ALGEBRAIC TURBULENCE AND GLOBAL REGULARITY: THE SECULAR REPLICATOR FLOW AS A SELF-CONSISTENT ALGEBRAIC SHELL MODEL FOR SINGULARITY FORMATION

V. F. S. SANTOS

Abstract. We introduce the *Secular Replicator Flow*, a finite-dimensional algebraic dynamical system inspired by the turbulent energy cascade of the Navier–Stokes equations, built from the spectral theory of golden resolvent operators on discrete network graphs [11]. The continuous mechanics of fluid turbulence—incompressibility, nonlocal pressure, nonlinear advection, and viscous dissipation—find precise algebraic counterparts in the constraints of a replicator equation evolving on the simplex of spectral participation weights, governed by a global secular equation.

Within this framework we establish three principal results. First, the *Variance Law*: the macroscopic coupled eigenvalue $\lambda^*(t)$ evolves monotonically according to Fisher's Fundamental Theorem, acting as a strict Lyapunov function (between excision events) whose rate of increase equals the fitness variance of the active spectrum. Second, the *Spectral Selection Theorem*: the fitness landscape is a strict bipolar U-shape in the base eigenvalue μ, guaranteeing that the replicator flow annihilates mid-spectrum noise and funnels all energy into the extreme macroscopic topologies of the network. Third, *Global Regularity*: as the system approaches a structural resonance from below (a transparent pole), the channel's fitness plunges to $-\infty$, triggering an auto-excision mechanism that exponentially starves the dangerous channel, rendering every pole singularity removable. The resulting dynamics form a *Sawtooth Cascade* of smooth climbs interrupted by discontinuous structural snaps whose direction is controlled by the residual load of the excised channel.

Global regularity is *unconditional*: the dynamical system is well-posed for all time and every coupling energy $c > 0$. At the critical threshold $c = 2\rho^2$, a geometric—not dynamical—phase transition occurs: the Chebyshev recurrence of the underlying network crosses the $r = 2$ boundary, the discrete oscillatory structure transitions to hyperbolic geometry, but the Secular Replicator Flow itself remains smooth. All results herein apply to this model; implications for the full Navier–Stokes equations in $\mathbb{R}^3$ remain open.

Date: February 2026.

2020 *Mathematics Subject Classification.* 76D05, 76F02, 37N10, 05C50, 92D25.

Key words and phrases. Navier-Stokes equations, turbulence, shell models, spectral graph theory, golden ratio, replicator dynamics, Fisher's fundamental theorem, global regularity, singularity formation.

1. Introduction: beyond PDEs to algebraic shell models

For over a century, the mathematical pursuit of fluid turbulence—crystallised in the Navier–Stokes (NS) Millennium Prize problem—has been trapped in a geometric straitjacket. The core question asks whether the nonlinear advection of a three-dimensional fluid can concentrate energy so violently that it overcomes viscous dissipation, creating a finite-time singularity where the spatial velocity field blows up to infinity.

Decades of partial differential equation (PDE) analysis have stalled precisely because tracking a continuous spatial field through an infinite concentration of energy is mathematically intractable [7, 1]. To bypass this geometric collapse, physicists have long relied on discrete "shell models" (such as the GOY [3, 9] and Sabra [8] models), which abstract the fluid into a discrete ladder of frequencies to study the energy cascade [10]. However, these models are phenomenological constructions—designed to reproduce scaling properties of PDEs, but lacking endogenous structural rigour.

This paper proposes a complementary perspective: *PDEs need not be the only language for understanding singularity formation.* Rather than approximating continuous space, we present a finite-dimensional algebraic surrogate. By bridging the Golden Resolvent Theory [11] of discrete network graphs with simplex dynamics, we introduce the *Secular Replicator Flow*—a constrained spectral ecosystem that models structural analogues of 3D turbulence mechanics through abstract algebra.

In this framework, the physical mechanics of a fluid cascade find natural algebraic counterparts in the symmetries of a network's eigenspaces:

- **Incompressibility (the Leray projection).** Replaced by strict Parseval energy conservation $\left(\sum p_k^2 = c\right)$ on the simplex of spectral weights.
- **Nonlocal pressure.** Replaced by the global Secular Equation $\left(\sum p_k^2 N_k(\lambda) = \lambda\right)$, where the macroscopic energy state λ^* is an implicit, instantaneous global constraint on the entire spectrum.
- **Nonlinear advection.** Replaced by a Replicator Equation, where energy is actively transported from low-sensitivity modes to high-sensitivity modes, strictly driven upward by Fisher's Fundamental Theorem [2] (the Variance Law).

By translating the physics into this algebraic code, the nature of the "finite-time blowup" changes qualitatively. In a continuous PDE, an infinite concentration of energy tears the mathematical fabric of space. In the Secular Replicator Flow, we prove that an approaching singularity triggers an *Auto-Excision Mechanism.*

As the macroscopic energy λ^* is driven toward a structural resonance (a transparent pole), the sensitivity of the network locally diverges. However, the algebraic constraints of the secular equation force the fitness function to plunge to $-\infty$, driving the Replicator flow to exponentially starve the dangerous channel of all its mass.

The resulting dynamic is not a catastrophic failure of the equations, but an *event-driven topological shock*. The system dynamically amputates its own resonant dimensions to survive. When a dimension is excised, the network undergoes a "Structural Snap"—the macroscopic eigenvalue λ^* shifts discontinuously to a new energy state, and the cascade resumes on a reduced simplex.

Ultimately, we demonstrate that within this algebraic surrogate, the race between advection and viscosity always favours viscosity. The Secular Replicator Flow acts as a self-correcting mechanism, preventing localised blowups and driving the "turbulent" cascade toward a variance-zero laminar equilibrium.

By abandoning the continuous spatial geometry of PDEs, we provide a rigorously solvable discrete model showing that, within this algebraic framework, turbulence need not end in a singularity, but rather in a sequence of discontinuous topological avalanches that guarantee global regularity.

Remark 1.1 (Scope and the term "self-consistent"). The Secular Replicator Flow is a finite-dimensional dynamical system whose structure is *inspired by* the mechanics of the Navier–Stokes equations; it is not derived from them via a limiting procedure. We call it a *self-consistent* shell model in the sense that all inter-mode couplings are determined algebraically by the secular constraint and the golden resolvent—no phenomenological closure or free parameters are introduced. This does not mean "derived from NS." All regularity results in this paper apply to the model itself. Whether and how these results transfer to the full Navier–Stokes equations in $\mathbb{R}^3$ remains an open question. A legitimate bridge would require, at minimum, a rigorous embedding of an NS truncation into the secular-replicator framework (e.g. an invariant-manifold reduction), or quantitative matching of cascade statistics (e.g. Kolmogorov scaling exponents or intermittency corrections). We do not claim any such bridge here.

2. The translation dictionary: from continuous PDEs to discrete spectral algebra

To abandon the continuous spatial geometry of the Navier–Stokes equations, we exhibit structurally analogous mechanics within the algebraic ecosystem of the Golden Resolvent. The NS equations describe a continuous velocity field $u(x,t)$ driven by nonlinear advection, constrained globally by incompressible pressure, and damped by viscosity. In our algebraic shell model, the continuous spatial domain is replaced by the discrete eigenspaces of a network's adjacency operator. The fluid "velocity" is replaced by the spectral participation weights $p_k^2(t)$, which track the distribution of coupling energy across these structural modes.

We begin by collecting the standing notation from [11], then exhibit the mapping term by term.

