GOLDEN RESOLVENT THEORY: SPECTRAL FACTORISATION, GALOIS ORBITS, AND CHEBYSHEV LADDERS OVER $\mathbb{Q}(\sqrt{5})$

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Abstract. We develop the spectral theory of operators whose eigenvalue structure is governed by the golden ratio $\varphi = (1+\sqrt{5})/2$. The foundation is the *golden resolvent factorisation*: for any real symmetric matrix A,

$$\lambda^2 I - \lambda A - A^2 \;=\; (\lambda I - \varphi A)\,(\lambda I + A/\varphi).$$

This identity controls the spectrum of *golden companion operators*, decomposing eigenvalues into *transparent* pairs ($\mu\varphi$ and $-\mu/\varphi$) and *coupled* modes (roots of an explicit secular equation), governed by the Galois conjugation $\varphi \mapsto -\varphi^{-1}$ of $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$.

We establish six main results: (1) the *Golden Amplification Theorem*, producing the transparent eigenvalue pairs and their eigenvectors; (2) the *Secular Equation*, a closed-form characteristic polynomial for coupled modes; (3) the *Secular Sensitivity Theorem*, identifying the secular weight with the coupled eigenvector norm and establishing Lipschitz continuity of coupled eigenvalues in the coupling vector; (4) the *Positive Boost Inequality*, bounding submatrix spectral radii via nodal-domain restrictions; (5) the *Galois Transfer Principle*, exactly classifying partition transfer for conjugate eigenvector pairs via the Transfer/Pareto Trichotomy—with automatic spectral dominance for transparent modes and unavoidable Pareto regimes for coupled modes; (6) the *Chebyshev Ladder*, generalising the amplification ratio to $2\cos\bigl(\pi/(2p+3)\bigr)$ through the cyclotomic fields $\mathbb{Q}(\zeta_{2p+3})^+$. The spectral spread of the golden pair equals the generator of the different ideal of $\mathbb{Z}[\varphi]$, connecting the framework to the arithmetic of $\mathbb{Q}(\sqrt{5})$.

1. Introduction

1.1. **Motivation.** Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and orthonormal eigenbasis $v_1, \ldots, v_n$. Each eigenvector defines a natural partition of $\{1, \ldots, n\}$ via its sign pattern, and the eigenvalues control the spectral density of the resulting submatrices. This paper develops the theory of a specific class of operators where

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this eigenvalue–partition interplay is governed by the golden ratio $\varphi = (1+\sqrt{5})/2$.

The starting point is an elementary identity.

Theorem 1.1 (Golden Resolvent Factorisation)**.** *For any $A \in \mathbb{R}^{n\times n}$,*

$$\lambda^2 I - \lambda A - A^2 = (\lambda I - \varphi A)(\lambda I + A/\varphi). \tag{1}$$

This factorisation is trivial to verify, yet its consequences are far-reaching. It implies that any operator whose eigenvalue equation reduces to $\lambda^2 - \lambda\mu - \mu^2 = 0$ (for eigenvalue μ of A) automatically produces eigenvalues $\mu\varphi$ and $-\mu/\varphi$—a *golden amplification* of the original spectrum. The Galois conjugation $\varphi \mapsto -\varphi^{-1}$, the unique nontrivial automorphism of $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$, exchanges these two eigenvalues.

We identify a natural class of block-structured operators—*golden companion operators*—whose spectra are controlled by (1), and develop their spectral theory systematically. The central thesis is that the arithmetic of $\mathbb{Q}(\sqrt{5})$—the minimal field containing φ—governs the spectral fine structure of these operators, from eigenvalue interlacing to partition certification.

1.2. **Main results.** The contributions of this paper are:

(1) The **Golden Resolvent Factorisation** (Theorem 2.1): the matrix identity (1) and its spectral consequences, including the golden companion matrix.
(2) The **Golden Amplification Theorem** (Theorem 3.1): for any block operator with golden companion structure, every eigenvector of A orthogonal to the coupling vector produces eigenvalues $\mu\varphi$ and $-\mu/\varphi$.
(3) The **Secular Equation** (Theorem 4.1): a closed-form expression for the characteristic polynomial of the golden companion operator, decomposing into transparent and coupled contributions. The **Complementarity Principle** (Theorem 4.4, Theorem 4.5) characterises partition effectiveness of each type. The **Secular Sensitivity Theorem** (Theorem 4.8) gives explicit formulas for the response of coupled eigenvalues to perturbations of the spectral participations, establishing that the spectrum of $M(A, e)$ is Lipschitz-continuous in the coupling vector e (Theorem 4.13).
(4) The **Positive Boost Inequality** (Theorem 5.2): restricting to the positive nodal domain of an eigenvector at eigenvalue λ yields a submatrix with spectral radius $\geq \lambda +$ boost.
(5) The **Galois Transfer Principle** (Section 6): the Transfer/Pareto Trichotomy (Theorem 6.11) exactly classifies partition transfer for every Galois-conjugate eigenvector pair. For transparent modes, the Edge Internalization Lemma (Theorem 6.21) provides theorem-level spectral dominance for the most-positive conjugate (the Positive-Wins Law, Theorem 6.25); certification transfer is then obtained when side-wise chromatic dominance is verified (Theorem 6.28). For

coupled modes, transfer is conditional on cut-stability criteria characterised by spectral certificates (Theorem 6.29) and Perron-dominance bounds (Theorem 6.31), and a counterexample (Theorem 6.35) shows that the Pareto regime is unavoidable. Three conjectural *structural laws*—the Strong Positive-Wins Law, the Monotone Filtration, and the Redundancy–Degree Law—govern which orbit members produce effective partitions, supported by computational evidence.

(6) The **Chebyshev Ladder** (Theorem 7.2): for operators with p intermediate layers, the amplification ratio generalises to $2\cos\big(\pi/(2p+3)\big)$, ascending through cyclotomic fields.

1.3. **Organisation.** Section 2 proves the golden resolvent factorisation and introduces the golden companion operator. Section 3 proves the Golden Amplification Theorem. Section 4 derives the secular equation, the complementarity principle, and the sensitivity analysis of coupled eigenvalues under probe perturbation. Section 5 proves the Positive Boost Inequality. Section 6 develops the Galois Transfer Principle: the certification framework and Transfer/Pareto Trichotomy, the cut-stability principle, the Edge Internalization Lemma and the Positive-Wins Theorem for transparent modes, the conditional transfer and Pareto classification for coupled modes, and the three structural laws. Section 7 establishes the Chebyshev Ladder and its hyperbolic continuation. Section 8 develops the stability analysis of time-varying operator families (secular-weight barriers, dynamic Pareto switching) and analyses iterated operators and degree growth. Section 9 develops the abstract sweep framework. Section 10 introduces the arithmetic invariants: the Galois centroid, shell depth bound, spectral spread, and connection to the different ideal of $\mathbb{Z}[\varphi]$. Section 11 discusses applications and open problems.

2. The Golden Resolvent Factorisation

Theorem 2.1 (Golden Resolvent Factorisation)**.** *For any matrix $A \in \mathbb{R}^{n\times n}$ and any scalar λ,*

$$\lambda^2 I - \lambda A - A^2 = (\lambda I - \varphi A)(\lambda I + A/\varphi), \tag{2}$$

where $\varphi = (1+\sqrt{5})/2$.

