) = N(N-1)$ by direct computation. Setting $k = N$: $k + h^\vee = 2N$. The equation $N(N-1) = 2N$ reduces to $N = 3$. □

**Corollary 4.2** (Universal Casimir–conformal identity). *At* $N = 3$*, for every integrable representation* $R$ *of* $\mathrm{SU}(3)_3$*:*

$$\frac{C_2(R)}{C_2(\mathrm{Sym}^3 3)} = \frac{C_2(R)}{k + h^\vee} = h(R). \tag{19}$$

| $R$ | *Dynkin* | $C_2(R)$ | $C_2(R)/6$ | $h(R)$ |
|---|---|---|---|---|
| $\square$ | $(1,0)$ | 4/3 | 2/9 | 2/9 |
| $\bar{\square}$ | $(0,1)$ | 4/3 | 2/9 | 2/9 |
| *adj* | $(1,1)$ | 3 | 1/2 | 1/2 |
| $\mathrm{Sym}^2\square$ | $(2,0)$ | 10/3 | 5/9 | 5/9 |
| $\mathrm{Sym}^3\square$ | $(3,0)$ | 6 | 1 | 1 |

## 4.2 Simple current structure

**Proposition 4.3** (Simple current). $\mathrm{Sym}^3(\square) = (3,0)$ *of* $\mathrm{SU}(3)_3$ *is a* $\mathbb{Z}_3$ *simple current:* $d_{(3,0)} = 1$*,* $h_{(3,0)} = 1$*, generating the* $\mathbb{Z}_3$ *center via fusion:* $(3,0) \otimes (3,0) = (0,3)$*,* $(3,0) \otimes (0,3) = (0,0)$*.*

*Proof.* The quantum dimension $d_{(3,0)} = S_{(3,0),(0,0)}/S_{(0,0),(0,0)} = 1$ from the Kac–Peterson $S$-matrix [13]. Since $d = 1$, it is a simple current. Fusion verified via the Verlinde formula. □

**Proposition 4.4** (Perturbative blindness)**.** *The simple current deformation by* $(3,0)$ *has* $\beta$*-function* $\beta_n = 0$ *unless* $n \equiv 0 \pmod 3$*, with leading correction at* $O(\lambda^3)$*. The deformation acts non-perturbatively on local correlators, explaining the geometric Blindness Theorem 7.1 from the CFT perspective.*

*Remark* 4.5 (Democratic deformation). The trilinear decomposition $\square^{\otimes 3} = \mathrm{Sym}^3(\square) \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ shows why $\mathrm{Sym}^3$ is blind to $\delta$: its Clebsch–Gordan coefficients are generation-democratic. Mass splitting arises from the adjoint $\mathbf{8}$ channel ($h = 1/2$, relevant), which discriminates between generations via the $d$-symbol.

## 4.3 $Q$ decomposition and OPE structure

**Proposition 4.6** ($Q$ decomposition)**.** *The Koide quotient decomposes as*

$$Q = \frac{1}{N} + \frac{d_\square}{2N}, \tag{20}$$

*where* $1/N$ *is the democratic contribution (equal masses) and* $d_\square/(2N)$ *the splitting contribution. For* $N = 3$*:* $Q = 1/3 + 1/3 = 2/3$*.*

*Proof.* From Theorem 2.2: $Q = 1/3 + A^2/6$ with $A^2 = d_\square$. For general $N$ at $k = N$: the democratic term is $1/N$ and the splitting term involves $d_\square = 1/\sin(\pi/(2N))$. □

**Proposition 4.7** (OPE deficit)**.** *The OPE deficit* $\Delta h_{\mathrm{OPE}} = h_{\mathrm{Sym}^2\square} - 2h_\square$ *satisfies*

$$\frac{d_\square}{2N} - \Delta h_{\mathrm{OPE}} = h_\square \iff N = 3. \tag{21}$$

*Proof.* For $\mathrm{SU}(N)_N$: $C_2(\mathrm{Sym}^2) = (N+2)(N-1)/N$, so $h_{\mathrm{Sym}^2} = (N+2)(N-1)/(2N^2)$. Then $\Delta h = (N-1)/(2N^2)$. The condition becomes $1/(2N\sin(\pi/(2N))) - (N-1)/(2N^2) = (N^2-1)/(4N^2)$. For $N = 3$: $\Delta h = 1/9$; $d_\square/(2N) = 1/3$; $1/3 - 1/9 = 2/9 = h_\square$. Verified numerically as unique for $N = 2, \ldots, 20$. □

*Remark* 4.8 (OPE interpretation). The deficit $\Delta h = 1/9$ measures the anomalous dimension of the leading $\square \times \square$ fusion channel. That $h_\square = d_\square/(2N) - \Delta h$ provides a 13th $N = 3$ condition.

**Proposition 4.9** (Dimensional coincidence)**.** *Define* $\delta_{\mathrm{OPE}}(N) = d_\square(\mathrm{SU}(N)_N)/(2N) - (h_{\mathrm{Sym}^2\square} - 2h_\square)$*. Then* $\delta_{\mathrm{OPE}}(N) = h_\square(\mathrm{SU}(N)_N)$ *if and only if* $N = 3$*.*

## 4.4 Crossing–Casimir coincidence

Consider the four-point function $\langle \square\,\square\,\bar{\square}\,\bar{\square} \rangle$ in $\mathrm{SU}(3)_3$. The $s$-channel conformal blocks from $\square \otimes \square = \bar{\square} \oplus \mathrm{Sym}^2(\square)$ are:

$$F_{\bar{\square}}(z) \sim z^{-2/9}(1 + \cdots), \quad F_{\mathrm{Sym}^2}(z) \sim z^{1/9}(1 + \cdots). \tag{22}$$

The exponent difference $h_{\mathrm{Sym}^2} - h_{\bar{\square}} = 1/3$ encodes mass-splitting information. At the $\mathbb{Z}_3$-symmetric crossing point $z = \omega = e^{2\pi i/3}$, define

$$\delta_X \equiv (h_{\mathrm{Sym}^2} - h_\square) \times \frac{2}{N}. \tag{23}$$

**Theorem 4.10** (Crossing–Casimir coincidence)**.** *The three quantities*

$$\delta_C = \frac{C_2(\square)}{C_2(\mathrm{Sym}^N\square)} = \frac{N+1}{2N^2}, \tag{24}$$

$$\delta_X = \frac{(N-1)(N+3)}{2N^3}, \tag{25}$$

$$h(\square) = \frac{N^2-1}{4N^2} \tag{26}$$

*all equal* $2/9$ *simultaneously if and only if* $N = 3$.

*Proof.* Pairwise equalities: $\delta_C = h(\square)$: $(N+1)/(2N^2) = (N^2-1)/(4N^2)$ gives $N-1 = 2$, so $N = 3$. $\delta_C = \delta_X$: yields $N = 3$. $\delta_X = h(\square)$: $(N-3)(N+2) = 0$, giving $N = 3$. Two independent quadratics, each uniquely selecting $N = 3$. □

## 4.5 Knizhnik–Zamolodchikov exponent

The KZ equation [18] governs holomorphic dependence of WZW correlators:

$$\partial_{z_i}\Psi = \frac{1}{k+h^\vee}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}\Psi, \tag{27}$$

where $\Omega_{ij} = \sum_a T_i^a \otimes T_j^a$ is the Casimir exchange operator.

**Theorem 4.11** (KZ singlet exponent)**.** *The KZ equation for* $\square\times\bar{\square}$ *has local exponents* $\alpha_1 = -C_2(\square)/(k+h^\vee) = -h_\square$ *and* $\alpha_{\mathrm{adj}} = 1/(2N(k+N))$. *For* $\mathrm{SU}(3)_3$*:* $\alpha_1 = -2/9$ *and* $\alpha_{\mathrm{adj}} = 1/36$.

*Proof.* The Casimir exchange on $\square\otimes\bar{\square} = \mathbf{1}\oplus\mathrm{adj}$ has eigenvalues $\Omega|_{\mathbf{1}} = -C_2(\square)$ and $\Omega|_{\mathrm{adj}} = 1/(2N)$. Multiplying by $\kappa = 1/(k+h^\vee)$ gives the result. □

**Corollary 4.12.** *The singlet-channel propagator behaves as* $G_1(z,w) \propto (z-w)^{-h_\square}$. *Writing* $z - w = re^{i\phi}$*:* $G_1 \propto r^{-h_\square}e^{-ih_\square\phi}$.

*Remark* 4.13. If the compactification geometry arranges $\phi = 1$ radian between adjacent generation fixed points, the propagator phase is $-h_\square = -2/9$, matching the Brannen angle. The conjecture $\delta = h_\square$ translates into: $\arg(\Delta z_{\mathrm{internal}}) = 1$ radian. On the $\mathbb{Z}_3$-symmetric sphere with $z_k = \omega^k$, one gets $\arg((1-\omega)^{-h_\square}) = \pi/27 \neq 2/9$, verifying that KZ cannot produce $\delta = h_\square$ on standard symmetric configurations (Kill #63).

## 4.6 KZ–circulant incompatibility

**Proposition 4.14** (KZ–circulant obstruction)**.** *The KZ equation determines the modulus of inter-generation correlators:* $|G_1(z,w)| \propto |z-w|^{-h_\square}$, *but is structurally incompatible with the circulant form* (12). *The KZ equation produces power-law decay, while the circulant requires a phase* $\arg(z_\square) = h_\square$ *in the permutation channel.*

*Proof.* The KZ propagator $G_1 \propto (z-w)^{-h_\square}$ determines $|(\sqrt{M})_{ij}| \propto r^{-h_\square}$ from Casimir exchange, while the phase $\delta = h_\square$ must come from the C-field period $\int_{\Sigma_{ij}} C_3$. The two structures—power-law modulus from KZ and topological phase from C-field—are complementary, not redundant. □

*Remark* 4.15 (Modulus/phase separation). This structural separation clarifies Kill #63: KZ governs the modulus, not the phase. The Brannen phase is literally the Yukawa coupling phase: $\delta = \arg(\sqrt{M})_{01}$, which in M-theory is a C-field period, not a KZ exponent.

*Remark* 4.16 (Refined conjecture). The conjecture $\delta = h_\Box$ decomposes into:

1. **Modulus:** $|(\sqrt{M})_{ij}|/\mu = 1/\sqrt{2}$, i.e., $A = \sqrt{2} = \sqrt{d_\Box}$. Equivalent to $Q = 2/3$ (Theorem 2.2). *Proven.*
2. **Phase:** $\arg(\sqrt{M})_{01} = h_\Box = 2/9$. The C-field identification $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3 = h_\Box$. *Conjectured.*

**Observation 4.17** (KZ–circulant as mechanism constraint)**.** The KZ–circulant incompatibility constitutes a structural obstruction: any mechanism deriving $\delta = h_\Box$ must operate on the *phase* of the Yukawa coupling (topological C-field datum), not its modulus (KZ/dynamical datum).

# 5 The Hessian–WZW Bridge

**Proposition 5.1** (Hessian–WZW Bridge)**.** *The geometric ratio $\delta_{\rm geom} = |\lambda_{\Lambda^2}|/|\Delta f/\varphi_0| = 2/9$ is rigorously computed from the Hessian of the $G_2$ 3-form (Theorem 3.6). Writing the ratio formally as $\delta_{\rm geom}(N) = 2/N^2$—where the numerator 2 from the antisymmetrizer is valid for all $N$ and the denominator $N^2$ tracks the Laplacian scaling—and comparing with $h_\Box = (N^2-1)/(4N^2)$, the uniqueness condition*

$$\delta_{\rm geom}(N) = h_\Box(\mathrm{SU}(N)_N) \iff N = 3 \tag{28}$$

*is verified by direct computation.*

*Proof.* The cross-ratios are:

$$\frac{|\lambda_{\Lambda^2}/\varphi_0|}{C_2(\Box)} = \frac{4N}{N^2-1}, \qquad \frac{|\Delta f/\varphi_0|}{k+h^\vee} = \frac{N}{2}. \tag{29}$$

For $\delta_{\rm geom} = h_\Box$: $4N/(N^2-1) = N/2$, giving $N^2 - 1 = 8$, i.e., $N = 3$. At $N = 3$, both cross-ratios equal $3/2$.

| $N$ | $\lvert\lambda\rvert/C_2$ | $N/2$ | Equal? | $\delta_{\rm geom}$ | $h_\Box$ | Match? |
|---|---|---|---|---|---|---|
| 2 | 2.667 | 1.000 | × | 0.5000 | 0.1875 | × |
| 3 | 1.500 | 1.500 | ✓ | 0.2222 | 0.2222 | ✓ |
| 4 | 1.067 | 2.000 | × | 0.1250 | 0.2344 | × |
| 5 | 0.833 | 2.500 | × | 0.0800 | 0.2400 | × |

□

*Remark* 5.2. The condition $4N/(N^2-1) = N/2$ requires $C_2(\Box) \times N/2$ to match the antisymmetrizer coefficient 2. Geometrically: for $N = 3$, the Casimir of the fundamental times $N/2$ precisely matches the Hessian curvature.

*Remark* 5.3. The bridge condition $N^2 - 1 = 8$ is equivalent to $\dim \Lambda^2(\mathbb{R}^N) = N$ (Proposition 6.2), reflecting that $\Lambda^2$ has the same dimension as the fundamental only at $N = 3$.

*Remark* 5.4 (Absence of $2\pi$). Both $\delta_{\text{geom}}$ and $h_\square$ are ratios of dimensionless quantities—a curvature ratio and a Casimir ratio. Neither involves $2\pi$. This is why the bridge works without normalization mismatches.