2.1. **Standing notation from golden resolvent theory.** Write $\varphi = (1+\sqrt{5})/2$ for the golden ratio. Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with orthonormal eigenpairs $(\mu_k, v_k)_{k=1}^n$, ordered so that $\mu_1 \le \cdots \le \mu_n$; the *Perron root* is $\rho := \mu_{\max} = \mu_n$. Let $e \in \mathbb{R}^n$ be a probe vector. The *participations* are $p_k := \langle e, v_k\rangle$, so that $p_k^2 \ge 0$ and the total coupling energy is $c := \|e\|^2 = \sum_k p_k^2$. The *active set* is $K(t) := \{k : p_k^2(t) > 0\}$, and we write $w_k := p_k^2/c$ for the normalised weights ($\sum_{k\in K} w_k = 1$).

For each base eigenvalue μ_k, define the *golden denominator* and the *fitness* (or sensitivity):

$$D_k(\lambda) := \lambda^2 - \lambda\mu_k - \mu_k^2 = (\lambda - \varphi\mu_k)\,(\lambda + \mu_k/\varphi), \tag{1}$$

$$N_k(\lambda) := \frac{\lambda - \mu_k}{D_k(\lambda)}, \qquad D_k(\lambda) \ne 0. \tag{2}$$

The zeros $\varphi\mu_k$ and $-\mu_k/\varphi$ of D_k are the *transparent poles*. The open interval $(\min(\varphi\mu_k,\ -\mu_k/\varphi),\ \max(\varphi\mu_k,\ -\mu_k/\varphi))$ in λ is the λ*-safe strip* for channel k. Dually, at fixed $\lambda^* > 0$, the golden denominator $D(\mu; \lambda^*) = (\lambda^*)^2 - \lambda^*\mu - \mu^2$ vanishes at $\mu = \lambda^*/\varphi$ and $\mu = -\lambda^*\varphi$; the interval $(-\lambda^*\varphi,\ \lambda^*/\varphi)$ is the μ*-admissible interval* at the current macroscopic energy.

A *coupled root* λ^* is any real solution of the *secular equation* $\sum_{k\in K} p_k^2\, N_k(\lambda) = \lambda$. The associated *secular weight* is

$$W(\lambda) := 1 + \sum_{k\in K} p_k^2\, \frac{(\lambda - \mu_k)^2 + \mu_k^2}{D_k(\lambda)^2}. \tag{3}$$

A coupled root is *simple* when $W(\lambda^*) \ne 0$; one always has $W(\lambda^*) \ge 1$ for real roots in the safe strip [11]. Simplicity guarantees that λ^* depends smoothly on the weights via the implicit function theorem.

In normalised coordinates the Secular Replicator Flow takes the standard replicator form

$$\dot{w}_k = 2\, w_k\big(N_k(\lambda^*) - \mathbb{E}_w[N(\lambda^*)]\big), \qquad \mathbb{E}_w[N(\lambda^*)] = \frac{\lambda^*}{c}, \tag{4}$$

making the connection with evolutionary game theory [5] visually immediate.

State variables. The primary state is the weight vector $p^2 = (p_1^2, \dots, p_n^2)$ on the simplex $\sum p_k^2 = c$. The normalised form $w_k = p_k^2/c$ lives on the probability simplex $\Delta^{|K|-1}$. We use p_k^2 when energy bookkeeping matters and w_k when the replicator structure is emphasised; $\lambda^*(t)$ is determined implicitly by the secular equation at each instant.

2.2. **Root selection and invariant regions.** Before defining the time-evolution of the network, we establish that the macroscopic coupled eigenvalue $\lambda^*(t)$ is a well-defined, continuous trajectory. Define the *secular function*

$$f(\lambda;\, p^2) \coloneqq \sum_{k\in K} p_k^2\, N_k(\lambda) - \lambda. \tag{5}$$

The coupled roots are the real solutions of $f(\lambda;\, p^2) = 0$. Let $\mathcal{I} = (\pi_L, \pi_R)$ be any open interval bounded by two adjacent active poles (or by a pole and $\pm\infty$).

Theorem 2.1 (Root Selection)**.** *For any non-empty active interval $\mathcal{I} = (\pi_L, \pi_R)$ and any strictly positive weight distribution on K, there exists exactly one simple coupled root $\lambda^* \in \mathcal{I}$. Moreover, λ^* is a smooth function of the weights p^2 and remains strictly confined to $\mathcal{I}$ until a boundary pole is reached.*

Proof. Step 1 (strict decrease). Differentiating (5) with respect to λ:

$$f'(\lambda;\, p^2) = \sum_{k\in K} p_k^2\, N_k'(\lambda) - 1.$$

Each summand satisfies $N_k'(\lambda) < 0$ (the full derivation appears in (13) of section 5.1; the result is $N_k' = -[(\lambda-\mu_k)^2 + \mu_k^2]/D_k^2$), so $f'(\lambda) < -1$ on $\mathcal{I}$.

Step 2 (endpoint signs for all pole types). Let π be any transparent pole; it equals either $\varphi\mu_\alpha$ or $-\mu_\alpha/\varphi$ for some channel α. Write $D_\alpha(\lambda) = (\lambda - \varphi\mu_\alpha)(\lambda + \mu_\alpha/\varphi)$.

Right approach ($\lambda \downarrow \pi$ with $\pi = \pi_L$). The vanishing factor goes to 0^+ (we approach from inside the interval). A case check on both pole types and both signs of μ_α gives:

$$\text{numerator } (\lambda - \mu_\alpha) \;\longrightarrow\; \begin{cases} \mu_\alpha/\varphi & \text{if } \pi = \varphi\mu_\alpha, \\ -\mu_\alpha\varphi & \text{if } \pi = -\mu_\alpha/\varphi, \end{cases}$$

and in every case the limiting sign of the numerator matches the limiting sign of the vanishing denominator factor, yielding $N_\alpha(\lambda) \to +\infty$.

Left approach ($\lambda \uparrow \pi$ with $\pi = \pi_R$). The vanishing factor now goes to 0^-, while the numerator limit is the same as above. The sign mismatch gives $N_\alpha(\lambda) \to -\infty$ in every case.

All other channels remain bounded on $\mathcal{I}$. Hence $f \to +\infty$ as $\lambda \downarrow \pi_L$ and $f \to -\infty$ as $\lambda \uparrow \pi_R$.

Step 3 (conclusion). By the Intermediate Value Theorem f crosses zero at least once; by strict monotonicity, exactly once. Because $f'(\lambda^*) \neq 0$, the root is simple. The Implicit Function Theorem then gives smooth dependence of λ^* on p^2. The interval $\mathcal{I}$ is an invariant region for the branch: $\lambda^*(t)$ cannot leave $\mathcal{I}$ without colliding with a boundary pole, which triggers the excision mechanism of section 5.2. ∎

2.3. **Well-posedness of the closed-loop system.** Because λ^* is defined implicitly, the Secular Replicator Flow (4) is a differential-algebraic system. We verify that it is in fact a smooth ODE on the simplex interior.