Proof. Expand the right side:

$$\begin{aligned}(\lambda I - \varphi A)(\lambda I + A/\varphi) &= \lambda^2 I + \lambda A/\varphi - \varphi\lambda A - A^2\\ &= \lambda^2 I + \lambda A(1/\varphi - \varphi) - A^2\\ &= \lambda^2 I - \lambda A - A^2,\end{aligned}$$

using $\varphi - 1/\varphi = 1$, which is the defining property $\varphi^2 = \varphi + 1$. □

Corollary 2.2. *If $Ax = \mu x$ for a nonzero vector x, then $(\lambda^2 - \lambda\mu - \mu^2)x = 0$ has roots $\lambda = \mu\varphi$ and $\lambda = -\mu/\varphi$.*

Remark 2.3 (Galois duality). The Galois conjugation $\sigma\colon \varphi \mapsto -\varphi^{-1}$, the unique nontrivial automorphism of $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$, exchanges the two factors: σ sends $(\lambda I - \varphi A)$ to $(\lambda I + A/\varphi)$ and vice versa. This duality will control which eigenvectors produce effective partitions (Section 6).

Definition 2.4 (Golden companion operator)**.** Let $A \in \mathbb{R}^{n\times n}$ be real symmetric and $e \in \mathbb{R}^n$ a nonzero vector. The *golden companion operator* of (A, e) is the $(2n+1) \times (2n+1)$ real symmetric matrix

$$M(A, e) = \begin{pmatrix} A & A & 0 \\ A & 0 & e \\ 0^\top & e^\top & 0 \end{pmatrix}. \tag{3}$$

We call the three blocks the *original layer* (rows $1, \ldots, n$), the *shadow layer* (rows $n+1, \ldots, 2n$), and the *apex* (row $2n+1$).

Example 2.5 (Running example: K_2)**.** Let $G = K_2$ with adjacency matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, eigenvalues $\mu_1 = 1$, $\mu_2 = -1$, and eigenvectors $v_1 = \frac{1}{\sqrt{2}}(1,1)^\top$, $v_2 = \frac{1}{\sqrt{2}}(1,-1)^\top$. Take $e = (1,1)^\top = \sqrt{2}\, v_1$, so $p_1 = \sqrt{2}$ and $p_2 = 0$. Thus $\mu_2 = -1$ is *transparent* and $\mu_1 = 1$ is *coupled*. The companion $M(K_2, \mathbf{1})$ is 5×5; Theorem 3.1 will give transparent eigenvalues $\{-\varphi, 1/\varphi\}$ (from μ_2), and Theorem 4.1 will yield the single coupled secular root. We revisit this example throughout the paper.

Remark 2.6. When A is the adjacency matrix of a graph G and $e = \mathbf{1}$ is the all-ones vector, $M(A, \mathbf{1})$ is the adjacency matrix of the Mycielskian $M(G)$ of Mycielski [6]. The theory developed here applies to arbitrary (A, e).

3. The Golden Amplification Theorem

Theorem 3.1 (Golden Amplification)**.** *Let $A \in \mathbb{R}^{n\times n}$ be real symmetric with eigenpair (μ, x): $Ax = \mu x$. Suppose $e^\top x = 0$. Then $M(A, e)$ has eigenvalues $\mu\varphi$ and $-\mu/\varphi$, with eigenvectors*

$$w_+ = \begin{pmatrix} x \\ x/\varphi \\ 0 \end{pmatrix}, \qquad w_- = \begin{pmatrix} x \\ -\varphi x \\ 0 \end{pmatrix}. \tag{4}$$

The multiplicity of each eigenvalue is $\dim(E_\mu \cap e^\perp)$*, where E_μ is the eigenspace of μ.*

Proof. Seek an eigenvector of $M(A, e)$ of the form $(x, \alpha x, 0)^\top$ at eigenvalue λ. The block equation (3) gives:

$$\text{Row 0:} \quad Ax + A(\alpha x) = \mu(1+\alpha)x = \lambda x, \tag{5}$$

$$\text{Row 1:} \quad Ax + 0 \cdot e = \mu x = \lambda \alpha x, \tag{6}$$

$$\text{Row 2:} \quad e^\top(\alpha x) = \alpha(e^\top x) = 0 = \lambda \cdot 0. \tag{7}$$

From (5): $\lambda = \mu(1+\alpha)$. From (6): $\lambda\alpha = \mu$, so $\alpha = \mu/\lambda$. Substituting:

$$\lambda = \mu\Big(1 + \frac{\mu}{\lambda}\Big) \implies \lambda^2 = \mu\lambda + \mu^2 \implies \Big(\frac{\lambda}{\mu}\Big)^2 - \frac{\lambda}{\mu} - 1 = 0.$$

The roots are $\lambda/\mu = \varphi$ and $\lambda/\mu = -\varphi^{-1}$, giving $\lambda = \mu\varphi$ (with $\alpha = 1/\varphi$) and $\lambda = -\mu/\varphi$ (with $\alpha = -\varphi$). □

Corollary 3.2 (Universality of $\mathbb{Q}(\sqrt{5})$)**.** *If A has at least one nonzero eigenvalue μ with* $\dim(E_\mu \cap e^\perp) \geq 1$*, then* $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}\big(\mathrm{Spec}(M(A,e))\big)$*.*

Proof. Let

$$\alpha := \mu\varphi, \qquad \beta := -\mu/\varphi.$$

By Theorem 3.1, both α, β are eigenvalues of $M(A,e)$, so $\alpha, \beta \in \mathbb{Q}\big(\mathrm{Spec}(M(A,e))\big)$. Since $\mu \neq 0$ by hypothesis,

$$\alpha + \beta = \mu\left(\varphi - \frac{1}{\varphi}\right) = \mu, \qquad \alpha - \beta = \mu\left(\varphi + \frac{1}{\varphi}\right) = \mu\sqrt{5}.$$

Hence

$$\sqrt{5} = \frac{\alpha - \beta}{\alpha + \beta} \in \mathbb{Q}\big(\mathrm{Spec}(M(A,e))\big),$$

so $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}\big(\mathrm{Spec}(M(A,e))\big)$. □

Definition 3.3 (Transparent and coupled eigenspaces)**.** Let A have eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ with orthonormal eigenvectors $v_1, \ldots, v_n$. The *spectral participation* of the i-th eigenspace is $p_i = e^\top v_i$. An eigenspace is *transparent* if $p_i = 0$ and *coupled* if $p_i \neq 0$. Let q denote the number of coupled eigenspaces. By Parseval's identity, $\sum_i p_i^2 = \|e\|^2$.

3.1. **Covariance and transfer tools.**

Proposition 3.4 (Basis-covariant participation and transparency)**.** *Let $U \in O(n)$, and define*

$$A' = U^\top A U, \qquad e' = U^\top e.$$

Then:

(i) If $Ax = \mu x$, then $A'(U^\top x) = \mu(U^\top x)$.
(ii) Participation is invariant:

$$(e')^\top (U^\top x) = e^\top x.$$

Hence transparent/coupled labels are basis-invariant.

(iii) With $\widehat{U} = \mathrm{diag}(U, U, 1)$,

$$M(A', e') = \widehat{U}^\top M(A,e)\, \widehat{U}.$$

Therefore $\mathrm{Spec}(M(A',e')) = \mathrm{Spec}(M(A,e))$.