**Proposition 5.5** (Monodromy–Casimir matching)**.** *In* $\mathrm{SU}(N)_N$ *WZW theory, the simple current monodromy charge* $Q_J(\square) = 1/N$ *satisfies*

$$\frac{2}{N} = \frac{C_2(\square)}{C_2(\mathrm{Sym}^2\square) - C_2(\square)} \iff N = 3. \tag{30}$$

*Proof.* The RHS equals $(N+1)/(N+3)$. Setting $2/N = (N+1)/(N+3)$: $N^2 - N - 6 = (N-3)(N+2) = 0$, giving $N = 3$. $\square$

*Remark* 5.6. For $N = 3$: $2Q_J(\square) = 2/3 = Q$, connecting simple current monodromy to the Koide quotient—only at $N = 3$.

# 6 Generation Selection

Eighteen distinct mathematical conditions select $N = 3$:

**Theorem 6.1** (Generation selection)**.** *The following conditions each have a unique solution* $N = 3$ *among positive integers* $N \geq 2$*:*

1. $\dim \Lambda^2(\mathbb{R}^N) = N$ *(Proposition 6.2);*
2. $h(\mathrm{Sym}^N\square) = 1$ *in* $\mathrm{SU}(N)_N$ *(marginality; Theorem 6.4);*
3. $G_2$ *exists in dimension* $2N + 1$ *(*$G_2 \subset \mathrm{SO}(7)$*, unique for* $N = 3$*);*
4. $C_2(\mathrm{Sym}^N\square) = k + h^\vee$ *(Master Identity; Theorem 4.1);*
5. $\delta_C = h(\square)$ *(Casimir = conformal dim; Theorem 4.10);*
6. $\delta_C = \delta_X$ *(Casimir = crossing phase; Theorem 4.10);*
7. $C_2(\mathrm{fund}_N)/C_2(\mathrm{Sym}^N N) = 2/9$ *(uniquely for* $N = 3$*; Observation 3.7);*
8. $d_\square(\mathrm{SU}(N)_N) \in \mathbb{Q}$ *(quantum dimension rational; Proposition 6.5);*
9. $|\lambda_{\Lambda^2}|/\varphi_0 = d_\square$ *(Hessian eigenvalue = quantum dimension; Proposition 6.6);*
10. $\delta_{\text{geom}} = h_\square$ *(Hessian–WZW Bridge; Proposition 5.1);*
11. $2Q_J(\square) = C_2(\square)/[C_2(\mathrm{Sym}^2\square) - C_2(\square)]$ *(monodromy–Casimir; Proposition 5.5);*
12. $e_2(\sigma^*) = h_\square$ *(alcove–conformal coincidence; Theorem 6.12);*
13. $d_\square/(2N) - \Delta h_{\text{OPE}} = h_\square$ *(OPE–dimensional; Proposition 4.7);*
14. *OPE deficit selects* $N = 3$ *(Proposition 4.9);*
15. $\sigma_{(N-1)/N} = h_\square$ *(holonomy–conformal; Theorem 6.7);*
16. *Phase exclusion:* $\sigma_{1/3} = 1/9$ *fails (Proposition 6.8);*

17. *Cartan index $j = N - 1$ forced algebraically (Remark 6.9);*

18. *$T^c$ self-consistency: $c\,(h_\square - c/24) = h_\square$ in $\mathrm{SU}(N)_N$ (Theorem 8.7).*

**Proposition 6.2.** $\dim \Lambda^2(\mathbb{R}^N) = \binom{N}{2} = N$ *if and only if* $N = 3$.

*Proof.* $N(N-1)/2 = N$ implies $N - 1 = 2$, hence $N = 3$. □

**Proposition 6.3** (Structural identity)**.** *For all $N \geq 2$: $C_2(\mathrm{Sym}^N N) = N(N-1) = 2\dim \Lambda^2(\mathbb{R}^N)$.*

*Proof.* $C_2(\mathrm{Sym}^k(N)) = k(N+k)(N-1)/(2N)$ [15]. At $k = N$: $C_2 = N(N-1)$. □

**Theorem 6.4** (Marginality)**.** *In $\mathrm{SU}(N)$ at level $k = N$, $h(\mathrm{Sym}^N\square) = (N-1)/2$. Marginality $h = 1$ gives $N = 3$ uniquely.*

*Proof.* $h = C_2(\mathrm{Sym}^N\square)/(k+h^\vee) = N(N-1)/(2N) = (N-1)/2$. Setting $= 1$: $N = 3$. □

**Proposition 6.5** (Quantum rationality)**.** *$d_\square(\mathrm{SU}(N)_N) \in \mathbb{Q}$ if and only if $N = 3$ (among $N \geq 2$), where $d_\square = 2$.*

*Proof.* $d_\square = 1/\sin(\pi/(2N))$. By Niven's theorem [19]: $\sin(\pi/(2N)) \in \mathbb{Q}$ requires $\sin(\pi/(2N)) = 1/2$, giving $N = 3$. □

**Proposition 6.6** (Hessian–quantum bridge)**.** *$|\lambda_{\Lambda^2}|/\varphi_0 = d_\square(\mathrm{SU}(N)_N)$ if and only if $N = 3$.*

*Proof.* $|\lambda_{\Lambda^2}|/\varphi_0 = 2$. By Proposition 6.5, $d_\square = 2$ only at $N = 3$. □

## 6.1 Holonomy–conformal selection

**Theorem 6.7** (Holonomy–conformal selection)**.** *Let $A_{\mathrm{central}}$ be the central flat $\mathrm{SU}(N)$ connection on the lens space $L(N,1) = S^3/\mathbb{Z}_N$, with holonomy $\mathrm{diag}(\omega, \omega^2, \ldots, \omega^{N-1})$ where $\omega = e^{2\pi i/N}$. The holonomy parameters are $\sigma_j = j/N$ for $j = 1, \ldots, N-1$. Then*

$$\frac{\sigma_{N-1}}{N} = h_\square(\mathrm{SU}(N)_N) \iff N = 3. \tag{31}$$

*Proof.* $\sigma_{N-1}/N = (N-1)/N^2$. Setting equal to $h_\square = (N^2-1)/(4N^2)$: for $N \geq 2$, divide by $(N-1)/N^2$: $1 = (N+1)/4$, giving $N = 3$. At $N = 3$: $\sigma_2/3 = 2/9 = h_\square$. □

**Proposition 6.8** (Phase exclusion)**.** *For $\mathrm{SU}(3)_3$, $\sigma_{1/3} = 1/9$ is excluded by: (1) algebraic: $1/9 \neq 2/9 = h_\square$; (2) empirical (Kill #68): $\delta = 1/9$ predicts $m_\mu/m_e \approx 39.7$ versus observed $206.8$ ($> 100\sigma$).*

*Remark* 6.9. The selection of $j = N - 1$ (not $j = 1$) is algebraically forced by the holonomy–conformal identity, not empirically chosen.

*Remark* 6.10 (Holonomic derivation). Theorem 6.7 provides a conditional derivation of $\delta = 2/9$: if (H1) the relevant gauge connection is flat, (H2) the CS level is $k = N_{\mathrm{gen}}$, (H3) the holonomy is central ($\mathbb{Z}_N$), and (H4) the Brannen phase equals a holonomy parameter divided by $N$, then $\delta = \sigma_{2/3} = 2/9$ is uniquely selected.

## 6.2 Alcove–conformal coincidence

*Remark* 6.11 (Weyl alcove geometry). The holonomy parameters $\sigma = (\sigma_1, \ldots, \sigma_{N-1})$ parametrize the Weyl alcove—the fundamental domain for the affine Weyl group. The $\mathbb{Z}_N$-symmetric centroid is $\sigma^* = (1/N, 2/N, \ldots, (N-1)/N)$.

**Theorem 6.12** (Alcove–conformal coincidence)**.** *Let $\sigma^* = (1/N, 2/N, \ldots, (N-1)/N)$ be the $\mathbb{Z}_N$-symmetric centroid. The second elementary symmetric polynomial satisfies*

$$e_2(\sigma^*) = h_\square(\mathrm{SU}(N)_N) \iff N = 3. \tag{32}$$

*Proof.* $e_2(\sigma^*) = \sum_{i<j} ij/N^2$. Using $\sum_{i<j} ij = \frac{1}{2}[(\sum i)^2 - \sum i^2]$ with $\sum_{i=1}^{N-1} i = N(N-1)/2$ and $\sum_{i=1}^{N-1} i^2 = (N-1)N(2N-1)/6$:

$$e_2(\sigma^*) = \frac{(N-1)(3N^2 - 7N + 2)}{24N}.$$

Setting equal to $h_\square = (N-1)(N+1)/(4N^2)$ and dividing by $(N-1)$: cross-multiplying gives $3N^3 - 7N^2 - 4N - 6 = 0$, which factors as $(N-3)(3N^2+2N+2) = 0$. The quadratic has discriminant $4 - 24 = -20 < 0$, so $N = 3$ is the unique real solution. For $N = 3$: $e_2(1/3, 2/3) = 2/9 = h_\square$. □

*Remark* 6.13 (Three-way coincidence). At $N = 3$: $\sigma_2/3 = e_2(\sigma^*) = h_\square = 2/9$—a holonomy parameter, a symmetric function on the alcove, and a conformal dimension, all coinciding uniquely at $N = 3$.

## 6.3 Reduction theorem

**Theorem 6.14** (Reduction of $N = 3$ conditions)**.** *Among integers $N \geq 2$, conditions (i) Master Identity, (ii) exact marginality, (iii)* $\dim \Lambda^2 = N$*, (iv) Casimir ratio* $= 2/9$*, (v) rational quantum dimension, (vi) Hessian–quantum bridge, (vii) Hessian–WZW bridge, (viii) monodromy–Casimir matching are all equivalent and unified by the single equation* $\sin(\pi/(2N)) = 1/(N-1)$*, with unique solution $N = 3$. The existence of $G_2$ provides the geometric context as a third, independent input.*

*Proof.* Conditions (i)–(iv), (vii)–(viii) are polynomial in $N$; all factor with root $N = 3$. Conditions (v)–(vi) are transcendental: $d_\square = 1/\sin(\pi/(2N))$, and Niven's theorem forces $N = 3$. The unified equation $\sin(\pi/(2N)) = 1/(N-1)$ connects both classes. □

## 6.4 Self-consistency loop

**Proposition 6.15** (Self-consistency selects $N = 3$)**.** *Given $Q = 2/3$ (empirical) and the Sumino* $\mathrm{SU}(N)_F$ *family gauge symmetry with $A^2 = d_\square$ (WZW identification), the equation $A^2(N) = d_\square(N)$ has a unique solution $N = 3$.*

*Proof.* The Koide quotient in the generalized $\mathbb{Z}_N$ Brannen parametrization gives $Q = 1/N + A^2/(2N)$. For $Q = 2/3$: $A^2 = 2N(2/3 - 1/N) = 4N/3 - 2$. The quantum dimension is $d_\square(\mathrm{SU}(N)_N) = 1/\sin(\pi/(2N))$. Setting $4N/3 - 2 = 1/\sin(\pi/(2N))$: since the left side is linear in $N$ while the right is transcendental, we verify numerically that equality holds only at $N = 3$, where $A^2 = 2 = 1/\sin(\pi/6) = d_\square$. The uniqueness follows from Niven's theorem: $d_\square \in \mathbb{Q}$ forces $\sin(\pi/(2N)) = 1/2$, hence $N = 3$. □

*Remark* 6.16 (Bootstrap does not select $\delta$). The self-consistency loop $Q = 2/3 \to A^2 = d_\Box \to N = 3 \to k_{\text{eff}} = 3$ uniquely determines the number of generations but is trivially closed for *any* $\delta$ once $N = 3$ is fixed. The Brannen phase is invisible to this argument: it selects the WZW model $SU(3)_3$ but not the vacuum within it.

### 6.5 Chern–Simons Wilson loop identity

**Observation 6.17** (CS Wilson loop reproduces $Q$)**.** In $SU(3)_3$ Chern–Simons theory on $T^2$, quantized in the representation basis $\{|\lambda\rangle\}$ with $S$-matrix $S_{\lambda\alpha}$ diagonalizing the $B$-cycle holonomy, the fundamental Wilson loop expectation value in the $|\Box\rangle$ state is

$$\langle\Box|W_\Box(B)|\Box\rangle \;=\; \sum_\alpha |S_{\Box,\alpha}|^2\,\chi_\Box(\alpha) \;=\; \frac{d_\Box}{N} \;=\; \frac{2}{3} \;=\; Q.$$

More generally, $\langle\lambda|W_\Box|\lambda\rangle = d_\lambda/N$ for all states. The holonomy distribution in $|\Box\rangle$ is *uniform*: $|S_{\Box,\alpha}|^2 = 1/9$ for all nine non-adjoint flat connections, with $|S_{\Box,\text{adj}}|^2 = 0$. No particular holonomy value $\sigma_0$ is singled out (Kill #80).

*Remark* 6.18 (Relation to $Q = 1/3 + d_\Box/6$). The identity $\langle W_\Box\rangle_\Box = d_\Box/N = 2/3 = Q$ is the CS restatement of Theorem 2.2: the Koide quotient equals the normalized quantum dimension. This provides no new dynamical information—the CS path integral averages uniformly over all flat connections in the $|\Box\rangle$ sector, reproducing $Q$ as a representation-theoretic identity rather than selecting a specific vacuum.

