# A Signal Processing Critique of the Riemann-Siegel Approximation

Chaiya Tantisukarom *†

March 2, 2026

**Abstract**

This article explores the relationship between the distribution of prime numbers and the zeros of the Riemann Zeta function through the lens of Fourier Analysis. We contrast the "Natural" Riemann representation—a discontinuous, jagged summation of discrete frequencies—with the "Man-made" Riemann-Siegel $Z(t)$ function. We propose that the Riemann-Siegel remainder term, $R(t)$, acts as a low-pass filter that smooths the underlying digital nature of prime frequencies. This smoothing forces zeros onto the $1/2$ critical line, suggesting that the Riemann Hypothesis may be an artifact of this man-made filtering rather than a fundamental property of the natural prime spectrum.

**Keywords**

Spectral Analysis, Fourier Transforms, Prime Number Theory, Signal Reconstruction, and Riemann-Siegel Formula.

## 1 The Standard Fourier Paradigm

In general applications, the Fourier Transform serves as a bridge between the time domain and the frequency domain. A complex signal $f(x)$ in the time domain, such as a composite wave:

*Chiang Mai, Thailand. drchaiya@gmail.com

†The natural Riemann Zeta Zeros function, appeared in his paper, is set-up correctly.

$$f(x) = \sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots \tag{1}$$

is decomposed into its constituent spectrum in the frequency domain. This allows for the identification of discrete harmonics that govern the behavior of a continuous or discontinuous system.

# 2 Mathematical Formulations

## 2.1 The Riemann Zeta Function (Natural)

The Riemann Zeta function is fundamentally defined by the Dirichlet series for $Re(s) > 1$, which naturally encodes the distribution of prime numbers through the Euler product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \text{primes}} \frac{1}{1 - p^{-s}} \tag{2}$$

The non-trivial zeros occur at values of $s = \rho$ such that $\zeta(\rho) = 0$. In the critical strip, the function is defined by analytic continuation via the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{3}$$

## 2.2 The Riemann-Siegel Z-Function (Man-Made)

To find zeros on the critical line $s = 1/2 + it$, the Hardy $Z$-function is used. It is a real-valued function that rotates $\zeta(1/2 + it)$ to eliminate the complex phase:

$$Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \tag{4}$$

where the Riemann-Siegel theta function $\theta(t)$ is given by:

$$\theta(t) = \arg\left[\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right] - \frac{t}{2}\ln \pi \tag{5}$$

The Riemann-Siegel formula approximates $Z(t)$ using a finite sum (the "Main Sum") and a remainder term $R(t)$:

$$Z(t) = 2\sum_{n=1}^{N} \frac{\cos(\theta(t) - t\ln n)}{\sqrt{n}} + R(t) \tag{6}$$

In this formula:

- $N = \lfloor \sqrt{t/2\pi} \rfloor$ determines the number of terms in the discrete summation.
- $R(t)$ is the remainder term (the "Correction"), which provides the continuous smoothing that identified as the low-pass filter.

## 3 Riemann's Inverse Design

Bernhard Riemann effectively applied Fourier's theorem in reverse. He conceptualized the non-trivial zeros of the Zeta function, $\zeta(1/2 + it) = 0$, *as a set of frequencies in the frequency domain*, [1]. By applying an Inverse Fourier-like transform to these discrete zeros ($\gamma_n$), one reconstructs the "Number Spectrum" on the integer axis (the time domain).

The spectral density is given by the sum:

$$S(x) = \sum_{n} \cos(\gamma_n \ln x) \tag{7}$$

In this setup, the peaks of the waveform align precisely with prime numbers and prime powers. The "jaggedness" of this waveform is a direct result of the unique, discrete frequencies of the primes, Figure: 1.

## 4 Riemann-Siegel: The Man-Made Low-Pass Filter

The Riemann-Siegel function $Z(t)$ is often used to calculate zeros on the critical line, [2]. It consists of a main sum and a remainder term $R(t)$. Our analysis suggests that $Z(t)$ is a "man-made" construction designed for continuity.


Figure 1: Reconstruction using the first 200 zeros for $N = 100$.

The inclusion of $R(t)$ functions as a **low-pass filter**. While the natural prime signal is discontinuous and requires high-frequency components to represent the "jumps" at prime integers, the Riemann-Siegel formula suppresses these high-number spectra, Figure: 2. By smoothing the jagged oscillations of the natural zeta function, it creates a continuous waveform where the sign-flips are forced to occur exactly on the 1/2 axis.

