# The Hessian of the $G_2$ 3-form on the Grassmannian, a geometric ratio matching the Koide angle, and two obstruction theorems

P. Music

## Abstract

We compute the Hessian of the $G_2$ 3-form $\varphi$ restricted to $\mathrm{Gr}(3, \mathbb{R}^6)$ at the flavour-symmetric point, obtaining eigenvalues $0^5, (-2\varphi_0)^3, (-3\varphi_0)^1$ on the SO(3)-decomposition $\mathrm{Sym}_0^2 \oplus \Lambda^2 \oplus \mathbb{R} \cdot I$. The geometric ratio $|\lambda_{\Lambda^2}|/|\Delta f/\varphi_0| = 2/9$ matches the empirical Brannen–Koide phase to 0.02%.

We prove two obstruction theorems constraining any dynamical derivation of $\delta = 2/9$. First, since $\cos(2/3)$ is transcendental (Lindemann–Weierstrass), no symmetric polynomial in the mass eigenvalues with algebraic coefficients can select $\delta = 2/9$ as an extremum. Second, any topological flux quantisation mechanism produces phases that are rational multiples of $\pi$, for which $\cos(3\delta)$ is algebraic; such mechanisms are therefore also excluded by the same transcendentality. These results establish that the Koide phase $\delta \approx 2/9$ — if exact — cannot arise from any polynomial potential or standard topological mechanism, severely constraining the space of viable explanations.

All analytic results are verified by independent numerical computation.

## Contents

# 1  Introduction

The charged lepton masses satisfy the Koide relation [1]

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = 0.666661 \approx \frac{2}{3} \tag{1}$$

to a precision of $6 \times 10^{-6}$. In the Brannen parametrisation [2], $\sqrt{m_k} = \mu(1 + \sqrt{2}\cos(\delta + 2\pi k/3))$ with $Q = 2/3$ holding identically for any $\delta$. The empirical phase is $\delta_{\exp} = 0.22227$ rad, remarkably close to $2/9 = 0.22222\ldots$ (deviation $\sim 0.02\%$). Here $\delta$ is defined on the principal branch $(0, \pi/3)$ by assigning $k = 0$ to $\tau$ (heaviest), $k = 1$ to $e$ (lightest), $k = 2$ to $\mu$; other assignments shift $\delta$ by $\pm 2\pi/3$.

Sumino [3] showed that a U(3) family gauge symmetry can protect $Q = 2/3$ from radiative corrections, but did not derive the value of $\delta$. Many approaches have been explored — Casimir ratios, exceptional algebraic structures, topological mechanisms — but none has produced a derivation of $\delta = 2/9$.

In this paper we provide two results that constrain any such derivation:

1. A complete computation of the Hessian of the $G_2$ 3-form on $\mathrm{Gr}(3, \mathbb{R}^6)$, yielding the geometric ratio $2/9$ (Section 3).

2. Two obstruction theorems showing that $\delta = 2/9$ cannot arise from (a) any symmetric polynomial in the mass eigenvalues with algebraic coefficients, nor (b) any topological flux quantisation mechanism (Section 5).

These obstructions do not prove that $\delta = 2/9$ is accidental; they prove that any mechanism producing it must be *neither polynomial nor topological* in the standard sense, which severely restricts the space of viable theories.

# 2  Setup

Let $V = \mathbb{R}^7$ carry the standard $G_2$-invariant 3-form $\varphi \in \Lambda^3(V^*)$, and let $\mathbb{R}^6 \subset \mathbb{R}^7$ be orthogonal to a fixed unit vector $e_7$. The restriction $\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega)$ where $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ is the holomorphic volume form on $\mathbb{C}^3$ via $z_k = e_k + ie_{k+3}$. Explicitly,

$$\varphi|_{\mathbb{R}^6} = e^{123} - e^{156} + e^{246} - e^{345}. \tag{2}$$

For an oriented 3-plane $W \in \mathrm{Gr}(3, \mathbb{R}^6)$ with frame $(v_1, v_2, v_3)$:

$$f(W) = \frac{\varphi(v_1, v_2, v_3)}{\mathrm{vol}(v_1, v_2, v_3)}, \qquad \mathrm{vol} = \sqrt{\det G}, \quad G_{ij} = \langle v_i, v_j \rangle. \tag{3}$$

The democratic point $V_0 = \mathrm{span}(e_1, e_2, e_3)$ has $\varphi_0 = f(V_0) = 1$. The tangent space $T_{V_0}\mathrm{Gr}(3, \mathbb{R}^6) \cong M_3(\mathbb{R})$ parametrises deformations $v_i(\varepsilon) = w_i + \varepsilon \sum_k A_{ik} u_k$ with $w_k = e_k$, $u_k = e_{k+3}$.