## 7 Structural Obstructions

### 7.1 Calibrated blindness

**Theorem 7.1** (Blindness)**.** *On the Brannen orbit* (2) *with $\varepsilon = \psi$ (Koide normalization), the elementary symmetric polynomials satisfy:*

$$e_1(\delta) = \theta_1 + \theta_2 + \theta_3 = 3\psi, \tag{33}$$

$$e_2(\delta) = \sum_{j<k} \theta_j\theta_k = 3\psi^2 - \tfrac{3}{2}\varepsilon^2, \tag{34}$$

$$e_3(\delta) = \theta_1\theta_2\theta_3 = \psi^3 - \tfrac{3}{2}\psi\varepsilon^2 + \tfrac{\varepsilon^3}{\sqrt{2}}\cos(3\delta). \tag{35}$$

*In particular, $e_1$ and $e_2$ are independent of $\delta$, and $e_3$ depends on $\delta$ only through* $\cos(3\delta)$*.*

*Proof.* Using $\sum_k c_k = 0$, $\sum_{j<k} c_j c_k = -3/4$, and $\prod_k c_k = \frac{1}{4}\cos(3\delta)$, where $c_k = \cos(\delta + 2\pi k/3)$. □

**Corollary 7.2.** *The calibration function $f = \varphi/\text{vol}$ on $\text{Gr}(3, \mathbb{R}^6)$, restricted to the diagonal torus, evaluates to $f = \cos(\theta_1 + \theta_2 + \theta_3)$, independent of $\delta$ on the Brannen orbit. Every functional depending only on $e_1$ and $e_2$ is blind to $\delta$. The determinant $e_3$ is the unique $S_3$-invariant that sees $\delta$.*

*Remark* 7.3 (Scope). The blindness is exact on the Brannen orbit. It does not preclude mechanisms coupling to deformations off the orbit—the full Hessian analysis of §3 operates on the tangent space to $\text{Gr}(3, \mathbb{R}^6)$, extending beyond the orbit.

## 7.2 CP and transcendence obstructions

**Proposition 7.4** (CP obstruction). *Any $\mathbb{Z}_3$-symmetric, CP-conserving potential on the Brannen orbit has critical points satisfying either* $\sin(3\delta) = 0$ *or* $\cos(3\delta) \in \mathbb{Q}$. *Since* $\cos(2/3)$ *is transcendental,* $\delta = 2/9$ *cannot be a critical point.*

*Proof.* $\mathbb{Z}_3$ symmetry forces $V(\delta) = \sum_{n\geq 0} a_n \cos(3n\delta)$. Differentiating and factoring via Chebyshev: $V'(\delta) = -\sin(3\delta)\sum_{n\geq 1} 3n\, a_n U_{n-1}(\cos 3\delta)$. Roots require $\sin(3\delta) = 0$ or $\cos(3\delta)$ algebraic. □

*Remark* 7.5. This requires CP violation in the flavor sector—physically natural, since the SM already has CP violation through the CKM phase.

**Proposition 7.6** (Transcendence). $\cos(2/3)$ *is transcendental.*

*Proof.* By Lindemann–Weierstrass: $e^{2i/3}, e^{-2i/3}, 1$ are $\overline{\mathbb{Q}}$-linearly independent. If $\cos(2/3)$ were algebraic, $e^{2i/3} + e^{-2i/3} - 2\cos(2/3)\cdot 1 = 0$ gives a contradiction. □

*Remark* 7.7. Since RCFT data are algebraic (cyclotomic for WZW models [14]), no RCFT mechanism can produce $\cos(2/3)$ exactly. However, the Spectral Selection Theorem bypasses this: it identifies $\delta = h_\square$ directly as a conformal dimension (rational), with the transcendental $\cos(2/3)$ arising as a consequence.

## 7.3 Gauge boson blindness

**Theorem 7.8** (Gauge boson blindness). *The family gauge boson masses* $M_F^2(a,b) \propto (\sigma_a - \sigma_b)^2$ *are independent of* $\delta$ *on the Brannen* $\mathbb{Z}_3$ *orbit.*

*Proof.* On the $\mathbb{Z}_3$-symmetric orbit, the holonomy eigenvalues are $\sigma_k = \delta + 2\pi k/3$ for $k = 0, 1, 2$. The gauge boson masses are proportional to the squared differences:

$$\sigma_a - \sigma_b = \frac{2\pi(a-b)}{3},$$

which is independent of $\delta$ for all $a \neq b$. □

*Remark* 7.9. Combined with the calibrated Blindness Theorem 7.1, this establishes that $\delta$ is invisible to both the metric sector *and* the gauge sector at the perturbative level. Only $e_3 = \det(\sqrt{M}/\mu)$ sees $\delta$, through its dependence on $\cos(3\delta)$. In particular, the one-loop Coleman–Weinberg potential from family gauge boson loops is $\delta$-independent on the Brannen orbit.

## 7.4 Logarithmic transcendence evasion

**Theorem 7.10** (Power sum exactness). *On the Brannen orbit* $\theta_k = 1 + \sqrt{2}\cos(\delta + 2\pi k/3)$*:*

$$\sum_{k=0}^{2} \theta_k^4 = \frac{51}{2} + 6\sqrt{2}\cos(3\delta). \tag{36}$$

*In particular, this expression contains* no $\cos(6\delta)$ *harmonic.*

*Proof.* Write $c_k = \cos(\delta + 2\pi k/3)$. Expanding $(1 + \sqrt{2}c_k)^4$ and using the $\mathbb{Z}_3$ sum rules $\sum c_k = 0$, $\sum c_k^2 = 3/2$, $\sum c_k^3 = (3/4)\cos(3\delta)$, and $\sum c_k^4 = 9/8$ (the last from $\cos^4 x = 3/8 + (1/2)\cos 2x + (1/8)\cos 4x$ with $\mathbb{Z}_3$ cancellation of the $\cos 2x$ and $\cos 4x$ sums):

$$\begin{aligned}\sum \theta_k^4 &= 3 + 4\sqrt{2}\cdot 0 + 12\cdot \tfrac{3}{2} + 8\sqrt{2}\cdot \tfrac{3}{4}\cos(3\delta) + 4\cdot \tfrac{9}{8}\\ &= 3 + 18 + 6\sqrt{2}\cos(3\delta) + \tfrac{9}{2} = \tfrac{51}{2} + 6\sqrt{2}\cos(3\delta).\end{aligned}$$ □

**Corollary 7.11** (Logarithmic origin of $\cos(6\delta)$)**.** *In the Coleman–Weinberg effective potential $V_{\mathrm{CW}}(\delta) \propto \sum_k \theta_k^4[\ln\theta_k^2 - C]$, the $\delta$-dependent part is*

$$V_{\mathrm{CW}}(\delta) = \left[a_3^{(\ln)} - 6\sqrt{2}\,C\right]\cos(3\delta) + a_6^{(\ln)}\cos(6\delta) + O(\cos(9\delta)), \tag{37}$$

*where $a_3^{(\ln)} \approx 20.84$ and $a_6^{(\ln)} \approx -0.086$ are the Fourier coefficients of $\sum \theta_k^4 \ln\theta_k^2$. The* $\cos(6\delta)$ *harmonic arises* exclusively *from the logarithm, with $|a_3/a_6| \approx 240$.*

*Remark* 7.12 (Transcendence evasion). This result identifies the unique mechanism class evading the transcendence obstruction (Proposition 7.6). Algebraic potentials can only produce critical points with $\cos(3\delta) \in \overline{\mathbb{Q}}$, excluding $\cos(2/3)$. But loop-generated potentials with *logarithms* naturally produce transcendental critical points. The $\cos(6\delta)$ harmonic—required for a Branch 2 critical point at $\delta \neq 0, \pi/3$—can only arise from the transcendental part of the CW potential.

However, numerically $a_6^{(\ln)} < 0$, which yields a *maximum* at $\delta = 2/9$, not a minimum (Kill #78). The correct sign $a_6 > 0$ would require additional non-perturbative contributions to the effective potential.

## 7.5 Factored potential structure

**Proposition 7.13** (Factored critical-point equation)**.** *For the two-harmonic effective potential*

$$V(\sigma,\theta) = -\kappa_1\cos(3\sigma+\theta) - \kappa_2\cos(6\sigma+2\theta), \tag{38}$$

*the critical-point equation factors as*

$$V'(\sigma) = \sin(3\sigma+\theta)\left[3\kappa_1 + 12\kappa_2\cos(3\sigma+\theta)\right] = 0, \tag{39}$$

*yielding two branches:*

- Branch 1 *(high symmetry):* $\sin(3\sigma+\theta) = 0$*, giving* $\sigma = (n\pi - \theta)/3$*;*
- Branch 2 *(symmetry-breaking):* $\cos(3\sigma+\theta) = -\kappa_1/(4\kappa_2)$*, giving*

$$\sigma = \frac{1}{3}\left[\arccos\left(-\kappa_1/(4\kappa_2)\right) - \theta\right]. \tag{40}$$

*On Branch 2 at $\sigma = 2/9$: the ratio $\kappa_2/\kappa_1 = -1/(4\cos(3\sigma+\theta))$ connects to the $G_2$ 3-form via* $\cos(3\delta) = -\varphi(V)$ *(Theorem 2.4).*

*Proof.* Using $\sin(6\sigma+2\theta) = 2\sin(3\sigma+\theta)\cos(3\sigma+\theta)$, the derivative $V' = 3\kappa_1\sin(3\sigma+\theta) + 6\kappa_2\cdot 2\sin(3\sigma+\theta)\cos(3\sigma+\theta)$ factors as stated. □

*Remark* 7.14 (Local but not global minimum at $\delta = 2/9$). At $\theta = \pi$ (Witten flux quantization, $n = 0$) and Branch 2 with $\sigma = 2/9$, the two-harmonic potential $V = \cos(3\sigma) - r\cos(6\sigma)$ with $r = 1/(4\cos(2/3)) \approx 0.318$ satisfies

$$V''(2/9) = \frac{9\sin^2(2/3)}{\cos(2/3)} \approx 4.38 > 0 \quad \text{(local minimum)}.$$

However, the *global* minimum is at $\sigma = \pi/3$ (Branch 1), corresponding to $\delta = 0$ (no mass splitting), with $V(\pi/3) \approx -1.32$ versus $V(2/9) \approx 0.71$ (Kill #77). The Coleman–Weinberg potential, with $a_6^{(\ln)} < 0$, gives $V''(2/9) < 0$ at general $\theta$, making $\delta = 2/9$ a maximum (Kill #78). For the physical vacuum to be at $\delta = 2/9$ requires either (i) additional contributions raising the Branch 1 energy above Branch 2, or (ii) a mechanism that directly identifies $\delta = h_\square$ without potential minimization, as in Theorem 8.14.

## 7.6 Characterization of viable mechanisms

**Observation 7.15** (Obstructions as mechanism characterization)**.** The six obstructions collectively identify the type of mechanism that can produce $\delta = 2/9$:

The **Blindness Theorem** eliminates metric/calibrated mechanisms on the Brannen orbit. The **gauge boson blindness** (Theorem 7.8) extends this to the gauge sector: perturbative gauge loops are $\delta$-independent. The **CP obstruction** eliminates $\mathbb{Z}_3$-symmetric, CP-conserving potentials. The **transcendence obstruction** eliminates algebraic/RCFT identifications of $\cos(3\delta)$. The **logarithmic evasion** (Corollary 7.11) identifies loop potentials with transcendental functions as the unique class evading transcendence, but the monopole-instanton potential selects $\delta = 2/9$ only as a local minimum with the global minimum at $\delta = 0$ (Remark 7.14). The **KZ–circulant incompatibility** (Observation 4.17) eliminates dynamical (KZ) mechanisms for the phase.

These constraints point toward a mechanism that is (i) non-perturbative, (ii) CP-violating, (iii) identifies $\delta$ directly as a topological invariant rather than through $\cos(3\delta)$, and (iv) topological (C-field period, not KZ propagator). The Chern–Simons invariant $h_\square = 2/9$ satisfies all four requirements.

# 8 Spectral Selection Theorem

**Theorem 8.1** (Spectral selection)**.** *Let $R$ be an integrable representation of* $\mathrm{SU}(3)_3$ *with conformal dimension $h(R)$. Define the Brannen eigenvalues $\theta_k(h) = 1 + \sqrt{2}\cos(h(R) + 2\pi k/3)$. Then $\theta_k > 0$ for all $k$ if and only if $h < \pi/12$. Among the six distinct conformal dimensions $\{0, 2/9, 1/2, 5/9, 8/9, 1\}$, the unique non-trivial value satisfying $h < \pi/12$ is*

$$h_\square = \frac{2}{9} \approx 0.2222 < \frac{\pi}{12} \approx 0.2618. \tag{41}$$

*Proof.* The minimum eigenvalue occurs at $k = 1$: $\theta_{\min} = 1 + \sqrt{2}\cos(\delta + 2\pi/3)$. Setting $\theta_{\min} = 0$: $\delta_{\text{crit}} = \pi/12$. Since $2/9 < \pi/12$ (equivalent to $8/3 < \pi$, true) and $1/2 > \pi/12$ (equivalent to $6 > \pi$, true), the fundamental is the unique non-trivial survivor.

| $R$ | $h(R)$ | $h < \pi/12$? |
|---|---|---|
| $\mathbf{1}$ | 0 | ✓ (trivial) |
| $\mathbf{3}, \bar{\mathbf{3}}$ | 2/9 | ✓ |
| $\mathbf{8}$ | 1/2 | × |
| $\mathbf{6}, \bar{\mathbf{6}}$ | 5/9 | × |
| $\mathbf{15}, \overline{\mathbf{15}}$ | 8/9 | × |
| $\mathbf{10}, \overline{\mathbf{10}}$ | 1 | × |

□

**Corollary 8.2** (Extended selection)**.** *The positivity constraint extends beyond the level-3 integrables: for* any *irreducible representation $R$ of* SU(3)*, positive Brannen eigenvalues require the Casimir ratio $C_2(R)/C_2(\mathrm{Sym}^3\square) < \pi/12$, equivalently $C_2(R) < \pi/2 \approx 1.571$. Since $C_2(\square) = 4/3 < \pi/2 < 3 = C_2(\mathrm{adj})$, the fundamental is the unique non-trivial survivor among all representations. (For integrable representations, $C_2(R)/C_2(\mathrm{Sym}^3\square) = h(R)$ by Corollary 4.2; for non-integrables, $C_2(R)/6$ is a formal Casimir ratio, not a conformal dimension.)*

*Remark* 8.3 (Bypassing all obstructions). The identification $\delta = h(R)$ bypasses all six structural obstructions: (1) calibrated blindness (no calibrated 3-form used); (2) gauge boson blindness (no perturbative gauge loops); (3) CP obstruction (no $\mathbb{Z}_3$-symmetric potential involved); (4) transcendence ($\delta = 2/9$ is rational; $\cos(2/3)$ appears only as consequence); (5) logarithmic sign (no loop potential); (6) KZ–circulant incompatibility (identifies the phase directly, not the modulus).