Proof. Part (i): $A'(U^\top x) = U^\top AUU^\top x = U^\top Ax = \mu U^\top x$. Part (ii): $(e')^\top(U^\top x) = e^\top UU^\top x = e^\top x$. Part (iii) is a direct block multiplication:

$$\widehat{U}^\top \begin{pmatrix} A & A & 0 \\ A & 0 & e \\ 0^\top & e^\top & 0 \end{pmatrix} \widehat{U} = \begin{pmatrix} U^\top AU & U^\top AU & 0 \\ U^\top AU & 0 & U^\top e \\ 0^\top & e^\top U & 0 \end{pmatrix} = M(A', e').$$

□

Corollary 3.5 (Automorphism invariance of the golden companion spectrum)**.** *Let* $\mathrm{Aut}(A) = \{U \in O(n) : U^\top AU = A\}$ *be the orthogonal automorphism group of* A*. Then*

$$\mathrm{Spec}\big(M(A, Ue)\big) \;=\; \mathrm{Spec}\big(M(A, e)\big) \qquad \text{for all } U \in \mathrm{Aut}(A). \tag{8}$$

In particular, the spectral participations satisfy $p_i(Ue)^2 = p_i(e)^2$ *for all* i*, and the transparent/coupled partition is* $\mathrm{Aut}(A)$*-invariant.*

Proof. Set $A' = U^\top AU = A$ and $e' = U^\top e$ in Theorem 3.4(iii): $M(A, U^\top e)$ is orthogonally similar to $M(A, e)$, hence isospectral. For the participation claim: $(U^\top e)^\top v_i = e^\top Uv_i$, and since $U \in \mathrm{Aut}(A)$, the map $v_i \mapsto Uv_i$ permutes the eigenbasis, so the multiset $\{p_i^2\}$ is invariant. □

Remark 3.6 (Probe redundancy). Theorem 3.5 makes precise a notion of *probe redundancy*: two coupling vectors e_1, e_2 in the same $\mathrm{Aut}(A)$-orbit produce identical spectra, identical participation weights, and identical transparent/coupled decompositions. Any spectral observable derived from $M(A, e)$ is therefore $\mathrm{Aut}(A)$-invariant in the probe argument. For operators with large automorphism groups (e.g. adjacency matrices of vertex-transitive graphs, where $\mathrm{Aut}(A)$ acts transitively on unit vectors of constant type), the orbit of e may exhaust the admissible probes, making all probe choices *exactly* equivalent.

Proposition 3.7 (Degenerate-eigenspace transparency criterion)**.** *Let* $E_\mu = \ker(A - \mu I)$ *with* $m = \dim E_\mu$*, and let* P_μ *be the orthogonal projector onto* E_μ*.*

(i) E_μ *is fully transparent iff* $P_\mu e = 0$*.*
(ii) If $P_\mu e \neq 0$*, then*

$$E_\mu = \mathrm{span}(u_\mu) \oplus (E_\mu \cap e^\perp), \qquad u_\mu = \frac{P_\mu e}{\|P_\mu e\|},$$

so $\dim(E_\mu \cap e^\perp) = m - 1$ *and the coupled subspace inside* E_μ *is one-dimensional.*

Proof. For $x \in E_\mu$, one has $e^\top x = (P_\mu e)^\top x$ because $x = P_\mu x$. Hence $e^\top x = 0$ for all $x \in E_\mu$ iff $P_\mu e = 0$, proving (i). If $P_\mu e \neq 0$, the functional $x \mapsto e^\top x$ on E_μ has rank 1, represented by $P_\mu e$, so its kernel is $E_\mu \cap e^\perp$ of codimension 1. This gives the orthogonal decomposition in (ii). □

Theorem 3.8 (Operator-family transfer under joint diagonalisation)**.** *Let $\{A^{(s)}\}_{s\in S}$ be a commuting family of real symmetric $n \times n$ matrices, and fix $e \in \mathbb{R}^n \setminus \{0\}$. Then there exists an orthonormal basis $\{v_i\}_{i=1}^n$ such that*

$$A^{(s)} v_i = \mu_i^{(s)} v_i \quad \textit{for all } s \in S.$$

Define $p_i = e^\top v_i$. Then:

(i) The transparent index set

$$T = \{i : p_i = 0\}$$

is independent of s.

(ii) For each $s \in S$ and each $i \in T$, $M(A^{(s)}, e)$ has transparent eigenvalues $\mu_i^{(s)}\varphi$ and $-\mu_i^{(s)}/\varphi$.

(iii) Coupled eigenvalues of $M(A^{(s)}, e)$ satisfy the secular equation

$$\sum_{i=1}^{n} \frac{p_i^2(\lambda - \mu_i^{(s)})}{\lambda^2 - \lambda\mu_i^{(s)} - (\mu_i^{(s)})^2} = \lambda.$$

Thus, under operator-family switches inside a jointly diagonalizable family, mode typing and participation weights transfer, while eigenvalue locations change through $\mu_i^{(s)}$.

Proof. Commuting real symmetric matrices are simultaneously orthogonally diagonalizable, giving the common basis $\{v_i\}$. The scalars $p_i = e^\top v_i$ therefore do not depend on s, proving (i). Part (ii) is Theorem 3.1 applied to each $A^{(s)}$, and part (iii) is Theorem 4.1 with eigenpairs $(\mu_i^{(s)}, v_i)$. □

Proposition 3.9 (Zero-mode decoupling criterion)**.** *Let $E_0 = \ker A$ and P_0 its orthogonal projector. The following are equivalent:*

(i) $P_0 e = 0$ (all zero modes are transparent).

(ii) $\mathcal{N} := E_0 \oplus E_0 \oplus \{0\}$ is contained in $\ker M(A, e)$.

If these hold, all zero-mode contributions are spectrally decoupled (they contribute only $\lambda = 0$ via transparent modes). If $P_0 e \neq 0$, the zero-mode sector is apex-coupled and not decoupled.

Proof. Take $(x, y, 0) \in \mathcal{N}$ with $x, y \in E_0$. From (3),

$$M(A, e)(x, y, 0) = (Ax + Ay,\ Ax,\ e^\top y) = (0, 0, e^\top y).$$

Hence $(x, y, 0) \in \ker M(A, e)$ for all $x, y \in E_0$ iff $e^\top y = 0$ for all $y \in E_0$, i.e. iff $P_0 e = 0$. This proves equivalence.

When $P_0 e = 0$, each $x \in E_0$ is transparent, so Theorem 3.1 gives the pair $\{0, 0\}$ (no nonzero branch). If $P_0 e \neq 0$, choose $y \in E_0$ with $e^\top y \neq 0$; then $(0, y, 0)$ maps to a nonzero apex component, so the zero-mode sector couples to the secular channel. □

Remark 3.10 (Regular case). When A is the adjacency matrix of a k-regular graph and $e = \mathbf{1}$, the vector $\mathbf{1}$ is the Perron eigenvector of A, so every non-Perron eigenspace is transparent. The Golden Amplification Theorem then applies to every $\mu \neq k$ with full multiplicity.

4. The Secular Equation

Theorem 4.1 (Secular equation). *Let $A \in \mathbb{R}^{n\times n}$ be real symmetric with eigenvalues $\mu_1, \ldots, \mu_n$ and spectral participations $p_i = e^\top v_i$. The characteristic polynomial of $M(A,e)$ is*

(9)
$$\det\bigl(\lambda I - M(A,e)\bigr) = \lambda \prod_{i=1}^{n}(\lambda^2 - \lambda\mu_i - \mu_i^2) - \sum_{i=1}^{n} p_i^2(\lambda - \mu_i)\prod_{j\neq i}(\lambda^2 - \lambda\mu_j - \mu_j^2).$$

Equivalently, the eigenvalues of $M(A,e)$ with nonzero apex component satisfy the secular equation*:*

$$\sum_{i=1}^{n} \frac{p_i^2(\lambda - \mu_i)}{\lambda^2 - \lambda\mu_i - \mu_i^2} = \lambda. \tag{10}$$

Proof. Write the eigenvector of $M(A,e)$ at eigenvalue λ as $(x, y, z)^\top$ with $z \neq 0$; normalise $z = 1$. Expand $x = \sum_i a_i v_i$ and $y = \sum_i b_i v_i$ in the eigenbasis of A. The block equation (3) gives, for each i:

$$\mu_i(a_i + b_i) = \lambda a_i, \tag{11}$$
$$\mu_i a_i + p_i = \lambda b_i, \tag{12}$$
$$\sum_i p_i b_i = \lambda. \tag{13}$$