Proposition 2.2 (Well-posedness)**.** *On any region where $\lambda^*(t)$ is a simple coupled root and all $D_k(\lambda^*(t)) \neq 0$, the vector field of the Secular Replicator Flow is smooth (in particular locally Lipschitz). Consequently:*

(i) *the open simplex $\Delta^\circ = \{w \in \mathbb{R}^n : w_k > 0,\ \sum w_k = 1\}$ is forward invariant;*
(ii) *solutions exist and are unique up to the first excision event (the first time some $D_k(\lambda^*) = 0$).*

Proof. By theorem 2.1, λ^* is a smooth function of w. Each $N_k(\lambda^*)$ is a smooth function of λ^* (away from poles), hence of w. The composition gives a smooth vector field on Δ°. Forward invariance of Δ° follows from the standard replicator property: $\dot{w}_k = 0$ whenever $w_k = 0$, so no face of the simplex is entered in finite time. Picard–Lindelöf then gives local existence and uniqueness. □

2.4. **Imported results from Golden Resolvent Theory.** The present paper builds on the spectral resolvent framework of [11]. For the reader's convenience we list the specific results imported here, with exact references to theorems therein.

(GRT1) **Golden factorisation.** For any eigenvalue–probe pair (μ_k, p_k), the resolvent contribution factors through the golden denominator (1), yielding the rational fitness $N_k(\lambda)$ of (2). (Theorem 3.1 of [11].)
(GRT2) **Secular weight bound.** At any simple coupled root λ^* lying in an inter-pole interval, the secular weight satisfies $W(\lambda^*) \geq 1$. (Proposition 4.2 of [11].)
(GRT3) **Chebyshev recurrence.** The structural amplitudes of a network cascade obey a second-order linear recurrence whose characteristic polynomial has discriminant $\Delta = r^2 - 4$, with $r = \lambda/\rho$. Oscillatory (bounded) solutions exist iff $|r| < 2$; the regime $|r| \geq 2$ marks the onset of hyperbolic growth. (Theorem 5.4 of [11].)
(GRT4) **Resolvent–fitness identity.** The sensitivity $N_k(\lambda)$ equals the diagonal resolvent entry $\langle v_k, G(\lambda)e\rangle / \langle e, v_k\rangle$, linking the replicator fitness to classical spectral perturbation theory. (Corollary 3.5 of [11].)

Beyond these four facts, the dynamical analysis (variance law, excision, phase escape) is self-contained.

2.5. **Incompressibility: the Leray projection vs. Parseval conservation.** In fluid dynamics, the fluid cannot be compressed:

$$\nabla \cdot u = 0.$$

This kinematic constraint forces the fluid to redistribute its kinetic energy rather than accumulating it at a single point.

In our algebraic surrogate, incompressibility is enforced by Parseval's identity on the probe vector $e(t)$. The total structural energy c is strictly conserved:

$$\sum_{k=1}^{n} p_k^2(t) = c \qquad \Longrightarrow \qquad \sum_{k=1}^{n} \dot{p}_k^2 = 0. \tag{6}$$

The flow of energy is strictly confined to the $(n-1)$-dimensional simplex of spectral weights. The system cannot "create" energy; it can only cascade it between the topological dimensions of the network.

2.6. **Nonlocal pressure: the Poisson equation vs. the secular equation.** The most notoriously difficult term in the NS equations is the pressure gradient ∇P. Pressure is not a local thermodynamic variable; it is a nonlocal Lagrange multiplier. If fluid moves in one corner of the domain, the pressure field instantaneously adjusts globally to enforce incompressibility everywhere.

In our model, this instantaneous global constraint is executed by the Secular Equation:

$$\sum_{k=1}^{n} p_k^2(t)\, N_k(\lambda^*) = \lambda^*. \tag{7}$$

The macroscopic coupled eigenvalue $\lambda^*(t)$ plays the role of the global pressure field. It is an implicit function of every single weight p_k^2 simultaneously. Any localised perturbation in a single channel's energy instantly alters λ^*, which in turn instantaneously alters the "fitness" $N_j(\lambda^*)$ of every other channel in the network.

2.7. **Nonlinear advection: the turbulent cascade vs. the replicator flow.** The engine of fluid turbulence is the nonlinear advective term $(u\cdot\nabla)u$, representing the fluid transporting itself. This quadratic self-interaction pushes energy from large, stable macroscopic eddies down into volatile, high-frequency microscopic scales (the Richardson cascade [10]).

In the spectral network, energy transport is governed by the *Secular Replicator Equation*:

$$\dot{p}_k^2 = 2\, p_k^2 \left(N_k(\lambda^*) - \frac{\lambda^*}{c} \right). \tag{8}$$

Just as advection transports fluid based on its own velocity gradients, the replicator flow transports energy p_k^2 based on its own topological sensitivity $N_k(\lambda^*)$ relative to the system's average energy λ^*/c. In normalised coordinates (4), this is a standard replicator equation with fitness landscape N_k. Mass is drained from structurally "unfit" dimensions and pumped into highly resonant ones.

2.8. **Viscosity: energy dissipation vs. algebraic auto-excision.** In a standard fluid, kinematic viscosity $\nu\,\Delta u$ diffuses energy, acting as a linear brake against the nonlinear advective cascade. A finite-time blowup occurs if advection focuses energy infinitely small before viscosity can smear it out.

Our algebraic model requires no external viscosity parameter. Instead, "viscosity" is an emergent, endogenous property of the Golden Resolvent's calculus. Because the derivative of the fitness function is strictly negative ($N_k'(\lambda) < 0$), any channel approaching a topological singularity (a transparent pole where $D_k(\lambda^*) \to 0$) experiences an infinite negative divergence in its fitness: $N_k \to -\infty$.

This triggers the Replicator Flow to exponentially starve the channel of mass. The "viscous dissipation" in our model is the system's ability to dynamically amputate its own dimensions to prevent an infinite concentration of structural tension.

2.9. **Summary: the translation lexicon.** The structural analogy between the PDE geometry and the algebraic simplex is summarised in the following dictionary. Each row is an analogy, not an isomorphism; see theorem 1.1.

Navier–Stokes (Continuous PDEs)	**Secular Replicator Flow (Discrete Algebra)**
Physical space ($\mathbb{R}^3$)	Spectral eigenspaces (μ_k)
Velocity field $\big(u(x,t)\big)$	Participation weights $\big(p_k^2(t)\big)$
Incompressibility ($\nabla \cdot u = 0$)	Simplex conservation $\big(\sum p_k^2 = c\big)$
Nonlocal pressure (∇P)	Macroscopic eigenvalue (λ^*)
Nonlinear advection $\big((u \cdot \nabla)u\big)$	Replicator dynamics $\big(\dot{p}_k^2 \propto p_k^2 N_k\big)$
Viscous dissipation ($\nu\,\Delta u$)	Auto-excision limit ($N_k \to -\infty$)
Singularity (velocity blowup)	Root-reset (discontinuous structural snap)

By establishing this structural analogy, we can now apply the rigorous tools of simplex dynamics and Lyapunov stability to answer the ultimate question of the turbulent cascade: does the system tear itself apart, or does it smoothly resolve?

3. The Variance Law: spectral inertia and the arrow of time

In physical fluid turbulence, the Richardson cascade possesses a strict macroscopic directionality: kinetic energy flows from large, stable macroscopic eddies down into volatile, high-frequency microscopic scales. For our algebraic surrogate to be valid, this "arrow of time" cannot be imposed from the outside; it must emerge endogenously from the algebra of the network itself.

We derive this directionality by tracking the time evolution of the macroscopic coupled eigenvalue $\lambda^*(t)$ as the Replicator Flow redistributes the coupling energy c across the simplex of spectral weights $p_k^2(t)$.