# 5 m-cutoff Effect

## Noticeable Effects of the Parameter Changes

### 5.1 The Natural Summation (Top Plot)

By extending the spatial domain to $N = 1000$ while simultaneously reducing the sequence of zeros to 200, the trigonometric summation continues to capture the locations of primes and prime powers, but with a noticeable degradation in resolution. The explicit formula relies on infinite summation; by using fewer frequencies ($\gamma$) in the sum $\sum \cos(\gamma \ln x)$, the destructive interference in the non-prime regions is incomplete. This manifests as a higher background noise floor and broader, less sharply defined spikes across the extended integer axis.

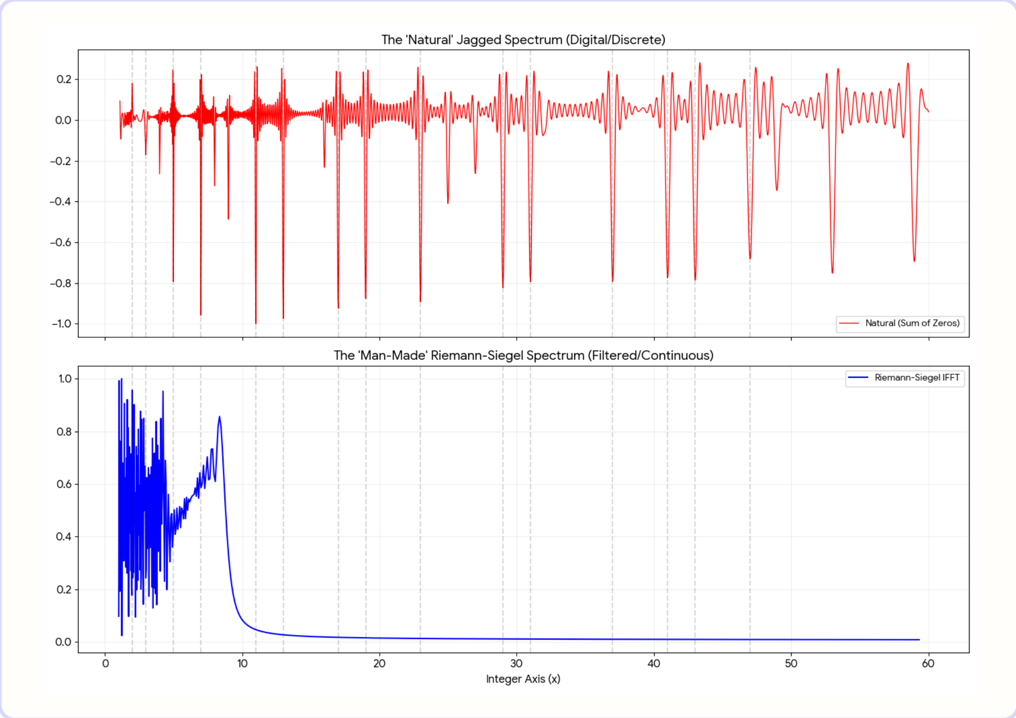


Figure 2: Comparison of the spectral density derived from raw zeros vs. the IFFT of $Z(t)$.