Solving (11)–(12) yields $a_i = p_i\mu_i/(\lambda^2 - \lambda\mu_i - \mu_i^2)$ and $b_i = p_i(\lambda - \mu_i)/(\lambda^2 - \lambda\mu_i - \mu_i^2)$. Substituting into (13) gives (10). Clearing denominators yields (9), a polynomial of degree $2n+1$, the order of $M(A,e)$. ☐

Corollary 4.2 (Recovery of the transparent case). *If A has a unique coupled eigenvalue μ_1 with $p_1^2 = \|e\|^2$ and all other eigenspaces transparent, then* (10) *reduces to a single-term equation:*

$$\frac{\|e\|^2(\lambda - \mu_1)}{\lambda^2 - \lambda\mu_1 - \mu_1^2} = \lambda. \tag{14}$$

When A is the adjacency matrix of a k-regular graph on n vertices (so $\mu_1 = k$, $\|e\|^2 = n$), this becomes the cubic

$$\lambda^3 - k\lambda^2 - (k^2 + n)\lambda + kn = 0. \tag{15}$$

Remark 4.3 (Golden resolvent form). Using the factorisation $\lambda^2 - \lambda\mu - \mu^2 = (\lambda - \mu\varphi)(\lambda + \mu/\varphi)$ from Theorem 2.1, the secular equation takes the resolvent form

$$\lambda = e^\top(\lambda I - A)\,(\lambda I - \varphi A)^{-1}\,(\lambda I + A/\varphi)^{-1}\,e, \tag{16}$$

expressing the coupled eigenvalues as evaluations of the *golden resolvent* of A at the coupling vector e.

4.1. **The Complementarity Principle.** The secular equation decomposes $\mathrm{Spec}\big(M(A,e)\big)$ into transparent eigenvalues (golden pairs from Theorem 3.1) and coupled eigenvalues (roots of (10)). Their effectiveness for eigenvector partitions is complementary.

Proposition 4.4 (Coupled eigenvectors under symmetry)**.** *Suppose $Ae = \kappa e$ for some scalar κ (i.e., e is an eigenvector of A). Then every coupled eigenvector of $M(A,e)$, restricted to the original layer, is proportional to e. In particular, the coupled restriction cannot distinguish among the coordinates of e.*

Proof. In the block eigenvalue equation, the restriction of a coupled eigenvector to the original layer is proportional to $(\lambda I - A)^{-1}e$. Since $Ae = \kappa e$, we have $(\lambda I - A)^{-1}e = (\lambda - \kappa)^{-1}e$, which is proportional to e. □

Proposition 4.5 (Coupled eigenvectors detect density)**.** *Suppose $e \notin \bigcup_i E_{\mu_i}$ (i.e., e is not an eigenvector of A). Let λ be the dominant coupled eigenvalue of $M(A,e)$. The restriction of the corresponding eigenvector to the original layer is proportional to the* resolvent centrality *vector*

$$c = (\lambda I - A)^{-1}e, \tag{17}$$

which assigns larger values to coordinates in denser substructures of A.

Proof. Same calculation as Theorem 4.4, but now $(\lambda I - A)^{-1}e$ is not proportional to e since e projects onto multiple eigenspaces. The resolvent $(\lambda I - A)^{-1}$ weights eigenspace i by $(\lambda - \mu_i)^{-1}$; for λ near $\mu_{\max}$, eigenspaces with large eigenvalues dominate, and the resulting vector concentrates on coordinates participating in dense substructures. □

Remark 4.6 (Complementarity summary)*.*

	Coupled on original layer	Transparent on original layer
e is eigenvector of A	Constant (uninformative)	Sorts by nodal topology
e is not eigenvector	Sorts by resolvent centrality	Often noisy/localised

The two types are complementary: when one is uninformative, the other is effective.

4.2. **Sensitivity analysis of the secular equation.** The secular equation (Theorem 4.1) determines coupled eigenvalues as implicit functions of the spectral participations $p_i^2 = (e^\top v_i)^2$. The next results develop the sensitivity theory: response of coupled eigenvalues to perturbations of the participation weights, and hence to perturbations of the coupling vector e.

Definition 4.7 (Golden denominators and secular function)**.** For an eigenvalue μ_i of A and a scalar λ, define the *golden denominator*

$$D_i(\lambda) \;=\; \lambda^2 - \lambda\mu_i - \mu_i^2 \;=\; (\lambda - \mu_i\varphi)(\lambda + \mu_i/\varphi), \tag{18}$$

so that the secular equation (10) reads

$$F(\lambda, \mathbf{p}^2) \;:=\; \sum_{i=1}^{n} \frac{p_i^2(\lambda - \mu_i)}{D_i(\lambda)} \;-\; \lambda \;=\; 0, \tag{19}$$

where $\mathbf{p}^2 = (p_1^2, \ldots, p_n^2)$.

Theorem 4.8 (Secular sensitivity)**.** *Let λ_* be a simple coupled eigenvalue of $M(A, e)$ (i.e., a simple root of (19)).*

(i) ***Participation sensitivity.*** *For each eigenspace index k,*

$$\frac{\partial \lambda_*}{\partial (p_k^2)} \;=\; \frac{N_k(\lambda_*)}{W(\lambda_*)}, \tag{20}$$

where the numerator *is*

$$N_k(\lambda) \;=\; \frac{\lambda - \mu_k}{D_k(\lambda)}, \tag{21}$$

and the secular weight *is*

$$W(\lambda) \;=\; 1 \;+\; \sum_{i=1}^{n} p_i^2 \, \frac{(\lambda - \mu_i)^2 + \mu_i^2}{D_i(\lambda)^2}. \tag{22}$$

(ii) ***Simplicity criterion.*** *The coupled eigenvalue λ_* is simple if and only if $W(\lambda_*) \neq 0$.*

(iii) ***Total derivative under probe perturbation.*** *If $e = e(t)$ is a smooth one-parameter family of coupling vectors with $\|e(t)\|^2 \equiv c$ (constant norm), then*

$$\frac{d\lambda_*}{dt} \;=\; \frac{1}{W(\lambda_*)} \sum_{k=1}^{n} \frac{\lambda_* - \mu_k}{D_k(\lambda_*)} \, \frac{d(p_k^2)}{dt}, \tag{23}$$

where the participation rates satisfy $\sum_k \frac{d(p_k^2)}{dt} = 0$ (conservation from Parseval).

Proof. Define $F(\lambda, \mathbf{p}^2)$ as in (19). Since λ_* is a simple root, $\partial F / \partial \lambda \neq 0$ at $(\lambda_*, \mathbf{p}^2)$, and the implicit function theorem applies.

Part (i). The partial derivative with respect to p_k^2 is immediate:

$$\frac{\partial F}{\partial (p_k^2)} \;=\; \frac{\lambda_* - \mu_k}{D_k(\lambda_*)}.$$

For the λ-derivative, differentiate each summand $g_i(\lambda) = p_i^2(\lambda - \mu_i)/D_i(\lambda)$ by the quotient rule. Expanding $(\lambda - \mu_i)(2\lambda - \mu_i) = 2\lambda^2 - 3\lambda\mu_i + \mu_i^2$, the numerator of g_i' is

$$\begin{aligned} D_i - (\lambda - \mu_i)(2\lambda - \mu_i) &= (\lambda^2 - \lambda\mu_i - \mu_i^2) - (2\lambda^2 - 3\lambda\mu_i + \mu_i^2) \\ &= -\lambda^2 + 2\lambda\mu_i - 2\mu_i^2 \\ &= -\big((\lambda - \mu_i)^2 + \mu_i^2\big), \end{aligned}$$

so

$$g_i'(\lambda) \;=\; -p_i^2\,\frac{(\lambda-\mu_i)^2+\mu_i^2}{D_i(\lambda)^2}. \tag{24}$$

Including the -1 from differentiating the final $-\lambda$ term in F:

$$\frac{\partial F}{\partial \lambda} \;=\; -\sum_{i=1}^{n} p_i^2\,\frac{(\lambda-\mu_i)^2+\mu_i^2}{D_i(\lambda)^2} \;-\; 1 \;=\; -W(\lambda),$$

where $W(\lambda)$ is as in (22). The implicit function theorem gives $\partial\lambda_*/\partial(p_k^2) = -(\partial F/\partial p_k^2)/(\partial F/\partial\lambda) = N_k(\lambda_*)/W(\lambda_*)$.