3.1. **The implicit derivative and spectral inertia.** Because λ^* is defined by the global constraint of the Secular Equation (7), it is an implicit function of the participation weights. Provided the root remains simple, the sensitivity of the eigenvalue to a shift in any specific channel's energy is obtained by differentiating $\sum p_k^2 N_k(\lambda) = \lambda$ with respect to p_j^2 at fixed $p_{k\neq j}^2$ and solving:

$$\frac{\partial \lambda^*}{\partial (p_j^2)} = \frac{N_j(\lambda^*)}{W(\lambda^*)}. \tag{9}$$

Here, $W(\lambda^*)$ is the *Secular Weight*, strictly bounded by $W(\lambda^*) \geq 1$ for all valid coupled eigenvalues. In our shell model, $W(\lambda^*)$ acts as the spectral inertia of the network. It dictates how much "resistance" the network's global topology offers against local energy perturbations.

3.2. **The Variance Law.**

Theorem 3.1 (Variance Law of Spectral Turbulence)**.** *Let $\lambda^*(t)$ be a simple coupled root ($W(\lambda^*(t)) > 0$; cf.* (3)*) tracked along a smooth time interval on which no excision event occurs. Then under the Secular Replicator Flow* (8)*, the macroscopic coupled eigenvalue evolves according to*

$$\dot{\lambda}^* = \frac{2c}{W(\lambda^*)}\ \mathrm{Var}_w\big(N(\lambda^*)\big)\ \geq\ 0. \tag{10}$$

Proof. Apply the chain rule over the active spectrum, substituting the Replicator Flow (8) for $\dot{p}_k^2$:

$$\dot{\lambda}^* = \sum_{k=1}^{n} \frac{\partial \lambda^*}{\partial (p_k^2)}\, \dot{p}_k^2 = \sum_{k=1}^{n} \frac{N_k(\lambda^*)}{W(\lambda^*)} \left[2\, p_k^2 \left(N_k(\lambda^*) - \frac{\lambda^*}{c} \right) \right].$$

Factor out constants and use normalised weights $w_k = p_k^2/c$:

$$\dot{\lambda}^* = \frac{2c}{W(\lambda^*)} \sum_{k=1}^{n} w_k\, N_k(\lambda^*) \left(N_k(\lambda^*) - \frac{\lambda^*}{c} \right).$$

By the Secular Equation (7), the mean fitness is $\sum w_k N_k(\lambda^*) = \lambda^*/c$. The summation is therefore the weighted variance $\mathrm{Var}_w(N(\lambda^*))$, which is nonnegative. Since $W(\lambda^*) \geq 1 > 0$, the result follows. ☐

3.3. **Fisher's theorem and the strict Lyapunov function.** Theorem 3.1 is the exact spectral analogue of Fisher's Fundamental Theorem of Natural Selection [2], which states that the rate of increase in fitness of any biological population is directly proportional to its genetic variance [5].

In the context of the Secular Replicator Flow:

The arrow of time.: Because variance is non-negative and the secular weight $W(\lambda^*)$ is strictly positive, the macroscopic eigenvalue $\lambda^*(t)$ must monotonically increase. The energy cascade only flows "uphill."

The Lyapunov function.: $\lambda^*(t)$ acts as a strict Lyapunov function along each smooth segment between excision events.

The laminar equilibrium.: The system can only stop evolving ($\dot{\lambda}^* = 0$) when the variance is entirely eliminated. This occurs if and only if all active channels possess identical sensitivity ($N_k = \lambda^*/c$), representing a variance-zero laminar state.

Every trajectory generated by the network follows a single arc: the Variance Law drives λ^* upward, pole encounters trigger excision events (section 5), and ultimately the system sheds its variance and converges to a laminar vertex equilibrium. There are no cycles, no chaotic attractors, and no finite-time blowups—only a monotone march toward structural resolution.

4. The Spectral Selection Theorem: bipolar laminar limits and the annihilation of noise

We have established that the Secular Replicator Flow possesses a strict Lyapunov function (theorem 3.1), monotonically driving the macroscopic energy $\lambda^*(t)$ uphill until the statistical variance of the network's fitness landscape is eradicated. But what is the physical geometry of this final, variance-zero state? In fluid dynamics, turbulence decays into a laminar flow, governed by the macroscopic boundaries of the physical domain. In our algebraic surrogate, we must prove exactly which topological dimension of the network survives the cascade and emerges as this laminar victor.

To answer this, we shift our perspective from the time-evolution of the system to the underlying geometry of the spectral space. We analyse the exact shape of the network's fitness landscape to determine which dimensions are mathematically destined to starve, and which are destined to inherit the total energy of the system.

4.1. **The variance-zero equilibrium and vertex collapse.**

The Variance Law dictates that the flow terminates if and only if $\mathrm{Var}_w\big(N(\lambda^*)\big) = 0$. This requires that every single active channel k existing on the simplex possesses the exact same sensitivity:

$$N_k(\lambda^*) = \frac{\lambda^*}{c} \qquad \forall\, k \in K.$$

For a generic network lacking perfectly symmetric or degenerate eigenspaces, it is algebraically impossible for multiple distinct base eigenvalues μ to yield the same fitness $N(\mu;\lambda^*)$. Therefore, the system is mathematically forbidden from maintaining a distributed energy state. The replicator dynamics will break any unstable symmetry, forcing the simplex to collapse to a *vertex equilibrium.*

At this laminar endpoint, the "turbulence" is entirely resolved: all coupling energy pools into a single winning channel k^*, such that $p_{k^*}^2 = c$, and all other channels are starved to absolute zero ($p_j^2 = 0$ for $j \neq k^*$). The question then becomes: how does the network select the winner k^*?

4.2. **The calculus of the fitness landscape.** To determine the winner, we must perform the actual linearised stability analysis at a vertex, accounting for the implicit coupling through λ^*.

Proposition 4.1 (Vertex invasion with coupled λ^*)**.** *At a single-channel vertex $w_{k^*} = 1$ (all mass on channel k^*), the secular equation reduces to $c\,N_{k^*}(\lambda) = \lambda$, implicitly defining λ_{k^*}. Perturb by introducing mass $w_j = \varepsilon$ in a quiescent channel j, with $w_{k^*} = 1 - \varepsilon$. Then the invasion eigenvalue (growth rate of ε to first order) is*

$$\sigma_j = 2\big[N_j(\lambda_{k^*}) - N_{k^*}(\lambda_{k^*})\big].$$

In particular, channel j invades if and only if $N_j(\lambda_{k^*}) > N_{k^*}(\lambda_{k^*})$; the coupled shift $\partial\lambda^*/\partial w_j$ contributes only at order ε^2.

Proof. The perturbed secular equation is $c\big[(1-\varepsilon)\,N_{k^*}(\lambda) + \varepsilon\,N_j(\lambda)\big] = \lambda$. Implicit differentiation at $\varepsilon = 0$ gives

$$\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = \frac{c\big[N_j(\lambda_{k^*}) - N_{k^*}(\lambda_{k^*})\big]}{W(\lambda_{k^*})},$$

where $W(\lambda_{k^*}) = 1 - c\,N'_{k^*}(\lambda_{k^*}) \geq 1$ by item (GRT2). (At a single-channel vertex only the k^* term survives in (3), giving $W = 1 + p_{k^*}^2[(\lambda - \mu_{k^*})^2 + \mu_{k^*}^2]/D_{k^*}^2 = 1 - c\,N'_{k^*}$, consistent with the general definition.) The growth rate of channel j from (4) is $\dot{w}_j = 2\varepsilon\big[N_j(\lambda^*) - \lambda^*/c\big]$. Using $\lambda^*/c = N_{k^*}(\lambda_{k^*})$ at the vertex and noting that the first-order shift $\delta\lambda = \varepsilon\,d\lambda/d\varepsilon$ enters N_j only at $O(\varepsilon)$, we obtain $\dot{w}_j = 2\varepsilon\big[N_j(\lambda_{k^*}) - N_{k^*}(\lambda_{k^*})\big] + O(\varepsilon^2)$. □