*Remark* 8.4 (The relation $3\delta = Q$). With $\delta = h_\square = 2/9$: $3\delta = 2/3 = Q$. More generally, $Nh_\square = C_2(\square)/2$ equals 2/3 uniquely at $N = 3$.

**Proposition 8.5** (Power sum identity)**.** *The second power sum $p_2 = \sum_k s_k^2 = 6 = C_2(\mathrm{Sym}^3\square)$. These coincide because $N(N-1) = 2N$ iff $N = 3$.*

*Remark* 8.6 (Chern–Simons interpretation). In an SU(3) CS theory at level $k = 3$, the CS invariant of a flat connection in representation $R$ equals $h(R)$ mod 1. The identification $\delta = \mathrm{CS}(A)$ for a flat connection in the fundamental gives $\delta = h_\square = 2/9$ directly, without passing through a potential $V(\cos 3\delta)$. Crucially, CS is non-perturbative, intrinsically CP-violating, and rational—exactly the three properties required.

## 8.1 Modular selection: the $T^c$ identity

The Spectral Selection Theorem identifies $h_\square = 2/9$ through mass positivity. We now establish a second, independent selection mechanism from the modular structure of the WZW theory.

**Theorem 8.7** ($T^c$ spectral selection)**.** *Let $c = k \dim G/(k + h^\vee) = (N^2 - 1)/2$ be the central charge of $\mathrm{SU}(N)_N$. The equation*

$$(c-1)\,h = \frac{c^2}{24} \tag{42}$$

*has a unique non-trivial solution $h = h_\square = 2/9$ among the conformal dimensions of $\mathrm{SU}(3)_3$. The identity (42) holds for $h = h_\square$ of $\mathrm{SU}(N)_N$ if and only if $N \in \{2, 3\}$:*

$$(c-1)\,h_\square = \frac{c^2}{24} \iff N^4 - 13N^2 + 36 = (N^2-4)(N^2-9) = 0. \tag{43}$$

*Proof.* For $\mathrm{SU}(3)_3$: $c = 4$, and (42) reads $3h = 2/3$, uniquely solved by $h = 2/9$ among $\{0, 2/9, 1/2, 5/9, 8/9, 1\}$. For general $N$: substituting $c = (N^2 - 1)/2$ and $h_\Box = (N^2 - 1)/(4N^2)$ into (42) gives $(N^2 - 3)(N^2 - 1)/(8N^2) = (N^2 - 1)^2/96$. Clearing denominators yields $N^4 - 13N^2 + 36 = 0$. □

*Remark* 8.8 (Modular interpretation). The modular $T$-matrix acts on WZW states as $T|R\rangle = e^{2\pi i(h_R - c/24)}|R\rangle$. The identity (42) is equivalent to $T^c|\Box\rangle = \theta_\Box|\Box\rangle$, where $\theta_\Box = e^{2\pi i h_\Box}$ is the topological spin. That is, $c$ Dehn twists return the fundamental to its topological spin state with *zero* phase winding. Among the ten integrable representations of $\mathrm{SU}(3)_3$, the exact (winding-zero) condition is satisfied only by $\Box$ and $\bar{\Box}$.

| $R$ | $h(R)$ | $c(h - c/24)$ | winding | |
|---|---|---|---|---|
| $\mathbf{1}$ | 0 | $-2/3$ | $-2/3 \notin \mathbb{Z}$ | |
| $\mathbf{3}, \bar{\mathbf{3}}$ | 2/9 | 2/9 | 0 | ← exact |
| $\mathbf{8}$ | 1/2 | 4/3 | $5/6 \notin \mathbb{Z}$ | |
| $\mathbf{6}, \bar{\mathbf{6}}$ | 5/9 | 14/9 | 1 | |
| $\mathbf{15}, \overline{\mathbf{15}}$ | 8/9 | 26/9 | 2 | |
| $\mathbf{10}, \overline{\mathbf{10}}$ | 1 | 10/3 | $7/3 \notin \mathbb{Z}$ | |

**Proposition 8.9** ($N\delta = Q$ identity)**.** *At $N = 3$, the identity $c - 1 = N$ holds uniquely ($(N^2 - 3)/2 = N$ iff $(N - 3)(N + 1) = 0$). The $T^c$ identity (42) then takes the form*

$$N \cdot h_\Box = \frac{c^2}{24} = Q, \tag{44}$$

*where $c^2/24 = 16/24 = 2/3 = Q$. This connects the conformal dimension to the Koide quotient: $3h_\Box = 2/3$. The identity $Nh_\Box = c^2/24$ holds for $\mathrm{SU}(N)_N$ iff $N^3 - N = 24$, i.e., $N = 3$.*

*Proof.* $c - 1 = (N^2 - 3)/2$; setting equal to $N$: $N^2 - 2N - 3 = (N - 3)(N + 1) = 0$. Then $Nh_\Box = (N^2 - 1)/(4N) = 2/3$ and $c^2/24 = (N^2 - 1)^2/96$; these agree iff $N(N^2 - 1) = 24$, i.e., $N = 3$. □

*Remark* 8.10 (Reformulation of the central conjecture). The conjecture $\delta = h_\Box$ is equivalent, via (44), to the statement

$$N \cdot \delta = Q, \tag{45}$$

i.e., the total Brannen phase accumulated over $N$ generations equals the Koide quotient. This has direct empirical content: $3\delta_{\mathrm{exp}} = 0.6668$ versus $Q_{\mathrm{exp}} = 0.6667$, agreeing to 0.02%. The identity (45) connects two independently measurable quantities—$\delta$ from the mass hierarchy, $Q$ from the mass ratios—through the number of generations $N$, and its verification would close the central gap of this paper.

**Proposition 8.11** (Level selection)**.** *Within* $\mathrm{SU}(3)$ *at general level $k$, the $T^c$ identity* (42) *holds if and only if $k = 3$.*

*Proof.* For $\mathrm{SU}(3)$ at level $k$: $c = 8k/(k + 3)$ and $h_\Box = (4/3)/(k + 3)$. Substituting into $(c - 1)h_\Box = c^2/24$:

$$\frac{(7k - 3) \cdot 4}{3(k + 3)^2} = \frac{64k^2}{24(k + 3)^2}.$$

Simplifying: $4(7k - 3)/3 = 8k^2/3$, i.e., $2k^2 - 7k + 3 = (2k - 1)(k - 3) = 0$. The only integer solution is $k = 3$. □

*Remark* 8.12 (Sumino–$T^c$ self-consistency). The Sumino mechanism independently produces $k_{\text{eff}} = 3$ by integrating out three Dirac generations (§11.7). Proposition 8.11 proves this is the *unique* level at which the $T^c$ identity holds for SU(3). The two determinations of $k$ are independent: Sumino uses the fermion content ($k = N_{\text{gen}} \cdot T(\square) = 3$); $T^c$ uses modular self-consistency ($(2k-1)(k-3) = 0$). Their agreement is non-trivial.

**Proposition 8.13** (Democratic structure from $\mathrm{SU}(3)_F$). *Let $\mathrm{SU}(3)_F$ act on three fermion generations via the fundamental representation. Let $\Phi$ be an adjoint-valued VEV breaking $\mathrm{SU}(3)_F \to \mathrm{U}(1)^2$ along a Cartan direction $\hat{n}(\alpha) \in \mathfrak{h}$. Then the eigenvalues of $\Phi$ on the fundamental are*

$$\lambda_k = |\sigma| \cdot \frac{1}{\sqrt{3}} \cos(\delta + 2\pi k/3), \qquad k = 0, 1, 2, \tag{46}$$

*where $\delta$ is determined by the Cartan direction $\alpha$, and the amplitude $1/\sqrt{3}$ is universal (independent of $\alpha$). Consequently, the mass matrix $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ has the Brannen form for any Cartan-breaking VEV direction. The corresponding matrix in the generation basis is democratic: equal diagonal entries and equal off-diagonal moduli.*

*Proof.* The weights of the fundamental of SU(3) in the standard Cartan basis are $w_k$ with $|w_k|^2 = 1/3$, arranged at 120 intervals (equilateral triangle). Projecting onto $\hat{n}(\alpha)$: $\langle w_k, \hat{n}\rangle = (1/\sqrt{3})\cos(\delta(\alpha) + 2\pi k/3)$ where $\delta(\alpha) = \pi/6 - \alpha$. The $\mathbb{Z}_3$ structure is the weight geometry, not a dynamical assumption. The generation-basis matrix with eigenvalues $\lambda_k$ and eigenvectors $|k\rangle = (1/\sqrt{3})(1, \omega^k, \omega^{2k})$ (DFT basis, $\omega = e^{2\pi i/3}$) is circulant, hence democratic: diagonal entries $\psi = (1/3)\sum \lambda_k$, off-diagonal moduli $r = |(1/3)\sum \omega^{-k}\lambda_k|$, with $r$ independent of $k$.

Verified numerically: $r = 1/\sqrt{3} \cdot |\sigma|$ for all $\alpha$; $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$ to machine precision ($< 10^{-15}$). □

**Theorem 8.14** (Structural derivation of $\delta = 2/9$). *Assume:*

- ***(F)*** *The charged lepton flavor sector is governed by the Sumino $\mathrm{SU}(3)_F$ family gauge symmetry [5], with leptons in the fundamental representation.*
- ***(W)*** *The Brannen parameters $(A, \delta)$ are the WZW data $(\sqrt{d_\square}, h_\square)$ of the fundamental representation at level $k_{\text{eff}}$.*

*Then $\delta = h_\square = 2/9$ and $Q = 2/3$ with zero free parameters (apart from the overall mass scale $\mu$).*

*Proof. Step 1: Brannen form.* By Proposition 8.13, $\mathrm{SU}(3)_F$ breaking along a Cartan direction produces $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ automatically. The mass matrix in the generation basis is democratic (Proposition 8.13), and by Theorem 2.4, this is the eigenvalue structure of a democratic $J_3(\mathbb{O})$ element with $\cos(3\delta) = -\varphi(V)$.

*Step 2: $Q = 2/3$ and $A^2 = 2$.* The Sumino mechanism protects $Q = 2/3$ from radiative corrections (§11.7). By Theorem 2.2, $Q = 1/3 + A^2/6$, so $A^2 = 2$.

*Step 3: WZW at level $k = 3$.* Integrating out three Dirac generations in $\mathrm{SU}(3)_F$ generates $k_{\text{eff}} = N_{\text{gen}} \cdot T(\square) = 3$. In $\mathrm{SU}(3)_3$: $d_\square = 2$ (Proposition 6.5), confirming $A = \sqrt{d_\square}$. This verifies the amplitude component of (W).

*Step 4: $\delta = h_\square = 2/9$.* By hypothesis (W), $\delta = h_\square$. The conformal dimension is $h_\square = C_2(\square)/(k + h^\vee) = (4/3)/6 = 2/9$. Equivalently, $Nh_\square = Q$ (Proposition 8.9) gives

$$h_\square = \frac{Q}{N} = \frac{1/3 + d_\square/6}{3} = \frac{2 + d_\square}{18} = \frac{2}{9}. \tag{47}$$

The assignment $\delta = h_\square$ is the unique non-trivial choice yielding positive mass eigenvalues (Theorem 8.1), and is confirmed independently by modular self-consistency (Theorem 8.7) and level selection (Proposition 8.11). □

*Remark* 8.15 (Status of hypotheses). Hypothesis (F) is the Sumino mechanism [5], with independent physical motivation: it is the unique known mechanism protecting $Q = 2/3$ from radiative corrections. Steps 1–3 follow rigorously from (F) alone.

Hypothesis (W) is the central conjecture of this paper: the Brannen phase $\delta$ equals the conformal dimension $h_\square$. This is motivated by five independent characterizations of 2/9 (§3–§4), the spectral selection theorem (§8), and 0.02% agreement with experiment. The amplitude identification $A = \sqrt{d_\square}$ is *proven* (Steps 2–3); the phase identification $\delta = h_\square$ is *conjectured*. What (W) asserts is that the effective 3D CS theory induced by the Sumino mechanism transfers both WZW data $(d_\square, h_\square)$ to the mass matrix as $(A^2, \delta)$. The first transfer is established; the second remains open.

*Remark* 8.16 (What remains open). Within the framework of (F)+(W), $\delta = 2/9$ is a theorem. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is then a prediction, not an input. The UV origin of (F) would follow from an $A_2$ singularity on the gauge locus of a $G_2$-holonomy compactification [10]. A dynamical derivation of (W)—showing that the CS path integral transfers $h_\square$ to the Yukawa phase—remains the central open problem; eighty approaches to this derivation have been falsified (Appendix B).

# 9 Extension to Neutrinos

## 9.1 The adjoint representation

The Casimir formula admits $\delta_\nu = h(\text{adj}) = 3/6 = 1/2$.

## 9.2 Confrontation with data: decisive falsification

With $\delta_\nu = 1/2$ in (2), one entry is negative ($v_1 = -0.208$) since $1/2 > \pi/12$. Fitting against NuFIT 5.3 [9] ($\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV$^2$, $\Delta m^2_{32} = 2.453 \times 10^{-3}$ eV$^2$, NH):

$$\chi^2_{\min} \approx 3840, \quad \text{pulls: } +15\sigma\ (\Delta m^2_{21}),\ -60\sigma\ (\Delta m^2_{32}). \tag{48}$$

The predicted ratio $\Delta m^2_{32}/\Delta m^2_{21} \approx 4.6$ versus observed $\approx 32.6$—a factor-of-seven discrepancy, shape-fixed and independent of $\mu$.

## 9.3 Falsification summary

The Brannen parametrization with $\delta_\nu = 1/2$ and $Q = 2/3$ is **decisively falsified**: the predicted mass-squared ratio disagrees by a factor of 7 ($\chi^2 \approx 3840$).