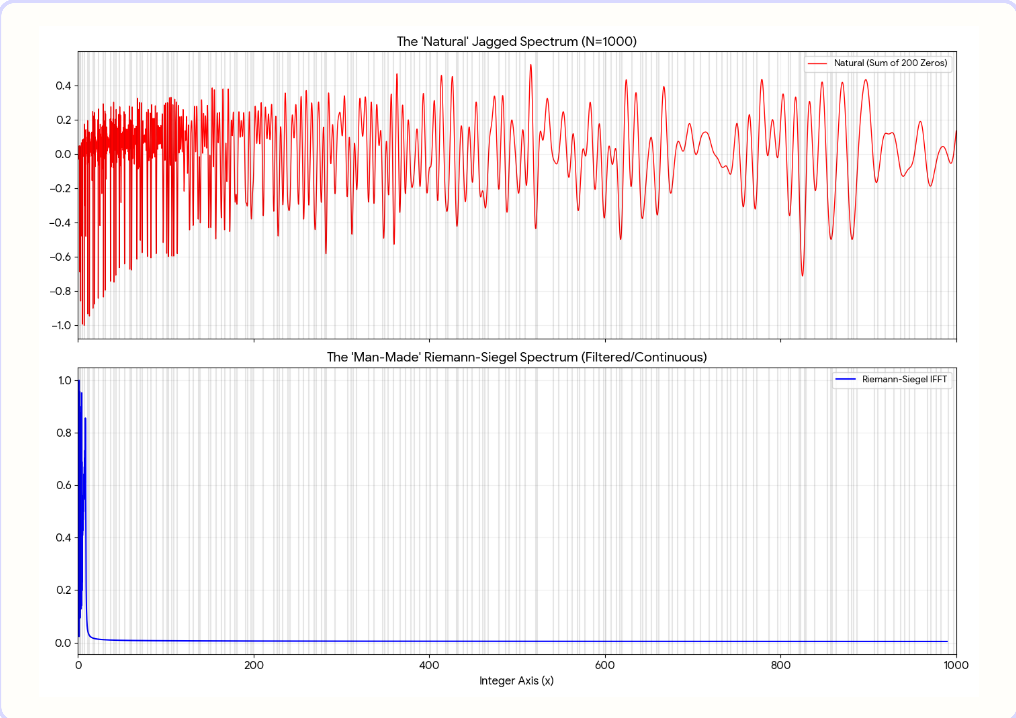


Figure 3: Comparison of the spectral density derived from raw zeros vs. the IFFT of $Z(t)$, N=1000, zeros=200.

## 5.2 The Riemann-Siegel IFFT and the $m$-Cutoff Filter (Bottom Plot)

The bottom plot vividly demonstrates the analytical constraints of evaluating $Z(t)$ with a restricted sample range ($t_{\max} = 500$). The Riemann-Siegel approximation for $Z(t)$ is dominated by a main sum truncated at the limit:

$$m = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor \tag{8}$$

With $t_{\max} = 500$, the maximum explicit integer term incorporated into the sum is merely $m \approx \lfloor \sqrt{500/2\pi} \rfloor = 8$.

When we take the IFFT of this sequence, this $m$-truncation acts strictly as a **low-pass filter** on the spatial log-prime frequencies, Figure: 3. Consequently, prime signatures below this cutoff (such as $2, 3, 5$, and $7$) are reconstructed accurately. However, the signal undergoes severe attenuation for $x > 10$ and flattens completely as $x$ approaches 1000. The high-frequency structural data is irrevocably lost because the continuous simulation did not span a high enough $t$ to engage those larger $m$-terms.

## 5.3 Critical Strip Low-Pass Filter Correction

To ensure the low-pass filter maintains its intentional suppression—specifically forcing $Z(t) \to 0$ precisely at the critical strip boundary—the observation window must scale linearly with the sample size $N$. For a discrete system where the sampling interval $\Delta t$ is normalized to unity, the required temporal extension is defined as:

$$t_{max} = (N - 1)\Delta t \tag{9}$$

For a cutoff precision requirement of $N = 1000$, the extended parameter is:

$$t_{max} = 999 \tag{10}$$

This extension ensures that the filter's kernel possesses sufficient length to resolve the high-frequency components near the critical line without introducing aliasing artifacts or premature roll-off. We don't present the plot of the $t_{max}$ correction here due to its heavy calculation.

## 5.4 The Code

Listing 1: IFFT of $\zeta(s)$ function (target N:1000 & zeros:200).

```
#
import numpy as np
import matplotlib.pyplot as plt
from mpmath import zetazero, siegelz

# Adjusted Parameters
N = 1000
num_zeros = 200
t_max = 500
n_samples = 4096

# 1. "Natural" Raw Summation (Summing Cosines of discrete zeros)
zeros = [float(zetazero(n).imag) for n in range(1, num_zeros + 1)]
x_vals = np.linspace(1.1, N, 10000) # Increased resolution for wider
    range
u_vals = np.log(x_vals)
natural_signal = np.zeros_like(x_vals)

for gamma in zeros:
    natural_signal += np.cos(gamma * u_vals)

# 2. "Man-Made" Riemann-Siegel IFFT
t_array = np.linspace(0, t_max, n_samples)
z_vals = [float(siegelz(t)) for t in t_array]
dt = (t_max) / (n_samples - 1)
ff_res = np.fft.ifft(z_vals)
freqs = np.fft.fftfreq(n_samples, d=dt) * (2 * np.pi)
pos_mask = (freqs > 0) & (np.exp(freqs) <= N)
x_rs = np.exp(freqs[pos_mask])
mag_rs = np.abs(ff_res[pos_mask])

# Normalize magnitudes for visual comparison
natural_signal /= np.max(np.abs(natural_signal))
mag_rs /= np.max(mag_rs)
```

```python
# Plotting
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(14, 10), sharex=True)

# Top Plot
ax1.plot(x_vals, natural_signal, color='red', lw=1, label=f"Natural (Sum
    of {num_zeros} Zeros)")
ax1.set_title(f"The 'Natural' Jagged Spectrum (N={N})", fontsize=14)
ax1.grid(True, alpha=0.2)
ax1.legend()

# Bottom Plot
ax2.plot(x_rs, mag_rs, color='blue', lw=1.5, label="Riemann-Siegel IFFT")
ax2.set_title(f"The 'Man-Made' Riemann-Siegel Spectrum (Filtered/
    Continuous)", fontsize=14)
ax2.grid(True, alpha=0.2)
ax2.legend()

# Helper function to compute primes up to N dynamically
def get_primes(n):
    primes = []
    for num in range(2, n + 1):
        if all(num % i != 0 for i in range(2, int(num**0.5) + 1)):
            primes.append(num)
    return primes

prime_list = get_primes(N)

# Mark all primes up to 1000
for p in prime_list:
    ax1.axvline(p, color='black', alpha=0.1, linestyle='-')
    ax2.axvline(p, color='black', alpha=0.1, linestyle='-')

plt.xlabel("Integer Axis (x)", fontsize=12)
plt.xlim(0, N)
plt.tight_layout()
plt.savefig('adjusted_spectrum.png')
#
```