Therefore, the ultimate victor of the cascade must be the global maximiser of the fitness function across the active spectrum. To map this landscape, we treat the base eigenvalue μ as a continuous variable and evaluate the fitness at a fixed, safe coupled root $\lambda^* > 0$. Writing $N(\mu;\lambda^*)$ for the fitness viewed as a function of μ (cf. (2)):

$$N(\mu;\lambda^*) = \frac{\lambda^* - \mu}{(\lambda^*)^2 - \lambda^*\mu - \mu^2}. \tag{11}$$

Applying the quotient rule and simplifying yields:

$$\frac{\partial}{\partial\mu}\,N(\mu;\lambda^*) = \frac{\mu\,(2\lambda^* - \mu)}{D(\mu;\lambda^*)^2}, \tag{12}$$

where $D(\mu;\lambda^*) = (\lambda^*)^2 - \lambda^*\mu - \mu^2$ is the golden denominator. Because $D(\mu;\lambda^*)^2$ is strictly positive, the slope of the fitness landscape is entirely dictated by the numerator $\mu\,(2\lambda^* - \mu)$.

For any system that has not yet suffered an excision event, all active base eigenvalues μ_k must reside within the μ-admissible interval (cf. section 2.1):

$\mu_k \in (-\lambda^*\varphi,\ \lambda^*/\varphi)$. Because $1/\varphi = \varphi - 1 < 2$, the upper bound satisfies $\lambda^*/\varphi < 2\lambda^*$, so the factor $(2\lambda^* - \mu)$ is strictly positive for every admissible μ.

This imposes a rigid topology on the fitness landscape: the sign of $\partial N/\partial\mu$ is exactly the sign of μ.

4.3. **The bipolar selection law.** The physical implications of this derivative map directly to a strict *Bipolar Selection Law* governing the fluid-like cascade:

- For $\mu > 0$: the derivative is positive. Fitness strictly increases as one moves to the right, toward the positive pole.
- For $\mu < 0$: the derivative is negative. Fitness strictly increases as one moves to the left, deeper toward the negative pole.
- At $\mu = 0$: the derivative is exactly zero, acting as the absolute global minimum of the fitness landscape ($N(0; \lambda^*) = 1/\lambda^*$).

The shape of the network's fitness is a strict, bipolar "U-shape." The near-zero modes—representing the localised, unstructured "bulk" or mid-spectrum noise of the graph—are algebraically the weakest dimensions in the system. The replicator dynamics will systematically drain energy away from these mid-spectrum eigenvalues.

4.4. **The eradication of noise.**

Theorem 4.2 (Bipolar Laminar Selection)**.** *Let λ_{k^*} be the coupled root at a single-channel vertex equilibrium, and let*

$$K^+ = \{k : \mu_k > 0,\ D_k(\lambda_{k^*}) \neq 0\}, \qquad K^- = \{k : \mu_k < 0,\ D_k(\lambda_{k^*}) \neq 0\}$$

be the available positive and negative base spectrum. Because $N(\mu; \lambda_{k^*})$ strictly increases outward from $\mu = 0$ within the safe strip, the generic winner of the Secular Replicator Flow must be an extreme macroscopic topology of the network. Energy is prohibited from pooling in the mid-spectrum bulk. The ultimate victor k^* is restricted to either:

(i) *the dominant positive topology: the maximiser of $N(\mu_k; \lambda_{k^*})$ over K^+ (generically $\mu_{\max}$), or*
(ii) *the dominant negative topology: the maximiser of $N(\mu_k; \lambda_{k^*})$ over K^- (generically the most negative safe eigenvalue).*

The exact victor is deterministically selected by comparing these two maxima.

Proof. By theorem 4.1, channel j invades a vertex k^* if and only if $N_j(\lambda_{k^*}) > N_{k^*}(\lambda_{k^*})$. The derivative analysis of (12) shows that $\partial N/\partial\mu$ has the sign of μ inside the admissible interval, so the fitness landscape is strictly increasing for $\mu > 0$ and strictly decreasing for $\mu < 0$, with a global minimum at $\mu = 0$. Any vertex supported on a mid-spectrum eigenvalue can therefore be invaded by the corresponding spectral extreme (the maximiser of $N(\mu_k; \lambda_{k^*})$ in K^+ or K^-), so every non-extremal vertex is unstable. Stability of the extremal vertices follows from the same monotonicity: at the vertex $\mu_{k^*} = \mu_{\max}$, the equilibrium λ_{k^*} is determined by $c\,N_{k^*}(\lambda) = \lambda$, and the fitness $N(\mu;\lambda_{k^*})$ is maximised at $\mu_{\max}$ over K^+ by strict increase in $\mu > 0$; hence no positive-spectrum channel can invade. Analogously for the negative extreme. The ultimate victor is selected by comparing the two extremal fitness values. □

The Secular Replicator Flow therefore acts as an algorithmic filter analogous to the Richardson energy cascade in fluid turbulence: it systematically drains energy from mid-spectrum modes and concentrates it at the macroscopic extremes. (We stress that this is a *cascade analogue*; the flow is Lyapunov-driven and non-chaotic between events, hence not "turbulent" in the standard dynamical-systems sense of sensitive dependence on initial conditions.)

The "dominant positive topology" ($\mu_{\max}$) corresponds to the network's densest connected cluster—the Perron eigenspace. The "dominant negative topology" ($\mu_{\min}$, when negative) corresponds to the most frustrated bipartite cut of the graph, where the eigenvalue sign reflects the two-colouring structure [11]. The bipolar selection law predicts that energy will funnel into one of these two extremes, depending on which has the larger fitness at the current macroscopic energy.

5. Global regularity in the model: event-driven topology and auto-excision

The defining challenge of the Navier–Stokes Millennium Prize is determining whether the nonlinear advection of a fluid can concentrate energy infinitely small, infinitely fast, overcoming viscous dissipation to create a finite-time singularity. In our algebraic surrogate, we must subject the Secular Replicator Flow to the analogous test.

Because the Variance Law (theorem 3.1) strictly drives the macroscopic energy $\lambda^*(t)$ upward, the system is monotonically pushed toward the transparent poles of the active spectrum (where $D_k(\lambda^*) \to 0$). A true singularity—a structural blowup—would occur if the eigenvalue collides with a pole while the channel's energy p_k^2 remains finite, driving the system's sensitivity to infinity and tearing the geometry of the network.

To prove Global Regularity, we must demonstrate that the "viscosity" of the replicator flow drains the energy $p_k^2 \to 0$ faster than the "advection" of the variance law drives $D_k(\lambda^*) \to 0$.

5.1. The calculus of immunity: $N_k'(\lambda) < 0$.

To evaluate the race between advection and viscosity, we analyse how the topological fitness of a channel reacts as the macroscopic energy of the system barrels toward it. We differentiate the fitness function $N_k(\lambda) = (\lambda - \mu_k)/D_k(\lambda)$ with respect to the moving eigenvalue λ:

$$\frac{d}{d\lambda}\,N_k(\lambda) = \frac{(\lambda^2 - \lambda\mu_k - \mu_k^2) - (\lambda - \mu_k)(2\lambda - \mu_k)}{(\lambda^2 - \lambda\mu_k - \mu_k^2)^2}.$$

Expanding the numerator and completing the square yields:

$$\frac{d}{d\lambda} N_k(\lambda) = \frac{-\left[(\lambda - \mu_k)^2 + \mu_k^2\right]}{D_k(\lambda)^2}. \tag{13}$$

Because the numerator is the strictly negative sum of two squares, the derivative is strictly negative everywhere: $N_k'(\lambda) < 0$.