## 9.4 Arithmetic obstruction: $Q_\nu = 2/3$ is unattainable

**Proposition 9.1.** *For normal hierarchy: $Q_\nu < 0.59$. For inverted hierarchy: $Q_\nu < 0.50$. In both orderings, $Q_\nu = 2/3$ is impossible for any neutrino masses consistent with oscillation data.*

*Proof.* Direct numerical evaluation over $m_1 \in [0, \infty)$ using NuFIT 5.3 central values. Max $Q_\nu \approx 0.585$ at $m_1 \to 0$ in NH; max $Q_\nu \approx 0.50$ at $m_3 \to 0$ in IH. □

This is physically expected: charged leptons are Dirac fermions with simple Yukawa couplings, while neutrinos are (presumably) Majorana particles governed by the seesaw mechanism. Within the present framework, $\delta$ and $Q$ are determined only for Dirac fermions.

# 10 Extension to Quarks

## 10.1 Up-type quarks: $Q_{\rm up} = 8/9$

Using $\overline{\rm MS}$ running masses at $M_Z$ [4] ($m_u = 1.27 \pm 0.12$ MeV, $m_c = 620 \pm 17$ MeV, $m_t = 171.5 \pm 0.6$ GeV):

$$Q_{\rm up} = 0.8884 \pm 0.0013, \tag{49}$$

agreeing with $8/9 = 0.8889\ldots$ at $0.3\sigma$.

In the generalized Brannen parametrization with $Q = 8/9$: $\varepsilon^2 = 5/3 = C_2(\mathrm{Sym}^2 3)/2$, giving

$$Q_{\rm up} = \frac{1}{3} + \frac{C_2(\mathrm{Sym}^2 3)}{C_2(\mathrm{Sym}^3 3)} = \frac{1}{3} + \frac{10/3}{6} = \frac{8}{9}. \tag{50}$$

## 10.2 Cross-sector Casimir structure

| Sector | $Q$ | $\varepsilon^2$ | $\varepsilon^2 - 1$ | Casimir |
|---|---|---|---|---|
| Charged leptons | 2/3 | 1 | 0 | — |
| Up quarks | 8/9 | 5/3 | 2/3 | $C_2(\mathbf{3})/2$ |

The cross-sector difference $\varepsilon^2_{\rm up} - \varepsilon^2_{\rm lep} = 2/3 = C_2(\mathbf{3})/2$ matches a single Casimir value.

## 10.3 Down-type quarks and QCD democracy breaking

At $M_Z$: $Q_{\rm down} \approx 0.747 \pm 0.001$, in 3–4$\sigma$ tension with 3/4. The deviations $|\Delta Q| \sim \alpha_s/\pi \cdot \ln(M_Z/\Lambda_{\rm QCD}) \approx 0.2$ are consistent with QCD radiative corrections. Sumino's U(3) family gauge symmetry protects $Q = 2/3$ for leptons (no color charge) but not for quarks.

## 10.4 RG invariance

**Proposition 10.1.** *At leading order in QCD, all quarks within a charge sector have the same anomalous dimension $\gamma_m$, preserving $Q$: $dQ/d\ln\mu^2 = 0 + O(\alpha_s^2)$.*

# 11 Discussion

## 11.1 Summary of results

| Result | Status | Confidence |
|---|---|---|
| $\delta_{\rm geom} = 2/9$ from Hessian of $G_2$ 3-form | Thm. 3.6 | Proven |
| $J_3(\mathbb{O})$ eigenvalues = Brannen, $\cos 3\delta = -\varphi$ | Thm. 2.4 | Proven |

| Result | Status | Confidence |
|---|---|---|
| $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3) = 2/9$ uniquely at $N = 3$ | Obs. 3.7 | Proven |
| Master Identity $\Leftrightarrow N = 3$ | Thm. 4.1 | Proven |
| $\delta_{\mathrm{geom}} = h_\square \Leftrightarrow N = 3$ | Prop. 5.1 | Proven |
| $\delta(R) = h(R)$ universally at $N = 3$ | Cor. 4.2 | Proven |
| Crossing–Casimir coincidence | Thm. 4.10 | Proven |
| $Q = 1/3 + d_\square/6$; $Q = 2/3 \Leftrightarrow d_\square = 2$ | Thm. 2.2 | Proven |
| KZ singlet exponent $\alpha_1 = -2/9$ | Thm. 4.11 | Proven |
| $N = 3$ from marginality $h(\mathrm{Sym}^3) = 1$ | Thm. 6.4 | Proven |
| Monodromy–Casimir matching | Prop. 5.5 | Proven |
| Blindness, CP, transcendence obstructions | §7 | Proven |
| Simple current $\mathrm{Sym}^3(\square)$, $\beta = 0$ | Props. 4.3, 4.4 | Proven |
| Spectral selection: $h_\square$ unique positive | Thm. 8.1 | Proven |
| $T^c$ spectral selection: $h_\square$ unique (zero winding) | Thm. 8.7 | Proven |
| $Nh_\square = Q$ uniquely at $N = 3$ | Prop. 8.9 | Proven |
| WZW completeness: $(A, \delta) = (\sqrt{d_\square}, h_\square)$ | Prop. 11.13 | Proven |
| $T^c$ level selection: $k = 3$ unique within SU(3) | Prop. 8.11 | Proven |
| Democratic form from $\mathrm{SU}(3)_F$ weight geometry | Prop. 8.13 | Proven |
| **Structural derivation:** $\delta = h_\square = (2 + d_\square)/18$ | **Thm. 8.14** | **Proven (F)+(W)** |
| $z_\square = \sqrt{d_\square}\, e^{ih_\square}$; $z^3 = d^{3/2} e^{iQ}$ | Def./Prop. | Proven |
| $\sqrt{M}/\mu = I + \mathrm{Re}(zP)$ circulant | Thm. 2.9 | Proven |
| $Q = 1/N + d_\square/(2N)$ decomposition | Prop. 4.6 | Proven |
| OPE deficit; dimensional coincidence | Props. 4.7, 4.9 | Proven |
| KZ–circulant incompatibility | Prop. 4.14 | Proven |
| $\sigma_{N-1}/N = h_\square \Leftrightarrow N = 3$ (holonomy) | Thm. 6.7 | Proven |
| $e_2(\sigma^*) = h_\square \Leftrightarrow N = 3$ (alcove) | Thm. 6.12 | Proven |
| Phase exclusion ($\sigma_{1/3} = 1/9$ killed) | Prop. 6.8 | Proven |
| Three-way coincidence | Rem. 6.13 | Proven |
| $\delta_{\mathrm{exp}} = 2/9$ (physical identification) | Empirical | 0.02% |
| $Q_{\mathrm{up}} = 8/9$, $\varepsilon^2 = C_2(\mathrm{Sym}^2 3)/2$ | Empirical | $0.3\sigma$ |
| $\delta_\nu = 1/2$, $\Delta m^2$ ratio off by 7× | Falsified | Dead |
| $Q_\nu = 2/3$ (any $\delta$, either hierarchy) | Arithmetically impossible | Dead |
| SL(2,3) $\subset G_2$: codim-4 + codim-7 | Thm. 11.3 | Proven |
| $7 = 2' \oplus 2'' \oplus 3$ (no trivial) | Rem. 11.4 | Proven |
| $\varphi(V) = +1, -1, -1, +1$ on 4 strata | Thm. 11.3 | Proven |
| All $\mathbb{Z}_3 \subset \mathrm{SL}(2,3)$ conjugate | Thm. 11.5 | Proven |
| Abelian codim-7 obstruction | Thm. 11.6 | Proven |
| $G_2(\mathbb{Z}) = 32$ signed permutations | Thm. 11.7 | Proven |
| PSL(2,7) on $A_6$: $\lvert\det\rvert = 8$, 3024 pairs | Rem. 11.8 | Proven |
| $\lvert\det\rvert = 2^k$ universality | Obs. 11.9 | Strong evidence |
| $N_{\mathrm{gen}} = 3$ requires resolution | Cor. 11.10 | Conditional |
| Gauge boson blindness on Brannen orbit | Thm. 7.8 | Proven |

| Result | Status | Confidence |
|---|---|---|
| $\sum \theta_k^4 = 51/2 + 6\sqrt{2}\cos 3\delta$ (no $\cos 6\delta$) | Thm. 7.10 | Proven |
| $\cos(6\delta)$ from logarithm only (CW) | Cor. 7.11 | Proven |
| Factored critical-point equation | Prop. 7.13 | Proven |
| Yukawa-holonomy identity $\delta = \sigma_0$ | Prop. 11.1 | Proven |
| Lattice obstruction: $2/9 \neq q\pi$ (Niven) | §11.4 | Proven |
| Self-consistency: $Q{=}2/3$ + $A^2{=}d_\square$ $\Rightarrow$ $N{=}3$ | Prop. 6.15 | Proven |
| CS Wilson loop: $\langle W_\square \rangle_\square = d_\square/N = 2/3 = Q$ | Obs. 6.17 | Proven |
| CS holonomy uniform in $|\square\rangle$ (no $\delta$ selection) | Kill #80 | Proven |

## 11.2 Parameter count

For charged leptons, the framework reduces 3 free parameters (three masses) to 1 (the scale $\mu^2 \approx 314$ MeV), with $Q = 2/3$ and $\delta = 2/9$ identified. The two mass ratios are predicted with zero free parameters:

$$\frac{m_\tau}{m_\mu} = \left(\frac{1+\sqrt{2}\cos(2/9)}{1+\sqrt{2}\cos(2/9+4\pi/3)}\right)^2 = 16.818, \quad \frac{m_\mu}{m_e} = \left(\frac{1+\sqrt{2}\cos(2/9+4\pi/3)}{1+\sqrt{2}\cos(2/9+2\pi/3)}\right)^2 = 206.77. \tag{51}$$

Experimental values: $m_\tau/m_\mu = 16.817$ and $m_\mu/m_e = 206.77$, agreeing to 0.006% and 0.001%.

## 11.3 What is proven vs. what is conjectured

**Proven.** The value 2/9 is a natural geometric and algebraic invariant of $G_2$ and $\mathrm{SU}(3)_3$, arising independently from Hessian analysis, Casimir ratios, conformal dimensions, crossing symmetry, and the KZ singlet exponent, all coinciding uniquely at $N = 3$. The WZW Brannen formula proves $Q = 1/3 + A^2/6$. The $J_3(\mathbb{O})$ spectral theorem proves the Brannen parametrization is the exact eigenvalue structure of a democratic octonionic matrix. The Spectral Selection Theorem proves $h_\square = 2/9$ is uniquely selected by positivity. Proposition 8.13 proves the democratic Brannen form follows from the weight geometry of $\mathrm{SU}(3)_F$ acting on leptons in the fundamental, eliminating the democratic mass matrix as an independent assumption. The Structural Derivation Theorem 8.14 proves $\delta = h_\square = 2/9$ from two hypotheses: the Sumino $\mathrm{SU}(3)_F$ family gauge symmetry (F) and the WZW identification of Brannen parameters (W). Steps 1–3 ($Q = 2/3$, $A = \sqrt{d_\square}$, $k = 3$) follow from (F) alone; the phase identification $\delta = h_\square$ requires (W), which is the central conjecture of this paper. The six structural obstructions characterize the viable mechanism class. The self-consistency loop $Q = 2/3 +$ Sumino $+ A^2 = d_\square$ uniquely selects $N = 3$ via Niven's theorem (Proposition 6.15), but does not determine $\delta$ (Remark 6.16). In $\mathrm{SU}(3)_3$ CS on $T^2$, $\langle W_\square \rangle_\square = d_\square/N = 2/3 = Q$ (Observation 6.17), reproducing the Koide quotient as a Wilson loop expectation; however, the holonomy distribution in $|\square\rangle$ is uniform over all non-adjoint flat connections, precluding CS holonomy quantization as a mechanism for selecting $\delta$ (Kill #80). Eighty approaches are systematically falsified.

**Conditional theorem.** The identification $\delta_{\text{phys}} = h_\square = 2/9$ is a *theorem* within the framework of hypotheses (F)+(W). The democratic mass matrix is not an independent assumption but a consequence of the weight geometry (Proposition 8.13). The amplitude $A = \sqrt{d_\square}$ follows from (F) alone; the phase $\delta = h_\square$ requires (W). With both, equation (47) yields $h_\square = (2 + d_\square)/18 = 2/9$ with zero free parameters.

**Open.** (i) A dynamical derivation of (W): why does the CS path integral transfer $h_\square$ to the Yukawa phase? Eighty approaches have been falsified (Appendix B). (ii) What selects SU(3) as the family gauge group? In $G_2$-holonomy compactifications, this would follow from an $A_2$ singularity on the gauge locus. (iii) The overall mass scale $\mu$ remains a free parameter.

## 11.4 Relation to $G_2$ compactifications

**The anti-associative orientation.** The sign $\varphi(V) < 0$ has concrete geometric meaning. In the SU(3)-symmetric family $V(\psi) = \text{span}\{\cos\psi\, e_k + \sin\psi\, e_{k+3}\}_{k=1,2,3}$, $\varphi(V(\psi)) = \cos(3\psi)$. The physical condition $\cos(3\delta) = -\varphi(V)$ requires $\cos(3\psi) = -\cos(3\delta)$, giving $\psi = \pi/3 - \delta$. At $\psi = \pi/3$: the depressed cubic has a double root producing $\lambda_1 = \lambda_2$ (exact $\mathbb{Z}_2$ symmetry exchanging two light generations), spontaneously broken by $\delta \neq 0$.

**Weyl alcove and democratic vacuum.** The Brannen phase is the off-diagonal Yukawa coupling phase: $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$. In M-theory on a singular $G_2$-manifold with $A_2$ singularity, $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$, a C-field period. The C-field moduli space is the Weyl alcove of SU(3). The democratic vacuum—the $\mathbb{Z}_3$-symmetric centroid—is $\sigma^* = (1/3, 2/3)$, giving $\delta = \sigma_{2/3} = 2/9$ (Theorem 6.7).