# 3  Hessian of $f$ at the democratic point

**Lemma 1** (Levi-Civita). *At $V_0$: (a) $\varphi(w_i, w_j, u_k) = 0$ for all $i, j, k$. (b) $\varphi(w_k, u_a, u_b) = -\varphi_0\,\varepsilon_{kab}$.*

*Proof.* Since $\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega)$ and $z_k = w_k + iu_k$, the expansion of $\mathrm{Re}(\Omega)$ contains only terms with an even number of $u$-type indices. Part (a) follows (one $u$-index). Part (b) follows by direct evaluation against (2). Verified computationally for all $3^3 = 27$ index combinations. $\qquad \square$

2

**Lemma 2** (Gram matrix). $G_{ij}(\varepsilon) = \delta_{ij} + \varepsilon^2(AA^T)_{ij}$, hence $\mathrm{vol}(\varepsilon) = 1 + \frac{\varepsilon^2}{2}\|A\|^2 + O(\varepsilon^4)$.

**Lemma 3** (Numerator expansion). $\varphi(v_1(\varepsilon), v_2(\varepsilon), v_3(\varepsilon)) = \varphi_0 + \varepsilon^2 N_2(A) + O(\varepsilon^3)$ with

$$N_2(A) = \frac{\varphi_0}{2}\big[\mathrm{Tr}(A^2) - (\mathrm{Tr}\,A)^2\big]. \tag{4}$$

*Proof.* The $O(\varepsilon)$ term vanishes by Lemma 1(a). The $O(\varepsilon^2)$ term uses Lemma 1(b) and the identity $\sum_k \varepsilon_{kij}\varepsilon_{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$. $\qquad\square$

**Theorem 4** (Hessian). *The Hessian of $f = \varphi/\mathrm{vol}$ on $\mathrm{Gr}(3, \mathbb{R}^6)$ at the democratic point is*

$$H_f(A) = -\varphi_0\big[(\mathrm{Tr}\,A)^2 + 2\|A_{\mathrm{anti}}\|^2\big], \tag{5}$$

*where $A_{\mathrm{anti}} = \frac{1}{2}(A - A^T)$.*

*Proof.* $f(\varepsilon) = (\varphi_0 + \varepsilon^2 N_2)(1 - \frac{\varepsilon^2}{2}\|A\|^2) + O(\varepsilon^4)$, so $H_f(A) = 2N_2 - \varphi_0\|A\|^2$. Decomposing $A = S + \Lambda$ (symmetric + antisymmetric): $\mathrm{Tr}(A^2) = \|S\|^2 - \|\Lambda\|^2$ and $\mathrm{Tr}(AA^T) = \|S\|^2 + \|\Lambda\|^2$, giving (5). $\qquad\square$

**Remark.** *Both contributions to the Hessian are essential. The numerator alone gives $N_2|_{\Lambda^2} = -\frac{\varphi_0}{2}\|A\|^2$ and $N_2|_{\mathrm{Sym}_0^2} = +\frac{\varphi_0}{2}\|A\|^2$, while the normalisation contributes $-\frac{\varphi_0}{2}\|A\|^2$ uniformly. On $\mathrm{Sym}_0^2$ they cancel; on $\Lambda^2$ they add; on $\mathbb{R}\cdot I$ the ratio is 2:1.*

**Corollary 5** (Eigenvalue decomposition). *Under $\mathrm{SO}(3)$, $M_3(\mathbb{R}) = \mathrm{Sym}_0^2(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \oplus \mathbb{R}\cdot I$ with eigenvalues:*

| Subspace | dim | Eigenvalue | Interpretation |
|---|---|---|---|
| $\mathrm{Sym}_0^2(\mathbb{R}^3)$ | 5 | 0 | flat directions |
| $\Lambda^2(\mathbb{R}^3)$ | 3 | $-2\varphi_0$ | mass splitting |
| $\mathbb{R}\cdot I$ | 1 | $-3\varphi_0$ | trace/scale |