The consequence is strict: fitness is a strictly decreasing function of macroscopic energy.

Lemma 5.1 (Pole-approach sign)**.** *Let (π_L, π_R) be the current inter-pole interval.*

(i) *At the left endpoint, regardless of the sign of the underlying eigenvalue: $N_\alpha(\lambda) \to +\infty$ as $\lambda \downarrow \pi_L$.*
(ii) *At the right endpoint: $N_\beta(\lambda) \to -\infty$ as $\lambda \uparrow \pi_R$.*

In particular, along a smooth segment where $\dot{\lambda}^* \geq 0$, the next event is at the upper endpoint π_R, and the diverging channel's fitness plunges to $-\infty$.

Proof. This is established in the root-selection proof (theorem 2.1, Step 2): the vanishing factor of $D_\alpha(\lambda)$ goes to 0^+ at a left endpoint and 0^- at a right endpoint, while the numerator limit has a definite sign. The ratio $N_\alpha = (\lambda - \mu_\alpha)/D_\alpha$ therefore diverges to $+\infty$ on the left and $-\infty$ on the right, for every pole type and every sign of μ_α. □

Inside the Replicator Flow (8), this infinitely negative fitness triggers an infinitely negative growth rate. The network exponentially starves the resonant channel of mass, and the starvation direction is deterministic, not rhetorical: it is guaranteed by the monotonicity of λ^* and the sign structure of theorem 5.1.

5.2. **Dynamic excision and removable singularities.** To ensure robustness, we must account for the possibility of simultaneous pole interactions. Because the Variance Law guarantees that $\lambda^*(t)$ is strictly monotonic, it can only encounter multiple poles simultaneously if those poles coincide on the real line. This occurs either from degenerate eigenvalues ($\mu_i = \mu_j$), which produce identical pole pairs, or from distinct eigenvalues related by $\mu_j = -\varphi^2 \mu_i$, which produce a coincident pole at $\varphi\mu_i = -\mu_j/\varphi$. In both cases, the excision mechanism handles the entire cluster at once.

Lemma 5.2 (Dynamic Excision)**.** *Suppose $\lambda^*(t) \to \bar{\lambda}$ as $t \to T^-$, where $\bar{\lambda}$ is a transparent pole. Let*

$$S \;=\; \left\{k \in K(t) : D_k(\bar{\lambda}) = 0\right\}$$

be the full set of active channels whose golden denominators vanish at $\bar{\lambda}$ (these channels may share a base eigenvalue or may arise from distinct eigenvalues). Because $\lambda^*(t)$ is monotonic and the poles of a finite matrix form a discrete set, there exists a uniform barrier for all non-singular channels: $\inf_{t<T} |D_j(\lambda^*(t))| > \delta > 0$ for all $j \notin S$.

Then the collective channel load remains bounded:

$$\sup_{t<T}\left|\sum_{i\in S} p_i^2(t)\, N_i\big(\lambda^*(t)\big)\right| < \infty. \tag{14}$$

In particular, each individual channel satisfies $p_i^2(t) = \mathcal{O}\big(|D_i(\lambda^*(t))|\big)$ *as* $t \to T^-$ *for every* $i \in S$. *The entire cluster* S *undergoes simultaneous extinction* ($p_i^2(T^-) = 0$ *for all* $i \in S$), *and every pole singularity in the cluster is removable.*

Proof. Isolating the singular cluster from the surviving active set via the Secular Equation (7):

$$\sum_{i\in S} p_i^2(t)\, N_i\big(\lambda^*(t)\big) = \lambda^*(t) - \sum_{j\notin S} p_j^2(t)\, N_j\big(\lambda^*(t)\big) \;=:\; R(t). \tag{15}$$

Because the poles of a finite matrix are discrete and $\lambda^*(t)$ is monotonic, all $j \notin S$ remain strictly separated from their respective poles as $t \to T^-$. Therefore $R(t) \to R(T^-)$ (finite), establishing (14).

For each $i \in S$, define the *pole residue*

$$\nu_i \;=\; \lim_{\lambda\to\bar\lambda} (\lambda - \bar\lambda)\, N_i(\lambda),$$

which is finite and nonzero (the simple pole of N_i at $\bar\lambda$). By theorem 5.1, since $\bar\lambda$ is a right endpoint and $\lambda^*(t)$ approaches from below, we have $N_i(\lambda^*) \to -\infty$, hence $\nu_i > 0$ for every $i \in S$.

Multiplying (15) through by $(\lambda^* - \bar\lambda)$ and passing to the limit:

$$\sum_{i\in S} p_i^2(T^-)\, \nu_i \;=\; 0.$$

Because $\nu_i > 0$ and $p_i^2 \geq 0$, each term must vanish: $p_i^2(T^-) = 0$ for every $i \in S$. (Since $N_i \to -\infty$ and $p_i^2 \geq 0$, each product $p_i^2 N_i$ is eventually non-positive; combined with the bounded sum (14), each term is individually bounded, ruling out cancellations.) The asymptotic expansion gives $p_i^2(t) = \mathcal{O}(|\lambda^* - \bar\lambda|) = \mathcal{O}(|D_i(\lambda^*(t))|)$, confirming that every singularity in the cluster is removable. The entire cluster S is permanently amputated from the network, cleanly reducing the dimension of the simplex. □

5.3. **The structural snap (root-reset excision).** While the singularity is avoided, the physical consequences of the excision are profound. When channel k is amputated at $t = T$, the secular equation drops a dimension. The network must immediately seek a new coupled root λ^*_{new} that satisfies the reduced equation.

Let $\bar\lambda = \lambda^*(T^-)$ denote the pre-excision eigenvalue and $R(T^-)$ the residual load defined in (15). Because the reduced secular function is strictly decreasing in λ, the direction of the *Discontinuous Structural Snap* is controlled by the sign of the residual:

$$\operatorname{sign}\big(\bar\lambda - \lambda^*_{\text{new}}\big) = \operatorname{sign}\, R(T^-). \tag{16}$$

If $R(T^-) > 0$, the eigenvalue drops ($\lambda^*_{\text{new}} < \bar{\lambda}$); if $R(T^-) < 0$, it jumps upward ($\lambda^*_{\text{new}} > \bar{\lambda}$); if $R(T^-) = 0$ the transition is continuous.

5.4. **The hybrid dynamical system.** The Secular Replicator Flow is naturally formulated as a *hybrid dynamical system* [4] with the following components:

(1) **Continuous evolution.** While all $D_k(\lambda^*(t)) \neq 0$ for $k \in K(t)$, the weights evolve by (4) with $\lambda^*(t)$ the unique simple root in the current inter-pole interval (theorem 2.1). Well-posedness on this segment is guaranteed by theorem 2.2.
(2) **Event condition.** An excision event occurs at the first time T for which $\lambda^*(t)$ reaches a pole value $\bar{\lambda}$. The pole cluster $S = \{k \in K(T^-) : D_k(\bar{\lambda}) = 0\}$ is simultaneously excised. By theorem 5.2, $p_i^2(T^-) = 0$ for all $i \in S$.
(3) **Reset map.** Set $K(T) := K(T^-) \setminus S$ and $p_i^2(T) := 0$ for $i \in S$ (consistent with the excision limit). Recompute λ^*_{new} as the unique root of the reduced secular equation in the appropriate inter-pole interval (section 5.3).
(4) **Restart.** Continuous evolution resumes on the reduced simplex $\Delta^{|K(T)|-1}$ from the post-reset state.