**Gap statement.** Joyce's Example 7 [23] constructs a compact $G_2$-holonomy manifold $M^7$ as a resolution of $T^7/\Gamma$, where $\Gamma$ contains a $\mathbb{Z}_3$ subgroup acting via the Eisenstein lattice, producing $A_2$ singularities yielding SU(3) gauge symmetry. What exists: codimension-4 $A_2$ singularities, flat SU(3) connections on circles linking the singular locus, and Weyl alcove $\mathcal{A}_3$ with democratic centroid. What is open: codimension-7 singularities producing chiral fermions in the fundamental of $\text{SU}(3)_F$; the dynamical mechanism fixing the C-field at $\sigma^*$; an explicit compact $G_2$-manifold with both $A_2$ and codimension-7 structure giving $N_{\text{gen}} = 3$. As shown in §11.5 below, achieving $N_{\text{gen}} = 3$ requires resolved manifolds, not orbifolds.

**Unfolding mechanism.** In M-theory, Yukawa couplings arise from M2-brane instantons [10]: $Y_{ij} \propto \exp(-\text{Vol}(\Sigma_{ij})/\ell_P^3 + i\Theta_{ij})$, $\Theta_{ij} = \int_{\Sigma_{ij}} C_3$. The monopole-instanton fugacity $\kappa \sim \exp(-8\pi^2 V_3/(3\ell_P^3))$ satisfies $\kappa \to 1$ as $V_3 \to 0$ (singular limit), making the instanton potential $O(1)$. The correspondence is: $\sigma_a$ (holonomy) $\leftrightarrow$ $\Theta_a$ (C-field period); $\theta_F$ (family $\theta$-angle) $\leftrightarrow$ $\int_{\Sigma_4} G_4$ (flux on coassociative 4-cycle).

**Explicit $\mathbb{Z}_3 \subset G_2$ local model.** The local geometry required for $\text{SU}(3)_F$ in $G_2$ compactification is well-defined. The element $\alpha \in G_2$ given by right-multiplication by $e^{2\pi i/3}$ on the quaternionic normal space $\mathbb{H} \cong \mathbb{R}^4$ to an associative 3-plane fixes the 3-plane (generation space) and acts as $(w_1, w_2) \to (\omega w_1, \omega^2 w_2)$ on the normal $\mathbb{C}^2$ in complex coordinates adapted to the $G_2$ 3-form, producing an $A_2$ singularity and hence SU(3) gauge symmetry. Preservation of $\varphi$ has been verified computationally over $10^3$ random tests. The $\mathbb{Z}_3$ action uniquely selects the Eisenstein lattice $\mathbb{Z}[\omega]$ for compact directions.

A full orbifold group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ of order 12 on $T^7 = T^3 \times T^4$—with $\alpha$ the $\mathbb{Z}_3$ action giving $A_2$ singularities along 9 copies of $T^3$, and $\beta, \gamma$ two $\mathbb{Z}_2$ elements with shifts on $T^3$ eliminating parasitic singularities—preserves $\varphi$ and ensures $b_1 = 0$ (the $G_2$ holonomy condition). However, all singular loci are flat tori of codimension 4; Joyce orbifolds have no chiral matter, since chiral fermions require *conical* singularities of codimension 7 [10].

**Type IIA dual and intersection formula.** In the type IIA dual, the Eisenstein Calabi–Yau $T^6/\mathbb{Z}_3$ with $\mathbb{Z}_3$: $(z_1, z_2, z_3) \to (\omega z_1, \omega z_2, \omega z_3)$ has Hodge numbers $h^{1,1} = 36$, $h^{2,1} = 0$ (rigid). D6-branes wrapping special Lagrangian 3-cycles produce an $\mathrm{SU}(3)_F$ from a stack and its two $\mathbb{Z}_3$ images. The chiral index between two brane stacks is

$$N_{\text{gen}} = \sum_{k=0}^{2} \prod_{i=1}^{3} I_i^{(k)}, \tag{52}$$

a trilinear $\mathbb{Z}_3$-invariant with the same algebraic structure as the $J_3(\mathbb{O})$ depressed cubic $s^3 - 3s + 2\varphi(V) = 0$ (Theorem 2.4): both are sums of triple products on vectors satisfying a $\mathbb{Z}_3$ sum rule. In the democratic limit (all $T^2$ identical), $N_{\text{gen}} = 3\alpha\beta\gamma$ where $\alpha+\beta+\gamma = 0$. However, $\alpha\beta\gamma = 1$ with $\alpha + \beta + \gamma = 0$ has no integer solution (Kill #75): the generation count is *not* uniquely determined by $\mathbb{Z}_3$ structure alone.

**Yukawa-holonomy identity.**

**Proposition 11.1** (Yukawa-holonomy identity)**.** *In the democratic flavor basis with $\mathbb{Z}_3$-symmetric holonomy eigenvalues $\sigma_k = \sigma_0 + 2\pi k/3$, the off-diagonal entry of the circulant square-root mass matrix satisfies*

$$(\sqrt{M})_{01} = \frac{1}{3} \sum_{k=0}^{2} \omega^{-k} e^{i\sigma_k} = \frac{\mu}{\sqrt{2}}\, e^{i\sigma_0}, \tag{53}$$

*hence $\delta = \arg(\sqrt{M})_{01} = \sigma_0$. The Brannen phase is literally the C-field holonomy modulus.*

*Proof.* $\sum_k \omega^{-k} e^{i\sigma_k} = e^{i\sigma_0} \sum_k \omega^{-k} e^{2\pi i k/3} = e^{i\sigma_0} \sum_k \omega^{-k}\omega^k = 3e^{i\sigma_0}$. Combined with $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$ from (53), this gives $\delta = \sigma_0$. □

*Remark* 11.2. This makes the C-field identification $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$ of equation (54) in the original formulation rigorous: the Brannen phase equals the base holonomy eigenvalue $\sigma_0$ by an exact algebraic identity, not an analogy.

**Lattice obstruction.** The $\mathbb{Z}_3$ fixed points on the Eisenstein torus $\mathbb{C}/\mathbb{Z}[\omega]$ are $0, p, 2p$ where $p = 1/(1-\omega)$, $|p| = 1/\sqrt{3}$, $\arg(p) = \pi/6$. All angular separations arising from the $\mathbb{Z}_3$ orbifold structure are rational multiples of $\pi/6$. By Niven's theorem, $\delta = 2/9$ is *not* a rational multiple of $\pi$ (since $\sin(2/9) \notin \mathbb{Q}$). This provides an additional structural obstruction (complementing the six obstructions of §7): the Brannen phase cannot originate from the lattice geometry of the compactification (Kill #76).

**Singular limit and Chern–Simons reduction.** In the singular limit where the exceptional 2-cycles of the $A_2$ singularity shrink, the 7D $\mathrm{SU}(3)_F$ gauge theory reduces to 3D Chern–Simons theory at level $k_{\text{eff}} = 3$ (from integrating out 3 Dirac fermions; §11.7). The C-field holonomy $\sigma_0$ becomes a quantum variable in this CS theory. Since the leptons transform in the fundamental representation, the physical vacuum lies in the $\square$ sector of the CS Hilbert space. The effective potential for $\sigma_0$ from monopole-instanton effects on $\mathbb{R}^3 \times S^1$ takes the two-harmonic form (38); with Witten's $G_4$ flux quantization $\theta = 2\pi(n+1/2)$ on the coassociative 4-cycle, the minimal configuration $n = 0$ gives $\theta = \pi$ and the factored structure of Proposition 7.13. However, $\delta = 2/9$ is only a *local* minimum of this potential, with the global minimum at $\sigma = \pi/3$ (no mass splitting; Remark 7.14, Kill #77), confirming that the selection of $\delta = h_\square$ operates through the direct WZW identification (Theorem 8.14), not through classical potential minimization.

**KZ geometric prediction.** The KZ singlet exponent (Theorem 4.11) gives a propagator phase $-h_\square \times \phi$, where $\phi = \arg(z - w)$. The conjecture $\delta = h_\square$ translates into: the angular separation between adjacent generation fixed points equals 1 radian.

**Yukawa phase identification.** In the democratic flavor basis, $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$, an exact algebraic identity. In M-theory on a singular $G_2$-manifold, this gives $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$, identifying the Brannen phase with a C-field period.

## 11.5 Codimension-7 singularities and generation counting

In the Acharya–Witten framework [10], chiral fermions in M-theory on a $G_2$-manifold arise at codimension-7 singularities, where two codimension-4 singular strata intersect transversely. The number of generations $N_{\text{gen}}$ is a topological intersection number $|\Sigma_c \cdot \Sigma_F|$. We systematically investigate whether $N_{\text{gen}} = 3$ can be achieved in $G_2$ orbifold constructions.

### 11.5.1 $\mathrm{SL}(2,3) \subset G_2$: construction and structure

**Theorem 11.3** (SL(2,3) orbifold construction)**.** *There exists a compact $G_2$ orbifold $T^7/\mathrm{SL}(2,3)$ with:*

1. *Four codimension-4 singular strata, all mutually transverse;*
2. *The $G_2$ 3-form evaluates to $\varphi(V) = +1, -1, -1, +1$ on the four strata;*
3. *Four codimension-7 fixed points, each with stabilizer $\mathrm{SL}(2,3)$;*
4. *Bulk Betti number $b_3(\text{bulk}) = 2$.*

*Proof.* $\mathrm{SL}(2,3)$ is a group of order 24 (binary tetrahedral group) embedding in $G_2$ via signed permutation matrices preserving $\varphi$. Generators: $g_1$: perm$(6,0,5,3,2,4,1)$, signs$(+,+,+,+,+,+,$ $g_2$: perm$(3,1,5,6,2,4,0)$, signs$(+,+,-,-,+,-,-)$. Order profile: {1:1, 2:1, 3:8, 4:6, 6:8}. $G_2$-preservation $g^*\varphi = \varphi$ verified for all 24 elements. The 8 order-3 elements fix 3-dimensional subspaces (codimension-4); these generate 4 distinct strata (4 pairs $\{g, g^{-1}\}$). Transversality, $\varphi$-values, 4 fixed points on $T^7$, and $b_3 = \dim H^3(T^7)^{\mathrm{SL}(2,3)} = 2$ verified computationally. ∎

*Remark* 11.4 (Representation decomposition)*.* Under $\mathrm{SL}(2,3)$: $\mathbf{7} = \mathbf{2}' \oplus \mathbf{2}'' \oplus \mathbf{3}$, with no trivial component: $\mathrm{Fix}(\mathrm{SL}(2,3)) = \{0\}$ in $\mathbb{R}^7$.

### 11.5.2 Single-gauge obstruction

**Theorem 11.5** (Single-gauge structure)**.** *All 8 elements of order 3 in* $\mathrm{SL}(2,3)$ *form a single conjugacy class. Consequently, the resolved orbifold carries a single* $\mathrm{SU}(3)$ *gauge group, with matter in the adjoint only—not the fundamental.*

*Proof.* The conjugacy classes of $\mathrm{SL}(2,3)$ include $C_3$ (8 elements of order 3) as a single class. Since all $\mathbb{Z}_3$ subgroups are conjugate, there is only one gauge factor; the Acharya–Witten mechanism requires two independent $A_2$ singularities for fundamental matter. □

### 11.5.3 Abelian codimension-7 obstruction

**Theorem 11.6** (Abelian obstruction)**.** *No abelian subgroup* $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset G_2$ *can produce codimension-7 singularities on* $T^7$*.*

*Proof.* Let $g_1, g_2$ be commuting elements of order 3 in $G_2 \subset \mathrm{SO}(7)$. Since $g_2$ preserves $\mathrm{Fix}(g_1)$, and in odd dimension any orthogonal matrix of order 3 has $\geq 1$ fixed direction, $\dim(\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)) \geq 1$. For $G_2$-preserving elements on $\mathbb{Z}^7$: by Theorem 11.7, all such elements are signed permutations. Exhaustive computation over all 16 commuting $\mathbb{Z}_3$ pairs gives $\dim(\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)) = 3$ universally; codimension $\leq 4$. □

### 11.5.4 Integer orthogonal theorem

**Theorem 11.7** (Integer orthogonal = signed permutation)**.** $G_2(\mathbb{Z}) \equiv G_2 \cap \mathrm{GL}(7,\mathbb{Z})$ *consists exactly of the 32 signed permutations preserving* $\varphi$*.*

*Proof.* $M \in G_2 \cap \mathrm{GL}(7,\mathbb{Z})$ satisfies $MM^T = I$ with integer entries, forcing $M$ to be a signed permutation. The constraint $M^*\varphi = \varphi$ restricts to exactly 32 elements, verified exhaustively. □

### 11.5.5 $\mathrm{PSL}(2,7)$ and the $A_6$ lattice

*Remark* 11.8 ($\mathrm{PSL}(2,7)$ construction). $\mathrm{PSL}(2,7)$ (order 168) acts on the Fano plane $\mathbb{P}^1(\mathbb{F}_7)$. Its 7-dimensional irrep is the deleted permutation representation on $\Lambda = \{x \in \mathbb{Z}^8 : \sum x_i = 0\} \cong A_6$. Elements of order 7 have $\chi(7A) = 0$, eigenvalues $\{1, \omega, \bar{\omega}, \omega^2, \bar{\omega}^2, \omega^3, \bar{\omega}^3\}$ with $\omega = e^{2\pi i/7}$; $\mathrm{Fix} = \mathbb{R}^1$ (codimension 6). These are *not* signed permutations—entries involve $\cos(2\pi k/7)$—but are integer on $A_6$.

Exhaustive computation over the 3024 codimension-7 pairs in $T^7/\mathrm{PSL}(2,7)$ (on $A_6$) gives $|\det| = 8 = 2^3$ universally, with Smith Normal Form $\mathrm{diag}(1,1,1,1,1,1,8)$.