**Remark.** *The full $12 \times 12$ Hessian on $\mathrm{Gr}(3, \mathbb{R}^7)$ is observed numerically to have spectrum $(-3)^4, 0^8$. Restricting to $\mathrm{Gr}(3, \mathbb{R}^6) \subset \mathrm{Gr}(3, \mathbb{R}^7)$ yields the universal spectrum $0^5, (-2)^3, (-3)^1$, independent of which codimension-1 restriction is chosen (verified for all four choices).*

## 4 The geometric ratio and structural identities

### 4.1 The ratio $2/9$

**Theorem 6** (Geometric ratio). *For $N = 3$, the ratio*

$$\delta_{\mathrm{geom}} = \frac{|\lambda_{\Lambda^2}|}{|\Delta f/\varphi_0|} = \frac{2}{5 \times 0 + 3 \times 2 + 1 \times 3} = \frac{2}{9} \tag{6}$$

*matches the empirical Brannen–Koide phase to $0.02\%$.*

### 4.2 Structural identity

**Proposition 7.** *For all $N \geq 2$, $C_2(\mathrm{Sym}^N(N)) = N(N-1) = 2\dim\Lambda^2(\mathbb{R}^N)$.*

*Proof.* $C_2(\mathrm{Sym}^k(N)) = k(N+k)(N-1)/(2N)$. Setting $k = N$: $N \cdot 2N \cdot (N-1)/(2N) = N(N-1)$. Since $\dim\Lambda^2(\mathbb{R}^N) = \binom{N}{2} = N(N-1)/2$, the identity follows. Verified numerically for $N = 2, \ldots, 7$. $\qquad\square$

**Remark.** *For $N = 3$, the Hessian trace on $\Lambda^2$ satisfies $|\mathrm{Tr}(H_f|_{\Lambda^2})|/\varphi_0 = 3 \times 2 = 6 = C_2(\mathrm{Sym}^3 3)$. Both the Hessian eigenvalue 2 and the centre charge $c(\Lambda^2) = 2$ originate from the second exterior power $\Lambda^2$: the former via the antisymmetriser $\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$, the latter via $c(\Lambda^k) = k \bmod N$. This shared algebraic origin is suggestive but does not constitute a proven structural link between the variational and representation-theoretic constructions.*

## 4.3 Dimensional uniqueness of $N = 3$

Two independent conditions single out $N = 3$:

1. $\dim \Lambda^2(\mathbb{R}^N) = N$ iff $N = 3$ (since $\binom{N}{2} = N$ gives $N = 3$).

2. The Casimir ratio $C_2(\overline{N})/C_2(\mathrm{Sym}^N(N)) = (N+1)/(2N^2) = 2/9$ iff $N = 3$.

# 5 Obstruction theorems

## 5.1 Polynomial obstruction

**Theorem 8** (Polynomial obstruction). *Let $F : \mathbb{R} \to \mathbb{R}$ be any polynomial with algebraic coefficients. On the Koide surface (parametrised by $\delta$ with $Q = 2/3$ fixed), every symmetric polynomial in the three mass eigenvalues (equivalently, every polynomial invariant under the Weyl group $S_3$) depends on $\delta$ only through $\cos(3\delta)$. Therefore $F$ can select $\delta = \delta_0$ as an extremum only if $\cos(3\delta_0)$ is algebraic. Since $\cos(2/3)$ is transcendental by the Lindemann–Weierstrass theorem, no such $F$ can select $\delta = 2/9$.*

*Proof.* On the Koide surface, define $x_k = \sqrt{m_k}/\mu = 1 + \sqrt{2}\cos(\delta + 2\pi k/3)$. The elementary symmetric polynomials of $\{x_k\}$ satisfy:

$$\sigma_1 = \sum x_k = 3, \quad \sigma_2 = \sum_{i<j} x_i x_j = \tfrac{3}{2}, \quad \sigma_3 = \prod x_k = -\tfrac{1}{2} + \tfrac{\sqrt{2}}{2}\cos(3\delta), \tag{7}$$

where $\sigma_1$ and $\sigma_2$ are constant (using $\sum \cos(\delta + 2\pi k/3) = 0$ and $\sum \cos^2 = 3/2$), and only $\sigma_3$ varies with $\delta$.