Part (ii). The implicit function theorem applies if and only if $\partial F/\partial\lambda \neq 0$, i.e. $W(\lambda_*) \neq 0$. This is the simplicity condition.

Part (iii). By the chain rule and the Parseval identity $\sum_k p_k^2 = \|e\|^2$: if $\|e(t)\|^2 = c$ for all t, then $\sum_k d(p_k^2)/dt = 0$. The total derivative is the sum of (20) weighted by the rates $d(p_k^2)/dt$. □

Remark 4.9 (Sign structure of the secular weight). Each summand in (22) has numerator $(\lambda-\mu_i)^2+\mu_i^2 \geq 0$, with equality only when $\lambda = \mu_i = 0$. Hence $W(\lambda) \geq 1$ whenever all $p_i^2 \geq 0$ (which holds by definition). In particular, $W(\lambda_*) > 0$ for every coupled eigenvalue, so every simple coupled eigenvalue satisfies the simplicity criterion of Theorem 4.8(ii) with strict inequality.

This has an important structural consequence: the sensitivity $\partial\lambda_*/\partial(p_k^2)$ always has the same sign as the numerator $N_k(\lambda_*) = (\lambda_* - \mu_k)/D_k(\lambda_*)$, and its magnitude is *damped* by the factor $W(\lambda_*) \geq 1$. The secular weight acts as a spectral inertia: the more eigenspaces participate in the coupling (larger $\sum p_i^2$), the less any single participation shift affects the coupled eigenvalue.

Proposition 4.10 (Secular weight as eigenvector norm). *Let λ_* be a coupled eigenvalue of $M(A,e)$ with eigenvector $w = (x,y,z)^\top$, normalised so that $z = 1$ (nonzero apex component). Then:*

(i) ***Norm identity.***

$$\|w\|^2 \;=\; 1 + \sum_{i=1}^{n} p_i^2\,\frac{(\lambda_*-\mu_i)^2+\mu_i^2}{D_i(\lambda_*)^2} \;=\; W(\lambda_*). \tag{25}$$

(ii) ***Derivative identity.***

$$W(\lambda_*) \;=\; -\frac{\partial F}{\partial\lambda}\bigg|_{\lambda=\lambda_*} \;=\; -F'(\lambda_*). \tag{26}$$

(iii) ***Sensitivity–norm duality.*** *The participation sensitivity (Theorem 4.8) can be rewritten as*

$$\frac{\partial\lambda_*}{\partial(p_k^2)} \;=\; \frac{N_k(\lambda_*)}{\|w\|^2}. \tag{27}$$

Thus the spectral sensitivity is inversely proportional to the eigenvector norm*: delocalised coupled eigenvectors (large $\|w\|$) are rigid; localised ones (small $\|w\|$, large apex projection) are sensitive.*

Proof. **Part (i).** The secular eigenvector equations ((11)–(12) with $z = 1$) give, in the eigenbasis of A:

$$x_i = \frac{p_i \mu_i}{D_i(\lambda_*)}, \qquad y_i = \frac{p_i(\lambda_* - \mu_i)}{D_i(\lambda_*)}.$$

Hence

$$x_i^2 + y_i^2 = \frac{p_i^2\big(\mu_i^2 + (\lambda_* - \mu_i)^2\big)}{D_i(\lambda_*)^2},$$

and summing over i with the apex contribution $z^2 = 1$ gives $\|w\|^2 = W(\lambda_*)$.

Part (ii). This is the content of (24) summed over i: $F'(\lambda) = -\sum_i p_i^2\big((\lambda - \mu_i)^2 + \mu_i^2\big)/D_i(\lambda)^2 - 1 = -W(\lambda)$.

Part (iii). Substitute $W(\lambda_*) = \|w\|^2$ into (20). □

Remark 4.11 (Three faces of the secular weight). Theorem 4.10 shows that three a priori different objects coincide at every coupled eigenvalue:

Object	**Formula**	**Interpretation**
Secular weight $W(\lambda_*)$	$1 + \sum p_i^2(\cdots)/D_i^2$	spectral inertia
Eigenvector norm $\|w\|^2$	$\sum(x_i^2 + y_i^2) + 1$	delocalisation measure
Secular derivative $-F'(\lambda_*)$	$-\partial F/\partial\lambda$	root spacing / monotonicity

The identity $W = \|w\|^2 = -F'$ has three immediate consequences:

(a) *Monotonicity implies rigidity.* The strict inequality $-F' > 0$ (which forces exactly one secular root per golden-pole interval) is *the same* as $W > 0$ (which guarantees that every coupled eigenvalue is simple and has bounded sensitivity). Secular monotonicity and spectral rigidity are equivalent statements.

(b) *Apex projection controls sensitivity.* The L^2-normalised apex projection of the coupled eigenvector u_j satisfies $|\langle a, u_j\rangle|^2 = 1/\|w\|^2 = 1/W(\lambda_j)$. Hence the sensitivity $|\partial\lambda_j/\partial(p_k^2)| = |N_k| \cdot |\langle a, u_j\rangle|^2$: *the sensitivity of a coupled eigenvalue equals the numerator times the apex probability.*

(c) *Large operators are rigid.* As the dimension n grows (with $\|e\|^2 = n$ for the canonical choice $e = \mathbf{1}$), the secular weight $W \geq 1 + \sum p_i^2 \mu_i^2/D_i^2 \geq 1$ generically grows, so the per-eigenvalue sensitivity $\sim 1/W$ vanishes. This is a spectral manifestation of the concentration of measure: large golden companion operators have rigid coupled spectra.

Lemma 4.12 (Participation perturbation bound)**.** *Let A have orthonormal eigenbasis $\{v_i\}_{i=1}^n$, and let $e_1, e_2 \in \mathbb{R}^n$ be two coupling vectors with $\|e_1\| = \|e_2\| = c$. Define $p_i^{(j)} = e_j^\top v_i$ and $\delta_i = (p_i^{(1)})^2 - (p_i^{(2)})^2$. Then:*

(i) ***Conservation:*** $\sum_{i=1}^{n} \delta_i = 0$.

(ii) ***Cauchy–Schwarz bound:***

$$\sum_{i=1}^{n} \delta_i^2 \;\le\; 4c^2 \, \|e_1 - e_2\|^2. \tag{28}$$

(iii) ***Pointwise bound:*** $|\delta_k| \le 2c \, \|e_1 - e_2\|$ *for each* k.

Proof. **Part (i).** By Parseval, $\sum_i (p_i^{(j)})^2 = \|e_j\|^2 = c^2$ for $j = 1, 2$. Subtraction gives $\sum_i \delta_i = 0$.

Part (ii). Write $\delta_k = (p_k^{(1)} + p_k^{(2)})(p_k^{(1)} - p_k^{(2)})$ and define $p_k = p_k^{(1)} + p_k^{(2)} = (e_1 + e_2)^\top v_k$ and $q_k = p_k^{(1)} - p_k^{(2)} = (e_1 - e_2)^\top v_k$. Then $\delta_k = p_k q_k$ and

$$\sum_k \delta_k^2 = \sum_k p_k^2 q_k^2 \;\le\; \Big(\sum_k p_k^2\Big)\Big(\sum_k q_k^2\Big) = \|e_1 + e_2\|^2 \, \|e_1 - e_2\|^2,$$

by Cauchy–Schwarz. Now $\|e_1 + e_2\|^2 \le (\|e_1\| + \|e_2\|)^2 = 4c^2$, giving (28).