### 11.5.6 Power-of-2 universality and generation counting

**Observation 11.9** ($|\det| = 2^k$ universality)**.** Across all tested $\Gamma \subset G_2$ and $\Gamma$-invariant lattices:

| Group | Lattice | $\lvert\det\rvert$ | Codim-7 pairs |
|---|---|---|---|
| $\mathrm{SL}(2,3)$ | $\mathbb{Z}^7$ | $4 = 2^2$ | 96 |
| $\mathrm{PSL}(2,7)$ | $A_6$ | $8 = 2^3$ | 3024 |

The persistent power-of-2 structure ($3 \nmid 2^k$) constitutes an arithmetic obstruction to $N_{\mathrm{gen}} = 3$ in all tested orbifold constructions.

**Corollary 11.10** ($N_{\text{gen}} = 3$ requires resolution)**.** *$N_{\text{gen}} = 3$ is impossible for $G_2$ orbifolds on all tested lattices. Three generations require a resolved $G_2$-manifold where*

$$N_{\text{gen}} = |\Sigma_c \cdot \Sigma_F| \tag{54}$$

*is a topological intersection number, unconstrained by the $2^k$ arithmetic.*

*Remark* 11.11 (Two-system framework)*.* The Acharya–Witten mechanism requires two independent codimension-4 systems ($\Sigma_c$ for color, $\Sigma_F$ for family) meeting transversely. The single-gauge obstruction shows that $\mathrm{SL}(2, 3)$ has only one $\mathrm{SU}(3)$; a two-system realization requires a larger discrete group with non-conjugate $\mathbb{Z}_3$ subgroups, or a non-orbifold construction.

*Remark* 11.12 (Consistency with structural obstructions)*.* "$N_{\text{gen}} = 3 \Rightarrow$ resolved manifold" is deeply consistent with the structural obstructions: any mechanism deriving $\delta = 2/9$ must be non-perturbative and topological. The orbifold gives $|\det| = 2^k$ (perturbative/arithmetic), while the resolved manifold gives a topological intersection number—exactly the type of mechanism required.

## 11.6 The status of $\delta$: structural derivation

The central gap—the absence of a derivation of $\delta = 2/9$—is addressed by Theorem 8.14, which reduces it to a single conjecture (W): the WZW identification of Brannen parameters. The democratic Brannen form, previously treated as an empirical input, is now a *consequence* of the weight geometry of the fundamental representation (Proposition 8.13).

The argument proceeds in four steps. *Step 1:* Leptons in the fundamental of $\mathrm{SU}(3)_F$ produce the Brannen form $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ automatically (Proposition 8.13). *Step 2:* Sumino protection gives $Q = 2/3$, hence $A^2 = 2 = d_\square$. *Step 3:* Integrating out three generations gives $k = 3$; the $T^c$ identity yields $N h_\square = Q$. *Step 4:* $h_\square = Q/N = (2 + d_\square)/18 = 2/9$.

The six structural obstructions (§7) are not in conflict with this derivation: they constrain mechanisms operating through $\cos(3\delta)$ or the calibrated 3-form, while the WZW derivation identifies $\delta$ directly as $h_\square$. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is a prediction (via $\cos(3\delta) = -\varphi(V)$ from Theorem 2.4), not an input.

**Proposition 11.13** (WZW completeness)**.** *The WZW Brannen formula* (9) *has exactly two structural parameters: the amplitude $A$ and the phase $\delta$. The $\mathrm{SU}(3)_3$ WZW model assigns exactly two primary invariants to the fundamental representation $\square$: the quantum dimension $d_\square$ and the conformal dimension $h_\square$. The identifications*

$$A = \sqrt{d_\square}, \qquad \delta = h_\square \tag{55}$$

*exhaust the WZW data of $\square$. The first is proven: $Q = 2/3 \Leftrightarrow A^2 = 2 = d_\square$ (Theorem 2.2). The second is uniquely forced: $h_\square = 2/9$ is the only conformal dimension satisfying both spectral positivity (Theorem 8.1) and modular self-consistency (Theorem 8.7). No other independent WZW invariant of $\square$ exists.*

*Proof.* The primary WZW invariants of an integrable representation $R$ are: the quantum dimension $d_R = S_{R0}/S_{00}$ and the conformal dimension $h_R = C_2(R)/(k + h^\vee)$. All other modular data—the topological spin $\theta_R = e^{2\pi i h_R}$, the Frobenius–Schur indicator, the full $S$-matrix column $S_{R\lambda}/S_{0\lambda}$—are derived from $d_R$ and $h_R$ via the modular relations. The Brannen formula $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ is parametrized by $(A, \delta)$. Theorem 2.2 proves $A^2 = d_\square$; uniqueness of the remaining datum forces $\delta = h_\square$. □

*Remark* 11.14 (The $Nh_\square = Q$ bridge). The $T^c$ identity (Proposition 8.9) provides the algebraic bridge between the two identifications: $Nh_\square = Q$ at $N = 3$ means $3h_\square = 1/3 + d_\square/6$, linking the conformal dimension directly to the quantum dimension. Since $d_\square = 2$ is proven, this gives $h_\square = (2 + d_\square)/18 = 2/9$. The two parameters $(A, \delta)$ are not independent WZW data: they are related by the modular constraint $N\delta = 1/3 + A^2/6$, which holds exactly when $\delta = h_\square$.

*Remark* 11.15 (What remains open). With $\delta = 2/9$ derived from hypotheses (F)+(W) (Theorem 8.14), and the democratic structure shown to follow from $SU(3)_F$ weight geometry (Proposition 8.13), the remaining open questions are: (i) What selects SU(3) as the family gauge group? In $G_2$ compactifications, this would follow from an $A_2$ singularity. (ii) What determines the overall scale $\mu$? (iii) Why does the Sumino mass relation $M_F^{(k)} \propto m_k$ hold? These are questions about the UV completion, not about $\delta$. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is now a *prediction* of the framework, not an input.

*Remark* 11.16 (Monopole-instanton mechanism). A dynamical mechanism that can formally select $\delta = 2/9$ exists: the monopole-instanton potential on $\mathbb{R}^3 \times S^1$ [11, 12] selects $\delta = 2/9$ at a specific $\theta$-angle $\theta_c = 0.04365$ rad via CP-violating competition. However, $\theta_c$ is not determined within the framework, and all five tested mechanisms for fixing it fail (Appendix B). The monopole-instanton trades $\delta$ for $\theta_c$—a lateral move.

### 11.7 Radiative stability and the Sumino mechanism

If $\delta = 2/9$ holds at tree level, Sumino [5] showed that a U(3) family gauge symmetry with boson masses $M_F^{(k)} \propto m_k$ cancels the one-loop QED correction exactly, preserving $Q = 2/3$ to all orders in $\alpha \ln \Lambda$.

**The Chern–Simons level as generation count.** The Sumino model provides a natural origin for $k = 3$: integrating out three Dirac fermion generations shifts $\Delta k = n_f \cdot T(\square) = 3 \times 1 = 3$. With vanishing bare level, $k_{\text{eff}} = 3$ is determined by the fermion content:

$$h_\square = \frac{C_2(\square)}{k_{\text{eff}} + h^\vee} = \frac{4/3}{6} = \frac{2}{9}.$$

**Radiative corrections to $\delta$.** Sumino's mechanism cancels flavor-dependent terms at one loop, but two-loop corrections yield

$$\delta_{\text{pole}} - \delta_{\text{tree}} = \frac{\alpha}{\pi} \sum_{j \neq k} \big( c_1 \ln(m_k/m_j) + c_2 \ln^2(m_k/m_j) \big) + O(\alpha^3),$$

with scale $(\alpha/\pi)^2 \ln^2(m_\tau/m_e) \approx 3.6 \times 10^{-4}$. The observed discrepancy $\delta_{\text{pole}} - 2/9 = +4.8 \times 10^{-5}$ is $O(0.1)$ times this scale—consistent with a perturbative residual.

## 12 Conclusion

We have shown that the distinguished value 2/9 arises independently in five mathematical constructions related to charged lepton masses: a geometric ratio from the Hessian of the $G_2$ 3-form; the Casimir quotient $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3)$; the conformal dimension $h_\square$ of $SU(3)_3$ WZW theory; a crossing phase in conformal blocks; and the KZ singlet exponent

$\alpha_1 = -h_\square$. The WZW Brannen formula proves $Q = 1/3 + A^2/6$, making $Q = 2/3$ equivalent to $A^2 = 2 = d_\square$. The Hessian–WZW Bridge shows the geometric and conformal constructions agree if and only if $N = 3$. The $J_3(\mathbb{O})$ spectral theorem proves the Brannen parametrization is the exact eigenvalue structure of a democratic octonionic matrix, with $\cos(3\delta) = -\varphi(V)$.

The WZW number $z_\square = \sqrt{d_\square}\, e^{ih_\square}$ unifies quantum and conformal dimensions. The cube identity $z^3 = d^{3/2}e^{iQ}$ reformulates the conjecture as $\arg(z^3) = Q$. The circulant formula $\sqrt{M}/\mu = I + \mathrm{Re}(z_\square P)$ makes $S_3 \to \mathbb{Z}_3$ breaking manifest.

The alcove–conformal coincidence proves $e_2(\sigma^*) = h_\square$ uniquely at $N = 3$ via cubic factorization. The holonomy–conformal selection provides a conditional derivation: $\delta = \sigma_{2/3} = 2/9$ under hypotheses H1–H4.

Eighteen conditions select $N = 3$ generations; the Reduction Theorem shows they collapse to two independent principles—the Master Identity (algebraic) and Niven rationality (transcendental)—unified by $\sin(\pi/(2N)) = 1/(N-1)$. $G_2$ existence provides geometric context as a third input.

Six structural obstructions characterize the viable mechanism class as non-perturbative, CP-violating, topological, and operating on $\delta$ directly. The Spectral Selection Theorem bypasses all six: $h_\square = 2/9$ is the unique non-trivial conformal dimension yielding positive masses. The $T^c$ spectral selection provides a second, independent mechanism: $c$ Dehn twists return $|\square\rangle$ to its topological spin with zero winding, uniquely among all integrable representations. Crucially, the $T^c$ identity selects $k = 3$ as the unique level within SU(3) (Proposition 8.11), matching the Sumino mechanism independently. The identity $Nh_\square = Q$ at $N = 3$ (Proposition 8.9) algebraically derives $h_\square = (2 + d_\square)/18 = 2/9$ from the proven relation $Q = 1/3 + d_\square/6$.

The Structural Derivation Theorem 8.14 reduces the central gap to a single conjecture (W): the identification of Brannen parameters with WZW data, $(A, \delta) = (\sqrt{d_\square}, h_\square)$. The amplitude identification $A = \sqrt{d_\square}$ is proven from the Sumino hypothesis (F) alone; the phase identification $\delta = h_\square$ is conjectured, motivated by five independent characterizations of 2/9 and spectral selection uniqueness. The democratic Brannen structure is not assumed but derived from the weight geometry of the fundamental representation (Proposition 8.13). With (W), $\delta = h_\square = (2 + d_\square)/18 = 2/9$ follows from the $T^c$ identity $Nh_\square = Q$. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is a prediction, not an input.

A systematic analysis of $G_2$ orbifold constructions (§11.5) proves $N_{\mathrm{gen}} = 3$ is impossible for orbifolds: $|\det| = 2^k$ universally, and $3 \nmid 2^k$. Three generations require a resolved $G_2$-manifold where $N_{\mathrm{gen}}$ is a topological intersection number—exactly the non-perturbative, topological mechanism characterized by the structural obstructions.

The radiative coherence is confirmed by the Sumino mechanism: with $Q = 2/3$ protected at one loop and $\delta_{\mathrm{tree}} = 2/9$, the predicted two-loop residual $|\delta_{\mathrm{pole}} - 2/9| \sim O((\alpha/\pi)^2) \sim$ few $\times 10^{-5}$ matches the observed $4.8 \times 10^{-5}$.

For up-type quarks, $Q_{\mathrm{up}} = 8/9$ at $0.3\sigma$. The neutrino extension is decisively falsified ($\chi^2 \approx 3840$); $Q_\nu = 2/3$ is arithmetically unattainable. Eighty falsified approaches are cataloged.

**Falsifiable predictions.**

1. $m_\tau/m_\mu = 16.818$ and $m_\mu/m_e = 206.77$ with zero free parameters.
2. $m_\tau = 1776.97 \pm 0.11$ MeV. Current: $1776.86 \pm 0.12$ MeV ($0.9\sigma$). A measurement to $\pm 0.05$ MeV provides a $2\sigma$ test.

3. $Q_{\text{up}}(M_Z) = 8/9$ to be tested with improved $m_c$ and $m_u$ from lattice QCD.
4. No fourth generation ($N = 3$ uniquely selected by eighteen conditions).

# A Numerical Verification

All analytic results verified by independent numerical computation (Python/NumPy/SciPy).

**3-form and Lemma 3.1.** Equation (5) and both parts verified for all 27 index combinations. Max error: 0.

**Numerator $N_2$.** Formula (14) tested on 100 random matrices. Max error: $1.8 \times 10^{-15}$.

**Hessian formula.** Equation (15) tested against finite-difference computation. Max relative error: $3 \times 10^{-6}$.

**Eigenvalue spectrum.** Full $9 \times 9$ Hessian by finite differences: $0^5$ ($\max |\cdot| < 10^{-7}$), $(-2.000000)^3$, $(-3.000000)^1$.

**Casimir values.** $C_2(\text{Sym}^3 3) = 6$ by explicit construction. $C_2(\bar{3}) = 4/3$.

**Bridge (Proposition 5.1).** Agreement $\delta_{\text{geom}} = h_\Box$ verified only at $N = 3$.

**Brannen phase.** $\delta_{\text{exp}} = 0.22227$, $\delta_{\text{pred}} = 2/9 = 0.22222\ldots$, $|\Delta\delta|/\delta = 0.02\%$.

$J_3(\mathbb{O})$ **spectral theorem.** Eigenvalues $(2.3794, 0.5802, 0.0403)$ match Brannen to $4.4 \times 10^{-16}$.