Since $m_k = \mu^2 x_k^2$, any symmetric polynomial in $\{m_k\}$ is a symmetric polynomial in $\{x_k^2\}$. By Newton's identities, this is a polynomial in the power sums $p_{2n} = \sum x_k^{2n}$, each of which is a polynomial in $\sigma_1, \sigma_2, \sigma_3$. With $\sigma_1, \sigma_2$ fixed, all dependence on $\delta$ enters only through $\sigma_3$, hence through $\cos(3\delta)$. Thus any $S_3$-symmetric polynomial $F$ in the mass eigenvalues restricts to a polynomial $P(\cos(3\delta))$ on the Koide surface.

At an interior extremum ($\delta \neq 0, \pi/3$), $P'(\cos(3\delta_0)) = 0$, which forces $\cos(3\delta_0)$ to be algebraic.

By Lindemann–Weierstrass, if $\alpha \neq 0$ is algebraic then $e^{i\alpha}$ is transcendental. Since $2/3$ is algebraic and nonzero, $e^{2i/3}$ is transcendental, so $\cos(2/3) = \mathrm{Re}(e^{2i/3})$ is transcendental (if it were algebraic, $e^{2i/3} = \cos(2/3) + i\sin(2/3)$ with $\sin(2/3) = \sqrt{1 - \cos^2(2/3)}$ algebraic would give a contradiction). Thus $\cos(3 \times 2/9) = \cos(2/3)$ is transcendental and cannot be a root of any polynomial with algebraic coefficients. $\square$

**Remark.** *This eliminates all polynomial approaches with algebraic coefficients to deriving $\delta = 2/9$, including: the $J_3(\mathbb{O})$ cubic norm, polynomial Casimir combinations, any Sumino-type polynomial potential $V(\Phi)$ with algebraic couplings, and the $G_2$ volume functional restricted to the Koide surface (verified numerically: $f(\Phi(\delta))$ is monotonic in $\cos(3\delta)$ with extrema only at $\delta = 0, \pi/3$).*

## 5.2 Topological obstruction

The polynomial obstruction has a powerful corollary for topological mechanisms.

**Theorem 9** (Topological obstruction). *Any mechanism that produces the Brannen phase $\delta$ (in radians) as a rational multiple of $\pi$ is excluded as a derivation of $\delta = 2/9$.*

*Proof.* In any topological flux mechanism (e.g. 't Hooft flux quantisation [4], discrete Wilson lines on resolved ADE singularities [5]), holonomy eigenvalues are of the form $e^{2\pi i\, p/q}$ for integers $p, q$. If such a mechanism determines the Brannen phase, then $\delta = r\pi$ for some $r \in \mathbb{Q}$. But then $3\delta = 3r\pi$ is a rational multiple of $\pi$, and $\cos(3r\pi)$ is algebraic — since $e^{2\pi i\cdot 3r/2}$ is a root of the

4

cyclotomic polynomial $\Phi_q(x)$ for some $q$, and real and imaginary parts of algebraic numbers are algebraic.

By contrast, $\cos(3 \times 2/9) = \cos(2/3)$ is transcendental (Theorem 8). Since $2/9 \neq r\pi$ for any $r \in \mathbb{Q}$ (equivalently, $2/(9\pi) \notin \mathbb{Q}$, which follows from the transcendence of $\pi$), no rational multiple of $\pi$ equals $2/9$. $\qquad\square$

**Remark.** *This theorem applies to all standard topological mechanisms in gauge theory: 't Hooft flux quantisation ($m \in \mathbb{Z}_N$), discrete Wilson lines, orbifold phases, and Chern–Simons levels. All produce phases that are rational multiples of $\pi$, which are precisely the values not excluded by the polynomial obstruction. There is thus a complementarity: polynomial functionals can only select algebraic $\cos(3\delta)$ (rational multiples of $\pi$ among them), while the empirical $\delta = 2/9$ gives transcendental $\cos(2/3)$, which lies outside both classes.*

### 5.3 Perturbative exclusion

We verify independently that perturbative dynamics also fail to select $\delta = 2/9$.