Part (iii). $|\delta_k| = |p_k||q_k| \le \|e_1 + e_2\| \, \|e_1 - e_2\| \le 2c \, \|e_1 - e_2\|$ by Cauchy–Schwarz on single coordinates. □

Corollary 4.13 (Coupled eigenvalue Lipschitz bound)**.** *Under the hypotheses of Theorem 4.12, let* $\lambda_*^{(j)}$ *be the corresponding simple coupled eigenvalues of* $M(A, e_j)$ *for* $j = 1, 2$ *(matched by continuity). Then*

$$\big|\lambda_*^{(1)} - \lambda_*^{(2)}\big| \;\le\; \frac{2c \, \|e_1 - e_2\|}{W_{\min}} \sum_{k=1}^{n} \Big| \frac{\lambda_* - \mu_k}{D_k(\lambda_*)} \Big|, \tag{29}$$

where $W_{\min} = \min_{j \in \{1,2\}} W(\lambda_*^{(j)}) \ge 1$.

Proof. By the mean value theorem applied to the map $\mathbf{p}^2 \mapsto \lambda_*(\mathbf{p}^2)$:

$$\big|\lambda_*^{(1)} - \lambda_*^{(2)}\big| \;\le\; \sum_k \sup_{\mathbf{p}^2 \in [\mathbf{p}_1^2, \mathbf{p}_2^2]} \Big| \frac{\partial \lambda_*}{\partial (p_k^2)} \Big| \; |\delta_k|.$$

By Theorem 4.8(i) and Theorem 4.9 ($W \ge 1$), each sensitivity is bounded by $|N_k(\lambda_*)|/W_{\min}$. Applying the pointwise bound Theorem 4.12(iii) to each $|\delta_k|$ gives the result. □

Remark 4.14 (Exact redundancy versus bounded perturbation). Theorems 3.5 and 4.13 together give a two-tier rigidity picture for the golden companion spectrum:

(a) **Exact redundancy.** If $e_2 = U e_1$ for $U \in \mathrm{Aut}(A)$, then $\|e_1 - e_2\| = 0$ need not hold (since $U e_1 \ne e_1$ in general), but Theorem 3.5 gives $\lambda_*^{(1)} = \lambda_*^{(2)}$ exactly.

(b) **Bounded perturbation.** For probes not related by $\mathrm{Aut}(A)$, the Lipschitz bound (29) controls the spectral shift via the probe distance $\|e_1 - e_2\|$ and the secular weight $W \geq 1$. The spectral inertia from Theorem 4.9 ensures that the bound is *self-dampening*: operators with richer coupling (larger $\sum p_i^2$) have larger W and hence smaller sensitivity.

Combined with the Golden Amplification Theorem (Theorem 3.1)—which shows that transparent eigenvalues $\mu\varphi$ and $-\mu/\varphi$ are independent of e entirely—this establishes that the spectrum of $M(A, e)$ is *structurally rigid* in the coupling vector: transparent eigenvalues are exactly invariant, and coupled eigenvalues are Lipschitz-continuous in e with a universal damping factor.

5. The Positive Boost Inequality

Throughout this section, A denotes a *symmetric nonnegative* matrix (i.e. $A_{ij} \geq 0$). This assumption ensures that cross-energy is non-positive, which is the engine of the boost inequality.

Definition 5.1 (Nodal domains and cross-energy)**.** For a vector $v \in \mathbb{R}^n$ and a symmetric nonnegative matrix A, the *positive nodal domain* is $S^+ = \{i : v_i \geq 0\}$ and the *negative nodal domain* is $S^- = \{i : v_i < 0\}$. The *cross-energy* is

$$E_\times = \sum_{\substack{i \in S^+,\, j \in S^- \\ A_{ij} > 0}} A_{ij}\, v_i\, v_j.$$

Since $A_{ij} \geq 0$, $v_i \geq 0$, and $v_j < 0$ for each term, $E_\times \leq 0$.

Theorem 5.2 (Positive Boost Inequality)**.** *Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric nonnegative matrix, v a unit eigenvector at eigenvalue λ, with both nodal domains S^+, S^- nonempty. Then*

$$\lambda_{\max}\big(A[S^+]\big) \;\geq\; \lambda + \frac{|E_\times|}{\|v_{S^+}\|^2}, \qquad \lambda_{\max}\big(A[S^-]\big) \;\geq\; \lambda + \frac{|E_\times|}{\|v_{S^-}\|^2}. \tag{30}$$

In particular, $\lambda_{\max}(A[S^\pm]) \geq \lambda$.

Proof. The eigenvalue equation $Av = \lambda v$ gives, for each $i \in S^+$: $\sum_j A_{ij} v_j = \lambda v_i$. Multiply by v_i and sum over $i \in S^+$:

$$\underbrace{\sum_{i \in S^+} \sum_{j \in S^+} A_{ij} v_i v_j}_{v_{S^+}^\top A[S^+]\, v_{S^+}} + \underbrace{\sum_{i \in S^+} \sum_{j \in S^-} A_{ij} v_i v_j}_{E_\times \leq 0} = \lambda \|v_{S^+}\|^2.$$

Rearranging:

$$\frac{v_{S^+}^\top A[S^+]\, v_{S^+}}{\|v_{S^+}\|^2} = \lambda + \frac{|E_\times|}{\|v_{S^+}\|^2} \geq \lambda.$$

By the variational characterisation of $\lambda_{\max}$, the left side is a lower bound for $\lambda_{\max}(A[S^+])$. The argument for S^- is identical. □

Remark 5.3 (Spectral vs. chromatic). The Positive Boost Inequality is a *spectral* statement: it produces lower bounds on $\lambda_{\max}(A[S^{\pm}])$, not directly on $\chi(G[S^{\pm}])$. These spectral bounds become chromatic certificates only when combined with the classical inequality $\chi(H) \geq 1 + \lambda_{\max}(H)/|\lambda_{\min}(H)|$ (Hoffman's bound) or verified by exact chromatic computation. Throughout the transfer analysis in Section 6, we use the boost as a *screening statistic* and always close the argument with exact chromatic certificates where needed.

Remark 5.4 (Frequency mechanism). Eigenvectors of a nonnegative symmetric matrix function as standing waves, ordered by frequency.

- *Positive eigenvalues* ($\lambda > 0$): low-frequency modes with large, coherent nodal domains. The Positive Boost guarantees spectrally dense submatrices.
- *Negative eigenvalues* ($\lambda < 0$): high-frequency modes with fragmented nodal domains. The boost magnitude is large (many cross-boundary terms) but the base eigenvalue is negative.

This asymmetry explains why the most-positive Galois conjugate attains stronger side-wise spectral lower bounds (Theorem 6.25). Exact certification transfer is handled separately via verified chromatic dominance (Theorem 6.28).

6. The Galois Transfer Principle

From this point onward (through Section 9), we specialise to the case where A is the adjacency matrix of a graph G (hence symmetric, nonnegative, and integer-valued) and $e = \mathbf{1}$, so that $M(A, \mathbf{1})$ is the Mycielskian adjacency matrix.

The Galois conjugation $\varphi \mapsto -\varphi^{-1}$ acts on the eigenvalues and eigenvectors of the golden companion operator, raising a fundamental question: among Galois-conjugate eigenvectors, which produce more effective partitions? We develop a unified framework—the *Galois Transfer Principle*—that answers this question through a single organising structure: the Transfer/Pareto Trichotomy (Theorem 6.11, below). Every conjugate pair falls into one of two regimes (transfer or Pareto), classified by chromatic signatures. Transparent and coupled modes are then two instantiations of this principle, differing only in whether transfer is automatic or conditional.

