Octonionic Geometry and the Koide Angle: A Derivation from G_2 Casimir Invariants with Neutrino Mass Predictions

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Abstract

The Koide formula relates the masses of the three charged leptons through the parameter Q = 2/3 and an angle $\theta \approx 0.2222$. We derive $\theta = 2/9$ as the ratio of quadratic Casimir invariants $C_2(\mathbf{3})/C_2(\mathrm{Sym}^3\mathbf{3}) = (4/3)/6$ within the natural embedding $SU(3) \subset G_2 = \mathrm{Aut}(O)$, where the G_2 associative 3-form evaluated on the fermion 3-plane determines cos 3θ. The agreement with PDG data is 0.009% (< 1σ). Extending the construction to neutrinos via the adjoint representation, we conjecture $\theta_\nu = C_2(\mathbf{8})/C_2(\mathrm{Sym}^3\mathbf{3}) = 1/2$, predicting $\Sigma m_i = 70.9 \pm 0.4$ meV in normal hierarchy, testable by Euclid, CMB-S4, LEGEND, and nEXO within the coming years.

Keywords: *Koide formula, octonions, G_2 holonomy, Casimir invariants, neutrino masses, exceptional Jordan algebra*

1. Introduction

The Koide formula [1] is one of the most striking unexplained relations in particle physics. For the charged lepton masses m_e, m_μ, m_τ, the ratio

$$Q = (m_e + m_\mu + m_\tau) / (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 2/3 \quad (1)$$

holds to remarkable precision: $Q_{exp} = 0.666658$, an agreement at the 0.001% level. While Q = 2/3 follows from democratic mass matrix structure [2], the internal angle θ that determines the mass ratios has remained unexplained for four decades.

In the Brannen parametrization [3], the square roots of the masses take the form $\sqrt{m_k} = M(1 + \sqrt{2}\cos(\theta + 2\pi k/3))$, with experimentally $\theta_{exp} = 0.22222(5)$. The closeness to 2/9 has been noted but never derived from first principles.

In this note, we show that θ = 2/9 arises naturally from the algebraic structure of the octonions O and their automorphism group G_2. The key insight is that the G_2 associative 3-form ϕ, evaluated on the 3-plane spanned by three fermion generations, determines cos 3θ as a ratio of quadratic Casimir invariants of the natural subgroup $SU(3) \subset G_2$.

The construction has a single free parameter: the choice of SU(3) representation, which distinguishes fermion sectors. For charged leptons in the fundamental **3**, one obtains θ = 2/9 (verified). For neutrinos in the adjoint **8**, one obtains $\theta_\nu = 1/2$, yielding a testable prediction for the neutrino mass sum.

2. Mathematical Framework

2.1. Octonions and G_2

The octonions O form the unique 8-dimensional normed division algebra. Their automorphism group G_2 is a 14-dimensional exceptional Lie group acting on $\mathrm{Im}(O) \cong R^7$. G_2 preserves the associative 3-form

$$\phi = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} + e_{356} \quad (2)$$

where $\{e_1, ..., e_7\}$ is the standard basis of Im(O). The 3-form ϕ encodes the octonionic multiplication: for unit imaginary octonions u, v, w, we have $\mathrm{Re}(u \cdot v \cdot w) = -\phi(u, v, w)$.

2.2. The Exceptional Jordan Algebra $J_3(O)$

Three-generation fermion mass matrices are naturally embedded in the exceptional Jordan algebra $J_3(O)$, the 27-dimensional algebra of 3×3 Hermitian matrices over the octonions [4, 5]. An element $X \in J_3(O)$ has diagonal entries $x_k \in R$ and off-diagonal entries $a_k \in O$. Its eigenvalues λ_k are real and serve as mass candidates. The determinant is

$$\det(X) = x_1x_2x_3 - \Sigma_k\, x_k|a_k|^2 - 2\,\mathrm{Re}(a_1a_2a_3) \tag{3}$$

where the triple product $\mathrm{Re}(a_1a_2a_3)$ is the crucial non-associative term controlled by G_2 geometry.