**Proposition 10.** *The one-loop effective potential for $U(3)$ family gauge bosons compactified on $S^1$, with holonomy eigenvalues $\theta_k = \delta + 2\pi k/3$ (matching the Brannen convention, $\delta$ in radians), is monotonic on the Brannen domain $\delta \in (0, \pi/3)$ and has no interior critical points. In particular, $\delta = 2/9$ is not a critical point.*

*Proof.* The adjoint contribution depends only on differences $\theta_i - \theta_j = 2\pi(i-j)/3$, independent of $\delta$. For fundamental matter, $\sum_{k=0}^{2} e^{in\theta_k} = e^{in\delta} \sum_k e^{2\pi i n k/3}$, which vanishes unless $3|n$. The surviving terms give $V_{\text{fund}}(\delta) \propto \sum_{m \geq 1} \cos(3m\delta)/m^4$, which equals $B_4(3\delta/(2\pi))$ (fourth Bernoulli polynomial, periodised) up to normalisation. Since $B_4'(x) = 2x(2x-1)(x-1)$ has roots at $x = 0, \frac{1}{2}, 1$, the critical points in $\delta$ are $\delta = 0, \pi/3, 2\pi/3$ — all on the boundaries of the Brannen domain $(0, \pi/3)$. No interior critical point exists. Verified numerically: $V_{\text{fund}}$ is strictly monotonic on $(0, \pi/3)$. $\qquad\square$

This provides an independent exclusion: the one-loop potential, with $\delta$ in the Brannen convention (radians), is monotonic on the physical domain $(0, \pi/3)$ and has no interior critical points.

## 6 Confrontation with experiment

We use PDG 2024 [6] central values: $m_e = 0.51099895$ MeV, $m_\mu = 105.6583755$ MeV, $m_\tau = 1776.86 \pm 0.12$ MeV.

Setting $\delta = 2/9$ and $Q = 2/3$, with $\mu$ fixed from $m_\mu$ alone (one mass as input), the remaining two masses are predicted with zero free parameters:

| Observable | Prediction ($\delta = 2/9$) | Data | Deviation |
|---|---|---|---|
| Brannen phase $\delta$ | $2/9 = 0.22222\ldots$ | $0.22227$ | $0.02\%$ |
| $m_\tau$ | $1776.97$ MeV | $1776.86 \pm 0.12$ MeV | $0.9\sigma$ |
| $m_e$ | $0.510994$ MeV | $0.510999$ MeV | $0.001\%$ |

A measurement of $m_\tau$ to $\pm 0.05$ MeV would provide a $2\sigma$ test of $\delta = 2/9$.

## 7 Discussion

### 7.1 What is proven

1. The Hessian formula (5) and eigenvalue decomposition (Theorem 4, Corollary 5).

2. The geometric ratio $\delta_{\text{geom}} = 2/9$ (Theorem 6).

3. The structural identity $C_2(\text{Sym}^N(N)) = 2 \dim \Lambda^2(\mathbb{R}^N)$ for all $N$ (Proposition 7).

4. The polynomial obstruction: no symmetric polynomial in the mass eigenvalues with algebraic coefficients selects $\delta = 2/9$ (Theorem 8).

5. The topological obstruction: no flux quantisation mechanism selects $\delta = 2/9$ (Theorem 9).

6. The one-loop potential is monotonic on the Brannen domain $(0, \pi/3)$ (Proposition 10).

## 7.2 What is conjectured

**Conjecture 11** (Koide phase). *The Brannen–Koide phase equals $\delta = 2/9$ exactly.*

No dynamical mechanism is proposed. The obstruction theorems show that if this conjecture is true, its derivation must involve *neither* a polynomial potential *nor* a standard topological quantisation.

## 7.3 What kind of mechanism could work?

The two obstructions together severely constrain viable approaches. A mechanism producing $\delta = 2/9$ exactly must satisfy:

- It cannot be the extremum of a polynomial functional with algebraic coefficients (Theorem 8).

- It cannot arise from a topological phase (rational multiple of $\pi$) (Theorem 9).

- It could, in principle, involve: transcendental functionals (exponential, logarithmic); non-perturbative effects whose phases are not rational multiples of $\pi$ (e.g. instanton actions involving $\ln 2$ or similar); or an exact cancellation producing $2/9$ from a combination of $\pi$-dependent and $\pi$-independent contributions.

The fact that $2/9$ is rational while $\cos(2/3)$ is transcendental — and that $2/(9\pi)$ is transcendental so $2/9$ is not a rational multiple of $\pi$ — places $\delta = 2/9$ in a very specific number-theoretic class. Any derivation must explain why the phase is a simple rational number that is *not* a rational multiple of $\pi$.