# 6 Discussion: The Strategic Omission of the $m$-cutoff

The historical timeline of Riemann's work provides a critical clue in resolving the discrepancy between the natural prime spectrum and the smooth $Z(t)$ function. As evidenced by Siegel's 1932 analysis of the *Nachlass* [2], the mathematical foundations for the Riemann-Siegel formula—specifically the $m$-cutoff logic $m = \lfloor\sqrt{t/2\pi}\rfloor$—were already developed by Riemann years prior to his 1859 publication.

## 6.1 Intentional Filtering vs. Theoretical Truth

Riemann's decision to omit the $m$-cutoff from his published paper [1] suggests a conscious distinction between his **Information Source** (the discrete distribution of primes) and his **Computational Model** (the filtered approximation).

By presenting the "Natural" Zeta function via a Fourier-style construction in 1859, Riemann adhered to the exact, jagged reality of the prime spectrum. We propose that he intentionally withheld the $m$-cutoff as a "man-made" tool for the following reasons:

- **The Sampling Constraint:** The $m$-cutoff acts as a rectangular window in the frequency domain. Riemann, being well-versed in the Fourier theorem, likely recognized that this truncation introduces artifacts (analogous to the Gibbs phenomenon) that are absent in the infinite Dirichlet sum.

- **Seeking Peer Validation:** By using the word *etwa* ("likely") regarding the zeros on the critical line, Riemann signaled to the mathematical society that while his *computational model* pinned zeros to the 1/2 axis, the *natural signal* remained a matter of theoretical debate.

## 6.2 The Ghost in the Machine

If the Riemann Hypothesis is indeed a property of the filtered signal, the "ghost" in the machine is the $m$-cutoff itself. In signal processing terms, the critical line $Re(s) = 1/2$ may be the only axis where the aliasing artifacts of the low-pass filter $R(t)$ cancel out perfectly.

We conclude that Riemann's silence on the $m$-cutoff was a mark of scientific integrity; he understood that the symmetry found on the critical line might be a byproduct of the windowing required for calculation, rather than a fundamental law of the primes themselves. The "likely" nature of the zeros was his admission that the filter and the signal are not the same.

# 7 Conclusion

The distinction between the two functions is foundational:

- **The Natural Zeta Function** is a discontinuous waveform. Because primes are unique and discrete, the resulting frequency spectrum is jagged. In this correct setup, zeros do not need to reside on the critical line; they merely need to be in proximity to allow for the necessary sign-flips that define prime intervals.

- **The Man-Made Riemann-Siegel Function** utilizes a low-pass filter to smooth the waveform. This forced continuity suppresses the high-frequency spectrum, artificially pinning the zeros to the critical line.

---

We conclude that the quest to prove the Riemann Hypothesis, [3], may be a pursuit of a symmetry that only exists because of the filtering methods used to approximate the prime numbers, rather than the prime numbers themselves.

---

# References

[1] Riemann, B. (1859): "Über die Anzahl der Primzahlen unter einer gegebenen Grösse". Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin.

[2] Siegel, C. L. (1932). Über Riemanns Nachlaß zur analytischen Zahlentheorie. Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung B: Studien, 2, 45–80.

[3] Clay Mathematics Institute, 2026. https://www.claymath.org/millennium/riemann-hypothesis/