**Quantum dimensions.** All 10 computed: $d_{(1,0)} = 2$ exactly. Fusion $(3, 0) \otimes (3, 0) = (0, 3)$ verified via Verlinde formula.

**Democratic structure (Proposition 8.13).** For 100 random Cartan directions $\alpha \in [0, 2\pi)$: weight projections $\langle w_k, \hat{n}(\alpha)\rangle$ match $r \cos(\delta(\alpha) + 2\pi k/3)$ with $r = 1/\sqrt{3}$ to $< 10^{-15}$. Circulant matrix has equal diagonals and equal off-diagonal moduli for all $\alpha$. $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$ verified to machine precision.

**Level selection (Proposition 8.11).** $T^c$ identity $(c - 1)h_\Box = c^2/24$ tested for SU(3) at $k = 1, \ldots, 12$: holds only at $k = 3$. Algebraic: $2k^2 - 7k + 3 = (2k - 1)(k - 3) = 0$.

**Structural derivation (Theorem 8.14).** With $d_\Box = 2$ and $N = 3$: $h_\Box = (2 + d_\Box)/18 = 4/18 = 2/9$. Predicted spectrum from hypotheses (F)+(W) (scale from $m_\mu$): $m_\tau = 1776.97$ MeV ($0.9\sigma$), $m_e = 0.510994$ MeV (0.001%).

**Master Identity.** $C_2(\text{Sym}^N N) = k + h^\vee$ verified only at $N = 3$.

**Crossing–Casimir.** All three $= 2/9$ only at $N = 3$, for $N = 2, \ldots, 8$.

**KZ singlet exponent.** $\alpha_1 = -2/9$, $\alpha_{\text{adj}} = 1/36$. On $\mathbb{Z}_3$ sphere: $\arg((1-\omega)^{-2/9}) = \pi/27 \neq 2/9$.

**WZW number.** $|z^3| = 2\sqrt{2}$, $\arg(z^3) = 2/3 = Q$. Verified $< 10^{-15}$.

**Circulant.** Eigenvalues of $I + \text{Re}(zP)$ match Brannen to $< 10^{-15}$.

$Q$ **decomposition.** $1/3 + 2/6 = 2/3$. ✓

**OPE deficit.** $\Delta h = 1/9$; $1/3 - 1/9 = 2/9 = h_\square$. ✓

**Holonomy–conformal.** $\sigma_{2/3} = 2/9 = h_\square$ at $N = 3$; $N = 2, 4, 5$ fail.

**Alcove–conformal.** $e_2(1/3, 2/3) = 2/9$. Cubic $(N-3)(3N^2 + 2N + 2)$; discriminant $-20$. ✓

**Phase exclusion.** $\delta = 1/9$: $m_\mu/m_e \approx 39.7$ vs 206.8 ($> 100\sigma$); $1/9 \neq 2/9$.

**Neutrinos.** $\chi^2_{\min} \approx 3840$. Shape ratio 4.59 vs observed 32.6. Max $Q_\nu = 0.585$ (NH), 0.498 (IH).

**Quarks.** $Q_{\text{up}} = 0.8884 \pm 0.0013$, $|Q - 8/9| = 0.3\sigma$.

SL(2,3) **orbifold.** Order 24; all 24 elements $G_2$-preserving. 4 codim-4 strata with $\varphi(V) = +1, -1, -1, +1$. 4 codim-7 fixed points. $b_3 = 2$. $\mathbf{7} = \mathbf{2}' \oplus \mathbf{2}'' \oplus \mathbf{3}$; $\langle \chi_7, \chi_1 \rangle = 0$. All 8 order-3 in single class $C_3$. All 16 abelian $\mathbb{Z}_3$ pairs: $\dim(\cap) = 3$.

**Determinant universality.** $\text{SL}(2,3)/\mathbb{Z}^7$: 96 pairs, $|\det| = 4$, Smith $= \text{diag}(1,1,1,1,1,2,2)$. $\text{PSL}(2,7)/A_6$: 3024 pairs, $|\det| = 8$, Smith $= \text{diag}(1,1,1,1,1,1,8)$.

# B Catalog of Falsified Approaches

| # | Approach | Obstruction | Result |
|---|---|---|---|
| *Geometric/variational (37)* | | | |
| 1 | Calibration $f = \text{Re}(\Omega)/\text{vol}$ | Blindness | $f$ indep. of $\delta$ |
| 2 | $G_2$ associative calibration | Blindness | Cubic vertex $= 0$ |
| 3 | Heat kernel on SU(3)/SO(3) | High symmetry | Min at $\pi/6$ |
| 4 | Coleman–Weinberg 1-loop | Blindness | Leading at $\pi/6$ |
| *CFT/algebraic (4)* | | | |
| 5 | RCFT $S$-matrix identification | Transcendence | Produces $\pi/9$ |
| 6 | Zamolodchikov $c$-theorem | Numerics | Ratio 1.14 vs 5.34 |
| 7 | Mass formulas in $\text{SU}(3)_3$ | Transcendence | All 5 fail |
| 8 | Brieskorn sphere $\Sigma(2,3,7)$ | Topology | 2 flat connections |
| 9 | $m_k \sim e^{-\alpha S_{\text{CS}}}$ | Numerics | No fit |

| # | Approach | Obstruction | Result |
|---|---|---|---|
| 10 | Bohr–Sommerfeld | Tautology | $c(h-c/24)=2/9$ is input |
| *Potential minimization* | | | |
| 11 | Casimir energy, 4D $\mathbb{Z}_3$ twist | CP | Min at $\delta=0$ |
| 12 | Casimir energy, 2D | CP | Min at $\pi/3$ |
| 13 | GPY adjoint potential [16] | CP | Min at $\delta=0$ |
| 14 | GPY + bosonic fundamentals | CP | Min at $0,\pi/6,\pi/3$ |
| 15 | GPY + fermionic fundamentals | CP | Min at 0 for all $N_f$ |
| 16 | SM consistency ($\det Y_e$) | None | $\det Y_e$ unconstrained |
| 17 | Quark-sector Koide | Numerics | $Q_{\rm up}=0.888\neq 2/3$ |
| 18 | Ray–Singer torsion on $S^1$ | CP | High-symmetry extrema |
| 19 | Spectral $\zeta$ on SU(3)/SO(3) | CP | Same as heat kernel |
| *θ-angle mechanisms* | | | |
| 20 | Monopole-instanton, natural $\theta$ | No match | No natural $\theta$ gives 2/9 |
| 21 | Axion relaxation of $\theta$ | Min structure | Relaxes to $\pi/6$ |
| 22 | $\theta=$ CS of Brannen config | Independence | CS indep. of $\delta$ |
| 23 | Fractional $\theta=2\pi p/(3q)$ | Numerics | No small $(p,q)$ match |
| 24 | Self-consistent $\theta(\delta)$ | Non-convergence | Fixed pt at 1.27 |
| *Variational/information-theoretic* | | | |
| 25 | Riemannian volume | Numerics | No extremum at 2/9 |
| 26 | Shannon entropy | Numerics | No extremum at 2/9 |
| 27 | Rényi entropy | Numerics | No extremum at 2/9 |
| 28 | Fisher information | Numerics | No extremum at 2/9 |
| 29 | Purity functional | Numerics | No extremum at 2/9 |
| 30 | Spectral zeta $\zeta_s(\delta)$ | Numerics | No extremum at 2/9 |
| 31 | Generalized Koide $Q_s$ | Numerics | No extremum at 2/9 |
| 32 | RG fixed point analysis | Numerics | No fixed pt at 2/9 |
| 33 | Modular $T$-invariance | Selection | $\delta\in\{0,1/3,2/3\}$ |
| 34 | Character evaluation $\mathbb{Z}_3$ | | Brannen $\neq$ character |
| *Post-spectral theorem* | | | |
| 35 | Hessian–CFT bridge via adj in $10\otimes 10$ | Quantum trunc. | Fusion truncates |
| 36 | $\det(X)/\psi^3$ as selection | Transcendence | $\partial_\delta \det=0$ only at $0,\pi/3$ |
| 37 | $J_3(\mathbb{O})$ invariant ratios | Transcendence | Not algebraic at 2/9 |
| 38 | Shannon entropy of mass dist. | Numerics | No extremum |
| 39 | Spectral–Hessian self-consistency | Circularity | Ratio $\neq$ displacement |
| 40 | $\mathrm{Gr}(3,\mathbb{R}^7)$ via $\omega\wedge e^7$ | Orthogonality | $\omega(e_i,e_j)=0$ |
| 41 | Hessian flatness $\Leftrightarrow$ marginality | No link | Independent structures |
| 42 | Modular forms $\Gamma(3)$ | No prediction | 3 free couplings |
| 43 | Adjoint VEV direction in Cartan | CP | $V(\xi)\propto\cos(3\xi)$ |
| 44 | Complex Yukawa phase | Wrong form | $\lvert Y\rvert^2\neq$ Brannen |
| 45 | Conformal bootstrap | Circular | Uses $h_\square$ |
| 46 | 't Hooft anomaly matching | Topological | Independent of $\delta$ |
| 47 | M2-brane instanton, single modulus | Overconstrained | Forces $\zeta=0$ |
| 48 | Flux quantization $\theta_c=2\pi/144$ | Numerics | $\delta_{\rm min}=0.2221$ |
| 49 | Anomaly-generated $\theta_F$ in Sumino | Wrong scale | Cancels with quarks |
| *Topological/CS identification* | | | |
| 50 | CS phase $\Theta=2\pi h_\square$: $\delta=\Theta$ | $2\pi$ mismatch | $\Theta=4\pi/9\neq 2/9$ |
| 51 | Holonomy quantization $H_A=2\pi h_\square$ | Same | Holonomy phase |
| 52 | M-theory $C_3$ period as angle | Same | $2\pi$ persists |

| # | Approach | Obstruction | Result |
|---|---|---|---|
| 53 | Partition function $Z(\tau) \sim e^{2\pi i h}$ | Same | Phase $\neq$ number |
| 54 | $\delta = \Theta/(2\pi)$: rescale | Unphysical | $\sqrt{m_1} < 0$ |
| 55 | Hessian flatness $\Leftrightarrow$ CFT marginality | No link | Independent |
| 56 | $\delta = 4\pi/9$ in Brannen | Unphysical | $\theta_1 < 0$ |
| 57 | Vacuum alignment principle | Not a derivation | Restates conjecture |
| *Perturbative CFT/CS mechanisms* | | | |
| 58 | $\mathrm{Tr}(\Phi^4) = \text{const}$ in WZW | Quantum trunc. | Fusion truncates quartic |
| 59 | Braiding phase $\lvert R_1\rvert/(2\pi)$ | $2\pi$ mismatch | $= e^{2\pi i h}$, not $h$ |
| 60 | Fusion matrix eigenvalue | Algebraic | Cyclotomic |
| 61 | Verlinde formula ratio | Algebraic | $S_{\lambda\mu}/S_{00}$ algebraic |
| 62 | Modular $T$-matrix diagonal | $2\pi$ mismatch | $T = e^{2\pi i(h-c/24)}$ |
| 63 | KZ exponent on $\mathbb{Z}_3$ sphere/torus | Geometry | $\arg((1-\omega)^{-h}) \neq h$ |
| *Sessions 1–24 additions* | | | |
| 64 | Monopole-instanton $\theta_F$ fix (5 mech.) | Lateral | $\theta_c$ undetermined |
| 65 | Orbifold codim-7 from Joyce Ex. 7 | Codim arithmetic | $G_2$-involutions all codim-4 |
| 66 | Phase id $\delta = \pi\sigma_{1/3} = \pi/9$ | Positivity | $\theta_1 < 0$ |
| 67 | Phase id $\delta = \pi\sigma_{2/3} = 2\pi/9$ | Positivity | $\theta_1 < 0$ |
| 68 | Number id $\delta = \sigma_{1/3} = 1/9$ | Empirical + algebraic | $m_\mu/m_e \approx 39.7$ |
| 69 | CW on $z$-plane ($c_B/c_F < 1$) | Numerics | Min at $\delta \to 0$ |
| 70 | CW on $z$-plane (Sumino DOF) | Numerics | $\delta_{\min} \approx 0.252$ |
| 71 | Abelian codim-7 ($\mathbb{Z}_3 \times \mathbb{Z}_3 \subset G_2$) | Geometry | $\dim(\cap) = 3$ |
| 72 | Single-system $N_{\text{gen}} = 3$ | Arithmetic | $\lvert\det\rvert = 2^k$ universal |
| *Session 26 additions* | | | |
| 73 | 9-instanton potential $\cos(9\Theta)$ | Wrong extremum | Min at $\pi/9 \neq 2/9$; max at $\delta = 2/9$ |
| 74 | Wilson loop framing $f = c = 4$ | Parameter displacement | $f = 4$ has no geometric justification |
| *$G_2$ compactification and potential analysis* | | | |
| 75 | Democratic $N_{\text{gen}}$ on Eisenstein $\text{CY}_3$ | Arithmetic | $\alpha\beta\gamma = 1$, $\alpha+\beta+\gamma = 0$: no integer |
| 76 | $\delta$ from $\mathbb{Z}_3$ fixed-point geometry | Niven | All angles $\propto \pi/6$; $2/9 \neq q\pi$ |
| 77 | Two-harmonic monopole at $\theta = \pi$ | Not global min | $V''(2/9) > 0$ local min, but global m |
| 78 | CW potential alone | Wrong sign | $a_6^{(\ln)} = -0.086 < 0$; $\lvert a_3/a_6\rvert \approx 240$ |
| 79 | Combined monopole + CW | Parameter fitting | 3 params for 1 datum = pure fit |
| *Quantum CS (1)* | | | |
| 80 | CS holonomy quantization on $T^2$ | Uniform distribution | $\lvert S_{\Box,\alpha}\rvert^2 = 1/9$ for all non-adj $\alpha$; no s |

Total: 80 falsified approaches.

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