## 7.4 On the identification $\Lambda^2(\mathbb{R}^3) \cong \bar{3}$

The Hessian decomposition is under SO(3), with $\Lambda^2(\mathbb{R}^3)$ as the spin-1 representation. The standard isomorphism $\Lambda^2(\mathbb{C}^3) \cong \bar{3}$ holds for SU(3), but $\Lambda^2(\mathbb{R}^3) \subset \Lambda^2(\mathbb{C}^3)$ is not SU(3)-invariant. Relating the SO(3) Hessian eigenvalues to SU(3) Casimir values requires additional structure (e.g. the $G_2$-invariant almost-complex structure on $\mathbb{R}^6$) that we do not provide.

## 7.5 Falsifiable predictions

1. If $\delta = 2/9$ exactly: $m_\tau = 1776.97 \pm 0.11$ MeV. Present data: $1776.86 \pm 0.12$ MeV ($0.9\sigma$ tension). A future measurement to $\pm 0.05$ MeV provides a $2\sigma$ test.

2. The construction is mathematically consistent only for $N = 3$ generations.

## 7.6 Limitations

- $Q = 2/3$ is not predicted (input from Sumino).

- The overall scale $\mu$ is not predicted.

- The ratio $\delta_{\text{geom}} = 2/9$ is a purely geometric observation; no dynamical mechanism connecting the $G_2$ Hessian spectrum to the lepton mass matrix is known.

- The neutrino sector is not addressed.

- The quark sector has $Q \neq 2/3$ ($Q_{\text{up}} \approx 0.85$, $Q_{\text{down}} \approx 0.73$), consistent with the Sumino mechanism being lepton-specific (QCD corrections not compensated).

## Acknowledgements

[To be added.]

## A  Numerical verification

All analytic results have been verified by independent numerical computation in Python.

**Hessian spectrum.** Full $9 \times 9$ Hessian matrix on $\text{Gr}(3, \mathbb{R}^6)$ computed by centred finite differences ($\varepsilon = 10^{-4}$). Eigenvalues: $0^5$ (max $|.| < 10^{-7}$), $(-2.000000)^3$, $(-3.000000)^1$. Full $12 \times 12$ Hessian on $\text{Gr}(3, \mathbb{R}^7)$: spectrum $(-3)^4, 0^8$.

**Symmetric polynomials.** On the Koide surface ($Q = 2/3$, 200 values of $\delta \in (0, \pi/3)$): $\sigma_1 = 3.000000$ (constant to $10^{-14}$), $\sigma_2 = 1.500000$ (constant to $10^{-12}$), $\sigma_3$ linear in $\cos(3\delta)$ with residual $< 10^{-12}$.

**Brannen surface scan.** $f(\Phi(\delta))$ computed for 200 values of $\delta \in [0, \pi/3]$. Monotonic in $\cos(3\delta)$, extrema only at boundaries $\delta = 0, \pi/3$. $\delta = 2/9$ is not an extremum.

**One-loop potential.** $V_{\text{eff}}(\delta) = \sum_{m \geq 1} \cos(3m\delta)/m^4$ computed by direct summation ($N_{\text{terms}} = 500$) for $\delta \in (0, \pi/3)$ in radians. $V_{\text{eff}}$ is strictly monotonic on this interval; no interior critical points. Critical points of $B_4(3\delta/(2\pi))$ at $\delta = 0, \pi/3, 2\pi/3$, all on boundaries of the Brannen domain.

**Casimir values.** $C_2(\text{Sym}^3 3) = 6$ and $C_2(\bar{3}) = 4/3$ verified by explicit construction of SU(3) generators. Structural identity $C_2(\text{Sym}^N(N)) = N(N-1) = 2\binom{N}{2}$ verified for $N = 2, \ldots, 7$.

**Code availability.** Complete Python scripts reproducing all verifications are provided as supplementary material.

## References

[1] Y. Koide, Lett. Nuovo Cim. **34**, 201 (1982).

[2] C. A. Brannen, "The lepton masses," hep-ph/0503112 (2005).

[3] Y. Sumino, "Family gauge symmetry as an origin of Koide's mass formula," Phys. Lett. B **671**, 477 (2009) [arXiv:0812.2103].

[4] G. 't Hooft, "A property of electric and magnetic flux in non-abelian gauge theories," Nucl. Phys. B **153**, 141 (1979).

[5] B. S. Acharya, "M theory, Joyce orbifolds and super Yang-Mills," Adv. Theor. Math. Phys. **3**, 227 (1999) [hep-th/9812205].

[6] S. Navas et al. (Particle Data Group), Phys. Rev. D **110**, 030001 (2024).