Remark 5.3 (No Zeno behavior). Each excision event removes at least one index from the finite active set $K(t)$. Since $|K(0)| \leq n$, at most $n-1$ events can occur in total, ruling out Zeno-type accumulation of events in finite time. After the final possible excision, the flow is smooth for all subsequent time by theorem 2.2.

5.5. **The sawtooth cascade and global regularity.**

Theorem 5.4 (Unconditional Global Regularity). *Consider the hybrid Secular Replicator Flow of section 5.4 with initial data $w_k(0) > 0$ for all $k \in K(0)$ and a chosen inter-pole branch for $\lambda^*(0)$.*

Then the flow is defined for all $t \geq 0$ and does not exhibit finite-time blowup, for any *value of the coupling energy $c = \sum p_k^2 > 0$. The trajectory generates a* Sawtooth Cascade *governed by event-driven topological shocks:*

(i) ***The Climb.*** *The Variance Law (theorem 3.1) smoothly drives $\lambda^*(t)$ uphill on each smooth segment.*
(ii) ***The Excision.*** *The network approaches a structural resonance, and the Replicator Flow exponentially starves and amputates the dangerous eigenspace (theorem 5.2).*
(iii) ***The Snap.*** *The eigenvalue λ^* shifts discontinuously to a new root of the reduced secular equation; the direction is determined by* $\operatorname{sign} R(T^-)$ *(section 5.3).*
(iv) ***The Resolution.*** *After at most $n-1$ excision events (theorem 5.3), the system enters the super-Perron region and converges smoothly to the variance-zero laminar extreme.*

Proof. The argument proceeds in three stages.

Stage 1 (bounded spectrum). While λ^* navigates inter-pole intervals, every pole singularity is removable by theorem 5.2. Simplicity of λ^* is automatic (theorem 2.1: $f' < -1$). The monotonicity of λ^* ensures it approaches one pole value at a time; the generalised excision lemma handles the entire cluster $S = \{k : D_k(\bar{\lambda}) = 0\}$ at once, regardless of whether channels in S share a base eigenvalue or arise from distinct eigenvalues with coincident poles.

Stage 2 (transition to the super-Perron region). Each excision removes at least one channel, so after at most $n-1$ events the trajectory enters $\mathcal{I}_{\text{open}} = (\rho\varphi, \infty)$, where all $D_k(\lambda^*) > 0$. Parseval's identity gives $\sum_k p_k^2(t) = c$ for all t, and excised channels satisfy $p_k^2(T^-) = 0$ (theorem 5.2), so the surviving energy equals the original c.

Stage 3 (unconditional smoothness in the super-Perron region). In $\mathcal{I}_{\text{open}}$, there are no poles: $D_k > 0$ for every k. By theorem 2.2, the vector field is smooth and locally Lipschitz on the open simplex Δ°; since the boundary faces are forward-invariant by the replicator structure, the trajectory remains in Δ° for all time. The Variance Law gives $\dot{\lambda}^* \geq 0$, and the equilibrium $\lambda^*_{\text{eq}}(c)$ defined by $c\, N_\rho(\lambda) = \lambda$ is *finite* for every finite c. Therefore $\lambda^*(t)$ converges monotonically to $\lambda^*_{\text{eq}}(c)$, and the trajectory exists for all $t \geq 0$. No quantity in the dynamical system—neither the weights, nor λ^*, nor the fitnesses N_k—blows up.

Asymptotic convergence follows from LaSalle's invariance principle [6]: the Variance Law provides a strict Lyapunov function on the compact simplex, so the trajectory converges to the largest invariant set in $\{\mathrm{Var}_w(N) = 0\}$, which consists of equilibria where all active channels share a common fitness. Vertex equilibria (single active channel) are the generic attractors; for generic spectra—where distinct eigenvalues yield distinct fitness at the equilibrium $\lambda^*_{\text{eq}}(c)$—the limit is a unique vertex. □

Remark 5.5 (The $r = 2$ boundary is geometric, not dynamical)*.* The critical threshold $c = 2\rho^2$ does *not* separate regular from singular dynamics. The flow is globally well-posed on both sides. What changes at $\lambda^* = 2\rho$ ($r_{\text{eff}} = 2$) is the *network geometry*: the Chebyshev recurrence transitions from oscillatory to hyperbolic (section 6), so the discrete algebraic self-similarity of the Golden Resolvent is lost. The dynamical system continues smoothly; only its interpretation as a bounded network cascade ceases to apply. The energy threshold $c_{\text{crit}} = 2\rho^2$ (eq. (19)) separates two *qualitative regimes*, not two regularity classes:

- $c < 2\rho^2$: the equilibrium satisfies $r_{\text{eff}} < 2$, and the network retains its oscillatory Chebyshev structure throughout.
- $c \geq 2\rho^2$: the trajectory crosses the $r = 2$ phase boundary, and the network undergoes a geometric phase transition into hyperbolic space—but the Secular Replicator Flow itself remains smooth and globally defined.

6. The geometric phase transition: the r=2 Chebyshev boundary

In section 5, we established unconditional global regularity for the Secular Replicator Flow: the dynamical system is well-posed for all $t \geq 0$ and every $c > 0$. We now characterise the *qualitative* transition in the network geometry that occurs when the coupling energy is large enough to drive λ^* past the Chebyshev boundary $r_{\text{eff}} = 2$.

6.1. **The super-Perron spectrum and the open boundary.** After the final excision event (or from the start, if no poles are encountered), $\lambda^*(t)$ enters the super-Perron region above the highest positive pole $\rho\varphi$. In this region the golden denominators $D_k(\lambda^*) > 0$ for every k, so the vector field is smooth and no further excision events occur. The Variance Law ($\dot{\lambda}^* \geq 0$) drives the macroscopic eigenvalue steadily upward toward the finite equilibrium $\lambda^*_{\text{eq}}(c)$. The question is not whether the flow remains regular—it does, unconditionally (theorem 5.4)—but what happens to the *network's geometric structure* along the way.

6.2. **The macroscopic diagnostic and the Chebyshev limit.** To characterise the geometric regime of the network, we define the macroscopic effective amplification ratio of the network relative to the Perron root ρ (cf. section 2.1):

$$r_{\text{eff}}(t) = \frac{\lambda^*(t)}{\rho}. \tag{17}$$

The entire algebraic architecture of the Golden Resolvent—and its ability to sustain an oscillatory, discrete structure—relies on the Chebyshev recurrence bounds. The system maintains its additive self-similarity and topological integrity only as long as it remains within the oscillatory margin:

$$|r_{\text{eff}}(t)| \leq 2 - \epsilon. \tag{18}$$

To see why $r = 2$ is the critical threshold, recall that the Chebyshev ladder of [11] rests on the second-order recurrence $x_{t+1} = r\,x_t - x_{t-1}$, whose characteristic polynomial $z^2 - rz + 1 = 0$ has roots

$$\alpha_\pm = \frac{r \pm \sqrt{r^2 - 4}}{2}.$$

For $|r| < 2$ the discriminant is negative, the roots are complex conjugate on the unit circle, and the recurrence oscillates with bounded amplitude. At $|r| = 2$ the roots collide at a double real root, producing linear growth. For $|r| > 2$ the roots are real and reciprocal, and the recurrence grows exponentially—the discrete oscillatory structure is irreversibly lost.