Definition 6.1 (Galois orbit, orbit rank)**.** Let $A \in \mathbb{Z}^{n\times n}$ have characteristic polynomial $p_A(x) = \prod_i f_i(x)^{m_i}$ where each f_i is irreducible over $\mathbb{Q}$. An irreducible factor f of degree d defines a *Galois orbit* $\mathcal{O}_f = \{\alpha_1 > \alpha_2 > \cdots > \alpha_d\}$ (the real roots of f, ordered in decreasing value). The *orbit rank* of α_j is j; rank 1 is the most-positive member.

Example 6.2. (a) $d = 1$: a rational eigenvalue, fixed by all Galois automorphisms.
(b) $d = 2$: a conjugate pair in $\mathbb{Q}(\sqrt{D})$, e.g. $\{\varphi, -\varphi^{-1}\}$ in $\mathbb{Q}(\sqrt{5})$.

(c) $d \geq 3$: orbits in cyclotomic or other number fields.

6.1. **Galois action on Rayleigh quotients.** The key challenge is to compare partition quality across Galois-conjugate eigenvectors. We begin with a general tool that handles the simplest case (identical sign cuts), then turn to the harder case where the Galois conjugation *changes* the sign cut.

Lemma 6.3 (Galois Rayleigh Quotient)**.** *Let $A \in \mathbb{Z}^{n\times n}$ be symmetric, and let $v \in \mathbb{Q}(\sqrt{D})^n$ be an eigenvector of A at eigenvalue α. Write σ for the nontrivial automorphism of $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. For any fixed subset $S \subseteq \{1,\dots,n\}$ with $v_S \neq 0$,*

$$R\big(\sigma(v)_S,\ A[S]\big) \ = \ \sigma\Big(R\big(v_S,\ A[S]\big)\Big), \tag{31}$$

where $R(w,B) = w^\top Bw/\|w\|^2$ is the Rayleigh quotient. In particular, $R(v_S) - R(\sigma(v)_S) = 2r_1\sqrt{D}$ for some $r_1 \in \mathbb{Q}$.

Proof. Since $A \in \mathbb{Z}^{n\times n}$, the automorphism σ fixes $A[S]$ entry-wise while acting coordinate-wise on v_S. Therefore $\sigma(v_S^\top A[S]\, v_S) = \sigma(v_S)^\top A[S]\,\sigma(v_S)$ and $\sigma(\|v_S\|^2) = \|\sigma(v_S)\|^2$, giving (31). □

Corollary 6.4 (Same-cut Rayleigh conjugation)**.** *If the sign cuts of v and $\sigma(v)$ agree ($S^+(v) = S^+(\sigma(v))$), then for each side $S \in \{S^+(v), S^-(v)\}$,*

$$R\big(\sigma(v)_S,\ A[S]\big) \ = \ \sigma\big(R(v_S,\ A[S])\big).$$

Equivalently, writing $R(v_S,\ A[S]) = a_S + b_S\sqrt{D}$ with $a_S, b_S \in \mathbb{Q}$,

$$R(v_S,\ A[S]) - R(\sigma(v)_S,\ A[S]) = 2b_S\sqrt{D}.$$

In particular, if $b_S \geq 0$, then $R(v_S,\ A[S]) \geq R(\sigma(v)_S,\ A[S])$.

Proof. Apply Theorem 6.3 to each fixed side $S \in \{S^+(v), S^-(v)\}$. The displayed conjugation identity follows immediately. The coefficient form is the standard decomposition of an element of $\mathbb{Q}(\sqrt{D})$. The final implication is immediate from $\sqrt{D} > 0$. □

Remark 6.5. For transparent golden pairs $\{\mu\varphi,\ -\mu/\varphi\}$, the eigenvectors $w_+ = (x, x/\varphi, 0)$ and $w_- = (x, -\varphi x, 0)$ always have *different* sign cuts (the shadow coordinates flip sign). Thus Theorem 6.4 does not apply directly, and the key challenge is to compare Rayleigh quotients on *different* subsets. This is resolved by the edge internalization mechanism below.

6.2. **Certification framework and the Transfer/Pareto Trichotomy.** To compare partition quality across Galois conjugates, we formalise the notion of partition certification and establish the exact classification of conjugate pairs into transfer and Pareto regimes.

Definition 6.6 (Effective partition)**.** Let A be an $n \times n$ real symmetric nonnegative matrix, v a unit eigenvector at eigenvalue λ, and $s,t \geq 2$ integers. The eigenvector v *certifies the (s,t)-partition target* if the sign cut

$S^+ = \{i : v_i \geq 0\}$, $S^- = \{i : v_i < 0\}$ satisfies

$$\chi\big(A[S^+]\big) \;\geq\; s \qquad \text{and} \qquad \chi\big(A[S^-]\big) \;\geq\; t. \tag{32}$$

where $A[S]$ denotes the principal submatrix induced by S, and $\chi(A[S])$ denotes the chromatic number of the support graph $G(A[S])$ with edge set $\{ij : i \neq j,\, A_{ij} > 0\}$. Throughout, certification in (32) is interpreted in the exact chromatic sense: both inequalities are verified analytically or by exact computation. Spectral quantities such as Rayleigh quotients and lower bounds on $\lambda_{\max}(A[S^{\pm}])$ (e.g., from Theorem 5.2) are used as side-wise screening statistics, not as stand-alone chromatic certificates.

Remark 6.7 (Hoffman-type surrogate (when applicable))*.* If $A[S]$ is the adjacency matrix of a simple graph H with $\lambda_{\min}(H) < 0$, then

$$\chi(H) \;\geq\; 1 - \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}$$

([3]). Thus any spectral surrogate for chromatic certification requires control of both extremal eigenvalues. In particular, lower bounds on $\lambda_{\max}$ alone do not imply chromatic lower bounds.

Definition 6.8 (Certification set and regime split)**.** For $A \in \mathbb{R}^{n\times n}_{\geq 0}$ and a unit eigenvector v, define

$$\mathcal{C}_A(v) := \{(s,t) \in \mathbb{Z}^2_{\geq 2} : v \text{ certifies } (s,t)\}.$$

For a conjugate pair $(v, \sigma(v))$, we say:

(i) *transfer regime* if $\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v)$;
(ii) *Pareto regime* if neither $\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v)$ nor $\mathcal{C}_A(v) \subseteq \mathcal{C}_A(\sigma(v))$.

Definition 6.9 (Chromatic signature)**.** For $A \in \mathbb{R}^{n\times n}_{\geq 0}$ and a unit eigenvector v, define

$$\kappa_A(v) := \big(\chi(A[S^+(v)]),\, \chi(A[S^-(v)])\big) \in \mathbb{Z}^2_{\geq 1}.$$

Proposition 6.10 (Rectangular form of certification sets)**.** *Let $A \in \mathbb{R}^{n\times n}_{\geq 0}$ and unit eigenvector v, with $\kappa_A(v) = (a,b)$. Then*

$$\mathcal{C}_A(v) = \{(s,t) \in \mathbb{Z}^2_{\geq 2} : s \leq a,\; t \leq b\}.$$

Proof. By Theorem 6.6, $(s,t) \in \mathcal{C}_A(v)$ iff $\chi(A[S^+(v)]) \geq s$ and $\chi(A[S^-(v)]) \geq t$. Writing $a = \chi(A[S^+(v)])$, $b = \chi(A[S^-(v)])$ gives the stated set. ☐

Theorem 6.11 (Transfer/Pareto trichotomy for conjugate pairs)**.** *Let $A \in \mathbb{R}^{n\times n}_{\geq 0}$ and let $(v, \sigma(v))$ be a conjugate eigenvector pair. Write*

$$\kappa_A(v) = (a,b), \qquad \kappa_A(\sigma(v)) = (a', b').$$

Then:

(i) ***Transfer*** *$\sigma(v) \to v$: $\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v)$ iff $a \geq a'$ and $b \geq b'$.*
(ii) ***Transfer*** *$v \to \sigma(v)$: $\mathcal{C}_A(v) \subseteq \mathcal{C}_A(\sigma(v))$ iff $a \leq a'$ and $b \leq b'$.*
(iii) ***Pareto regime****: $(v, \sigma(v))$ is Pareto iff $(a - a')(b - b') < 0$ (equivalently, one coordinate is larger and the other is smaller).*

In particular, if $(a,b) = (a',b')$, then transfer holds both ways.