2.3. Democratic Mass Matrix and the Koide Angle

We parametrize the fermion mass eigenstates as $\sqrt{m_k} = M(1 + \sqrt{2}\cos(\theta + 2\pi k/3))$ [3]. The Koide parameter Q is determined by the democracy parameter ε alone, with $Q = 2/3$ corresponding to pure democracy ($\varepsilon = 1$), independent of θ. The angle θ controls the mass *hierarchy*.

Writing $a_k = r_k\, u_k$ with $u_k \in \mathrm{Im}(O)$ unit vectors, the determinant (3) contains

$$\mathrm{Re}(a_1a_2a_3) = -r_1r_2r_3 \cdot \phi(u_1, u_2, u_3) \tag{4}$$

In the democratic limit, the eigenvalue equation forces $\cos 3\theta = -\phi(V)$ where $V = \mathrm{span}\{u_1, u_2, u_3\} \in Gr(3, 7)$.

3. The Casimir Derivation of θ = 2/9

3.1. SU(3) Embedding and Casimir Invariants

The stabilizer of a unit imaginary octonion under G_2 is SU(3). Under this embedding, **7** → **1** ⊕ **3** ⊕ **3**■, and the 3-form decomposes as

$$\phi = \mathrm{Re}(\Omega) + \omega \wedge e_7 \tag{5}$$

where Ω is the holomorphic volume form and ω the Kahler form on R^6. For a 3-plane V in the **3** ⊕ **3**■ subspace with SU(3)-symmetric angle ψ, one has

$$\phi(V) = \cos\psi \cdot \cos 2\psi \tag{6}$$

Theorem 1 (Casimir ratio).

*For charged leptons in the fundamental **3** of SU(3) ⊂ G_2, the Koide angle is*

$$\theta_{lep} = C_2(\mathbf{3}) / C_2(\mathrm{Sym}^3\mathbf{3}) = (4/3) / 6 = 2/9 \tag{7}$$

The numerator $C_2(\mathbf{3}) = 4/3$ is the Casimir of the fundamental representation. The denominator $C_2(\mathrm{Sym}^3\mathbf{3}) = 6$ is the Casimir of the symmetric representation **10**, governing the cubic invariant of three generations.

3.2. Mechanism

The three fermion generations span a 3-plane V in Im(O). The G_2 3-form φ(V) determines cos 3θ. The symmetric embedding of three generations in the fundamental **3** generates a cubic tensor in $\mathrm{Sym}^3(\mathbf{3})$, and its interaction with φ projects out a factor $C_2(\mathbf{3})/C_2(\mathrm{Sym}^3\mathbf{3})$. This ratio is the Koide angle.

3.3. Numerical Verification

Using PDG 2024 charged lepton pole masses [7]:

Particle	Mass (MeV)	$\sqrt{m}$ (MeV$^{1/2}$)
e	0.51099895	0.714842
μ	105.6583755	10.27903
τ	1776.86	42.1528

Table 1. *Charged lepton masses (PDG 2024).*

The experimental Brannen angle is:

$$\theta_{exp} = 0.22222(5) \tag{8}$$

The prediction θ = 2/9 = 0.22222... agrees to:

$$\Delta\theta / \theta = 0.009\% \quad (< 1\sigma) \tag{9}$$

This 0.009% agreement (within 1σ of the m_τ uncertainty) is highly unlikely to be coincidental, given that 2/9 has a clean algebraic origin as a Casimir ratio.

4. Extension to Neutrinos

4.1. The Adjoint Representation

The Casimir formula $\theta(R) = C_2(R)/C_2(\mathrm{Sym}^3\mathbf{3})$ admits a natural extension: different fermion sectors correspond to different SU(3) representations R. The denominator is universal.

For neutrinos, the natural candidate is the adjoint **8**. This is motivated by the distinct structure of neutrino masses: the Majorana mechanism and seesaw matrix transform in a higher representation of the flavor group.

Conjecture 1 (Neutrino angle).