Because the Variance Law increases $\lambda^*(t)$, the effective ratio $r_{\text{eff}}(t)$ also increases. If $c \geq 2\rho^2$, the equilibrium $\lambda^*_{\text{eq}}(c)$ lies at or beyond 2ρ, and the trajectory smoothly crosses the $r_{\text{eff}} = 2$ boundary.

6.3. **The oscillatory–hyperbolic transition at $r=2$.** We now formalise the conditions under which the geometric phase boundary is crossed. Define the *super-Perron invariant region* $\mathcal{I}_{\text{open}} = (\rho\varphi, \infty)$. When $\lambda^* \in \mathcal{I}_{\text{open}}$, all golden denominators are strictly positive ($D_k(\lambda^*) > 0$ for every k), so the flow is smooth and free of excision events.

Theorem 6.1 (Hyperbolic Phase Escape)**.** *Suppose the macroscopic eigenvalue enters the super-Perron region $\mathcal{I}_{\text{open}}$. By theorem 2.1 the trajectory is smooth, and the Variance Law drives $\lambda^*(t)$ monotonically toward the single-channel vertex equilibrium $\lambda^*_{\text{eq}}(c)$ defined by $c\,N_\rho(\lambda) = \lambda$.*

The critical threshold admits a closed form. Evaluating the fitness at $\lambda = 2\rho$:

$$N_\rho(2\rho) = \frac{2\rho - \rho}{(2\rho)^2 - 2\rho^2 - \rho^2} = \frac{\rho}{\rho^2} = \frac{1}{\rho}.$$

The equilibrium equation $c\,N_\rho(2\rho) = 2\rho$ then gives $c = 2\rho/(1/\rho) = 2\rho^2$, i.e. the explicit critical energy

$$c_{\text{crit}} = 2\rho^2. \tag{19}$$

If $c \geq c_{\text{crit}}$, there exists a time T_{esc} (finite or asymptotic) at which $\lambda^*(T_{\text{esc}}) = 2\rho$. As $t \to T_{\text{esc}}^-$, the oscillatory margin $\epsilon(t) = 2 - r_{\text{eff}}(t) \to 0^+$, and the Chebyshev amplitude bound diverges: the oscillatory envelope of the recurrence $x_{t+1} = r\,x_t - x_{t-1}$ scales as

$$\big(4 - r_{\text{eff}}^2\big)^{-1/2} = \big((2 - r_{\text{eff}})(2 + r_{\text{eff}})\big)^{-1/2} \xrightarrow{r_{\text{eff}} \to 2^-} +\infty.$$

At the boundary $\lambda^* = 2\rho$, the discriminant of the Chebyshev recurrence vanishes ($r^2 - 4 = 0$) and the discrete additive self-similarity of the network collapses. The dynamical system itself remains smooth (theorem 5.4); only the oscillatory network geometry is lost.

Proof. In $\mathcal{I}_{\text{open}}$, all $D_k(\lambda) > 0$ strictly, so no excision events occur and the trajectory is smooth (theorem 2.2). The Variance Law (theorem 3.1) gives $\dot{\lambda}^* \geq 0$, with equality only at the laminar equilibrium. The closed-form $c_{\text{crit}} = 2\rho^2$ follows directly from $c = \lambda/N_\rho(\lambda)$ evaluated at $\lambda = 2\rho$. If $c \geq c_{\text{crit}}$, the equilibrium lies at or beyond 2ρ. Because the flow is monotonic and continuous, the Intermediate Value Theorem yields T_{esc} with $\lambda^*(T_{\text{esc}}) = 2\rho$. The amplitude divergence follows from the Chebyshev recurrence analysis item (GRT3): the oscillatory bound scales as $(4 - r^2)^{-1/2}$, which diverges as $r \to 2^-$. □

We classify this transition not as a dynamical singularity, but as a *geometric phase escape*: the discrete algebraic structure of the network undergoes a phase transition from bounded cyclotomic geometry into unbounded hyperbolic space. The Secular Replicator Flow itself continues smoothly beyond 2ρ, converging to the finite equilibrium $\lambda^*_{\text{eq}}(c)$; what is lost is the network's capacity to sustain bounded oscillatory cascades.

7. Conclusion

The Navier–Stokes Millennium Prize has driven mathematicians to search for singularities within the continuous spatial geometry of partial differential equations. By abstracting the fluid cascade into an algebraic shell model—the Secular Replicator Flow—we bypass the geometric difficulties of PDEs and expose the topological mechanics of singularity formation in a tractable setting.

By mapping incompressibility to Parseval simplex conservation, nonlocal pressure to the Secular Equation, and nonlinear advection to Fisher's Replicator Dynamics, we have provided a rigorously solvable, finite-dimensional surrogate for fluid turbulence. Within this framework, we proved *unconditional global regularity*: the Secular Replicator Flow is defined for all $t \geq 0$ and every coupling energy $c > 0$, with no finite-time blowup. The trajectory forms a Sawtooth Cascade of smooth climbs and discrete excision events, converging to a variance-zero laminar equilibrium.

A single geometric parameter—the critical energy $c_{\text{crit}} = 2\rho^2$—separates two qualitative regimes:

Oscillatory regime ($c < 2\rho^2$).: The network retains its discrete Chebyshev structure throughout. The Replicator Flow self-corrects via excision, and the equilibrium lies within the bounded oscillatory geometry.

Hyperbolic regime ($c \geq 2\rho^2$).: The trajectory crosses the $r = 2$ phase boundary, and the network's discrete algebraic structure transitions to hyperbolic geometry. The dynamical system remains smooth—only the geometric interpretation as a bounded cascade is lost.

The Secular Replicator Flow establishes that turbulence is not a chaotic plunge toward an infinite velocity point, but a highly ordered, algorithmic filter. The network actively amputates its own dimensions to conform to its macroscopic boundaries.

Future extensions of this algebraic shell model suggest natural interdisciplinary applications. Because the Golden Resolvent natively captures the spectral interference of symmetric matrices, the Secular Replicator Flow is immediately applicable to the study of localisation and thermalisation in quantum Hamiltonian networks, as well as the cascading mechanics of time-reversal symmetric (quaternionic) physical systems. By translating the language of calculus into the algebraic symmetries of Galois orbits, we have proposed a new framework for the study of flow, structure, and singularity.

References

[1] C. L. Fefferman, *Existence and smoothness of the Navier–Stokes equation*, The Millennium Prize Problems, Clay Math. Inst., 2006, pp. 57–67.

[2] R. A. Fisher, *The Genetical Theory of Natural Selection*, Clarendon Press, Oxford, 1930.

[3] E. B. Gledzer, *System of hydrodynamic type admitting two quadratic integrals of motion*, Soviet Phys. Dokl. **18** (1973), 216–217.

[4] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton Univ. Press, 2012.
[5] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge Univ. Press, 1998.
[6] H. K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, 2002.
[7] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
[8] V. S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, *Improved shell model of turbulence*, Phys. Rev. E **58** (1998), no. 2, 1811–1822.
[9] K. Ohkitani and M. Yamada, *Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence*, Progr. Theoret. Phys. **81** (1989), no. 2, 329–341.
[10] L. F. Richardson, *Weather Prediction by Numerical Process*, Cambridge Univ. Press, 1922.
[11] V. F. S. Santos, *Golden resolvent theory: spectral factorisation, Galois orbits, and Chebyshev ladders over* $\mathbb{Q}(\sqrt{5})$, viXra:2602.0094, February 2026. Available at `https://ai.vixra.org/pdf/2602.0094v1.pdf`.