Recursive Parity Extraction and the Structural Exclusion of Non-Trivial Cycles in the $3n + 1$ Problem

Siqi Liu
siqi.liu2025@gmail.com

February 24, 2026

Abstract

This paper provides a structural proof for the non-existence of non-trivial cycles in the $3n+1$ (Collatz) conjecture within the positive integers. By utilizing a nested expansion of the accelerated map, we derive a master identity that isolates the 2-adic valuation of each step. We demonstrate that the parity constraints of a closed loop require a recursive descent of exponents, ultimately forcing the system into a unique integer solution at $n = 1$.

1 Introduction

The Collatz conjecture defines a sequence $(n_i)_{i\geq 0}$ for $n_0 \in \mathbb{Z}^+$ such that $n_{i+1} = n_i/2$ if n_i is even, and $n_{i+1} = 3n_i + 1$ if n_i is odd. A cycle of length k occurs if $f^k(n) = n$. While the trivial cycle $1 \to 4 \to 2 \to 1$ is well-documented, the existence of non-trivial cycles remains the central mystery of the problem. This paper treats the cycle as a static algebraic system governed by 2-adic valuations.

2 The Accelerated Map and Master Identity

We define the accelerated Collatz map $f : \mathbb{Z}^+_{\text{odd}} \to \mathbb{Z}^+_{\text{odd}}$ as:

$$f(n) = \frac{3n + 1}{2^a}$$

where $a = \nu_2(3n + 1)$ is the 2-adic valuation. For a cycle of length k, we iterate f to obtain the nested identity:

$$n = \frac{3\left(\frac{3\left(\cdots\frac{3(3n+1)}{2^{a_1}}\cdots+1\right)}{2^{a_{k-1}}} + 1\right)}{2^{a_k}}$$

Rearranging this equation to isolate the powers of 2 against the powers of 3, we derive the structural loop identity:

$$(3n + 1)3^{k-1} = 2^{a_1}\left[2^{a_2}\left(\dots(2^{a_k} n - 1) - 3\dots\right) - 3^{k-2}\right] \tag{1}$$

3 Inductive Parity Extraction

By the Fundamental Theorem of Arithmetic, the even and odd parts of both sides of Eq. (1) must be equal. Since $\gcd(3, 2) = 1$, we have:

$$\nu_2(3n+1) = a_1 \quad \text{and} \quad 3^{k-1} = \text{OddPart}\,(\text{RHS})$$

Extracting the first shell Φ_1, we have:

$$3^{k-1} = 2^{a_2}\Phi_2 - 3^{k-2}$$

Rearranging yields:

$$3^{k-1} + 3^{k-2} = 3^{k-2}(3+1) = 4 \cdot 3^{k-2} = 2^{a_2}\Phi_2$$

Since Φ_2 is odd, the 2-adic valuation forces $2^{a_2} = 4$, hence $a_2 = 2$.

3.1 Recursive Descent

This structure is self-similar across all shells $j \in \{2, \ldots, k-1\}$. The general recurrence relation is:

$$2^{a_j}\Phi_j = 3^{k-j+1} + 3^{k-j} = 4 \cdot 3^{k-j}$$

which necessitates $a_j = 2$ for all intermediate steps. This recursive "peeling" continues until the innermost core is reached.

4 The Innermost Constraint

After exhausting all similar structures, the system collapses to the innermost identity:

$$2^{a_k}n - 1 = 3 \implies 2^{a_k}n = 4 \tag{2}$$

Simultaneously, the first shell condition $3n+1 = 2^{a_1}$ implies $n \in \{\frac{2^{a_1}-1}{3}\}$. The intersection of these two sets, $n = 4 \cdot 2^{-a_k}$ and $n = (2^{a_1} - 1)/3$, is tested for $n \in \mathbb{Z}^+$:

- If $a_1 = 2$, then $n = 1$. Substituting into Eq. (2) gives $2^{a_k}(1) = 4 \implies a_k = 2$. This is the trivial loop.
- If $a_1 = 4$, then $n = 5$. Substituting into Eq. (2) gives $2^{a_k}(5) = 4 \implies 2^{a_k} = 0.8$, which has no integer solution.
- For all $a_1 > 2$, n grows exponentially while the required 2^{a_k} becomes a smaller fraction, resulting in an algebraic void.

5 Conclusion

The structural exclusion method proves that a closed loop in the Collatz map requires a perfect balance of 2-adic valuations that only exists at $n = 1$. The recursive descent forces all intermediate exponents to 2, creating a pincer divergence for any $n > 1$.

Theorem 1. *The only cycle in the Collatz sequence for $n \in \mathbb{Z}^+$ is the trivial cycle $\{1, 4, 2\}$.*