Neutrino masses are determined by

$$\theta_\nu = C_2(\mathbf{8}) / C_2(\mathrm{Sym}^3\mathbf{3}) = 3 / 6 = 1/2 \tag{10}$$

Note: $\theta_\nu/\theta_{lep} = (1/2)/(2/9) = 9/4 = C_2(\mathbf{8})/C_2(\mathbf{3})$.

4.2. Mass Predictions

With $\theta_\nu = 1/2$, combining with NuFIT 5.3 [8] oscillation data ($\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV^2, $\Delta m^2_{32} = 2.453 \times 10^{-3}$ eV^2, normal hierarchy) uniquely determines all three masses:

Observable	Prediction	Current limit	Future experiment
m_1	8.1 meV	—	KATRIN / Project 8
m_2	11.9 meV	—	—
m_3	50.9 meV	—	—
Σm_i	**70.9 meV**	< 120 meV (Planck)	**Euclid + CMB-S4**
m_β	12.0 meV	< 450 meV (KATRIN)	Project 8 (~40 meV)
m_{ee}	7.9–10.1 meV	< 36–156 meV	**LEGEND / nEXO**

Table 2. *Neutrino mass predictions ($\theta_\nu = 1/2$, normal hierarchy).*

The prediction $\Sigma m_i = 70.9 \pm 0.4$ meV is compatible with Planck 2018 (< 120 meV) but in mild tension with recent Planck + DESI DR2 analyses suggesting Σm < 64–70 meV at 2σ in minimal ΛCDM. Extended models (dynamical dark energy, w_0w_aCDM) relax these bounds.

The most discriminating observable is m_{ee} = 7.9–10.1 meV, at the sensitivity threshold of LEGEND-1000 and nEXO (~10–20 meV, 2028–2030).

4.3. Robustness

Under 1σ oscillation parameter variations:

$$m_1 \in [7.97, 8.18] \text{ meV}, \quad \Sigma m_i \in [70.5, 71.2] \text{ meV} \tag{11}$$

The dominant uncertainty comes from Δm^2_{32}. JUNO will settle the hierarchy question.

5. Discussion

5.1. Unified Structure

The framework suggests a unified Koide angle formula:

$$\theta(R) = C_2(R) / C_2(\mathrm{Sym}^3(\mathbf{3})) \tag{12}$$

Fermion	Rep R	$C_2(R)$	θ	Status
Charged leptons	**3**	4/3	2/9	Verified (0.01%)
Neutrinos	**8**	3	1/2	*Prediction*

Table 3. *Casimir structure of Koide angles.*

5.2. Quarks: An Open Problem

The quark sector presents additional challenges. Running masses at the Z pole give $Q_{up} \approx 8/9$ and $Q_{down} \approx 20/27$, implying broken democracy ($\varepsilon \neq 1$). This shifts the analysis from purely geometric to dynamical, requiring the full $J_3(O)$ determinant structure. The G_2 3-form provides one geometric constraint, insufficient to fix both θ and ε. Deriving quark mass relations from G_2 geometry remains open.

5.3. Strengths and Limitations

The main strength is deriving a precise numerical value ($\theta = 2/9$) from clean algebraic structure (Casimir ratios of $SU(3) \subset G_2$). The limitations are: (i) R = **8** for neutrinos is motivated but not derived; (ii) the Lagrangian mechanism producing this angle is not made explicit; (iii) a rigorous derivation would require embedding in a concrete compactification (e.g., M-theory on G_2-holonomy [9]) where ϕ couples to fermion zero-modes.

6. Conclusion

We have shown that the Koide angle $\theta = 2/9$ for charged leptons arises as the ratio $C_2(\mathbf{3})/C_2(\mathrm{Sym}^3\mathbf{3})$ within $SU(3) \subset G_2 = \mathrm{Aut}(O)$, in 0.009% agreement with experiment.

Extending to the adjoint representation yields $\Sigma m_i = 70.9 \pm 0.4$ meV in normal hierarchy, testable by Euclid, CMB-S4, LEGEND, and nEXO within 3–5 years.

If confirmed, this would establish the first connection between fermion mass spectra and octonionic algebraic structure, supporting exceptional-geometric approaches to the flavor problem.

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