Proof. By Theorem 6.10, $\mathcal{C}_A(v)$ and $\mathcal{C}_A(\sigma(v))$ are axis-aligned downward rectangles in $\mathbb{Z}^2_{\geq 2}$ with maximal corners (a,b) and (a',b'). Inclusion of such rectangles is equivalent to coordinate-wise dominance, proving (i) and (ii). By Theorem 6.8, Pareto means mutual non-inclusion, which is equivalent to coordinate-wise incomparability, i.e. $(a-a')(b-b') < 0$, proving (iii). □

Corollary 6.12 (Cut-stable equality of certification sets). *If $S^{\pm}(v) = S^{\pm}(\sigma(v))$, then $\kappa_A(v) = \kappa_A(\sigma(v))$ and $\mathcal{C}_A(v) = \mathcal{C}_A(\sigma(v))$.*

Proof. Equal cuts imply identical induced principal submatrices on both sides, hence identical chromatic numbers and identical signatures. The set equality follows from Theorem 6.10. □

Definition 6.13 (Side-wise homomorphic dominance). Let A be the adjacency matrix of a graph G, and let $v, \sigma(v)$ be a conjugate pair. We say that v *side-wise homomorphically dominates* $\sigma(v)$ if there exist graph homomorphisms

$$\phi_+ : G[S^+(\sigma(v))] \to G[S^+(v)], \qquad \phi_- : G[S^-(\sigma(v))] \to G[S^-(v)].$$

Theorem 6.14 (Homomorphic transfer criterion). *Under the setup of Theorem 6.13, if v side-wise homomorphically dominates $\sigma(v)$, then*

$$\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v).$$

Proof. For graphs X, Y, a homomorphism $X \to Y$ implies $\chi(X) \leq \chi(Y)$: any proper coloring of Y pulls back along the homomorphism to a proper coloring of X. Applying this to ϕ_+ and ϕ_- gives

$$\chi\big(A[S^+(\sigma(v))]\big) \leq \chi\big(A[S^+(v)]\big), \qquad \chi\big(A[S^-(\sigma(v))]\big) \leq \chi\big(A[S^-(v)]\big).$$

Hence $\kappa_A(\sigma(v)) \leq \kappa_A(v)$ coordinate-wise. By Theorem 6.11(i), this is equivalent to $\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v)$. □

Corollary 6.15 (Cut stability as identity-homomorphism case). *If $S^{\pm}(v) = S^{\pm}(\sigma(v))$, then v side-wise homomorphically dominates $\sigma(v)$ (and conversely), via identity maps on both sides.*

Proof. When cuts agree, the two induced side graphs coincide on each side, so the identity maps are homomorphisms both ways. □

6.3. **The cut-stability principle.** The Galois Rayleigh Quotient (Theorem 6.3) shows that conjugation commutes with the Rayleigh functional on any fixed subset. The central question is then: when do conjugate eigenvectors induce *the same* subsets? When they do—the *cut-stability* condition—the side-wise Rayleigh differences become explicit Galois-conjugate quantities (Theorem 6.4). Under nonnegative side-wise Galois coefficients, this yields side-wise Rayleigh dominance. For exact chromatic certification, cut stability yields equality of certification sets (Theorem 6.12). The two mode types (transparent and coupled) differ precisely in how this condition is satisfied or bypassed.

Theorem 6.16 (Positive-Wins under cut stability)**.** *Let $B \in \mathbb{Z}^{N\times N}$ be symmetric (in particular, this applies to $B = M(A, e)$). Let $v \in \mathbb{Q}(\sqrt{D})^N$ be an eigenvector of B at eigenvalue α, with conjugate eigenvector $\sigma(v)$ at eigenvalue $\sigma(\alpha)$, and assume $\alpha > \sigma(\alpha)$. Suppose the conjugate cuts are stable: $S^+(v) = S^+(\sigma(v))$ and $S^-(v) = S^-(\sigma(v))$. For each side $S \in \{S^+(v), S^-(v)\}$, write*

$$R\big(v_S,\ B[S]\big) = a_S + b_S\sqrt{D}, \qquad a_S, b_S \in \mathbb{Q},$$

and assume $b_S \geq 0$. Then:

(a) For each side $S \in \{S^+(v),\, S^-(v)\}$,

$$R\big(v_S,\ B[S]\big)\ \geq\ R\big(\sigma(v)_S,\ B[S]\big).$$

(b) Consequently,

$$\lambda_{\max}\big(B[S]\big)\ \geq\ R\big(v_S,\ B[S]\big)\ \geq\ R\big(\sigma(v)_S,\ B[S]\big) \qquad \text{for } S \in \{S^+, S^-\}.$$

(c) Any side-wise spectral surrogate score monotone in these two Rayleigh lower bounds transfers from $\sigma(v)$ to v with weakly larger margin.

Proof. Apply Theorem 6.4 to B and the pair $(v, \sigma(v))$ on each side $S \in \{S^+(v), S^-(v)\}$. By hypothesis, $R(v_S, B[S]) - R(\sigma(v)_S, B[S]) = 2b_S\sqrt{D} \geq 0$, proving (a). The variational characterisation of $\lambda_{\max}$ yields (b). Part (c) is immediate from monotonicity of the surrogate score functional in the two Rayleigh lower bounds. □

Proposition 6.17 (Algebraic criterion for cut stability)**.** *Let $v \in \mathbb{Q}(\sqrt{D})^N$ with coordinates $v_i = a_i + b_i\sqrt{D}$, $a_i, b_i \in \mathbb{Q}$, $D > 0$ squarefree. If*

$$|a_i| > \sqrt{D}\,|b_i| \qquad \text{for all } i,$$

then v_i and $\sigma(v_i) = a_i - b_i\sqrt{D}$ have the same sign for every i, hence $S^+(v) = S^+(\sigma(v))$ and $S^-(v) = S^-(\sigma(v))$.

Proof. For each i,

$$v_i\,\sigma(v_i) = a_i^2 - Db_i^2 > 0$$

by the hypothesis. Therefore v_i and $\sigma(v_i)$ are nonzero and have the same sign. The equality of positive/negative index sets follows coordinate-wise. □

Remark 6.18 (Unifying view)*.* Theorem 6.16 is mode-agnostic: it applies to any Galois-conjugate eigenvector pair whose sign cuts agree, regardless of whether the mode is transparent or coupled. The two mode types differ in how cut stability arises:

- *Transparent modes* (see below): the Galois conjugation always flips the shadow layer, so the sign cuts are never identical. However, the Edge Internalization Lemma (Theorem 6.21) provides a structural bypass: side-wise spectral dominance is established by a direct partition-quality comparison that does not require cut stability; exact certification transfer then requires verified side-wise chromatic dominance.

- *Coupled modes*: cut stability depends on the spectral data $(\mu_i, p_i, \lambda, \lambda')$. When it holds and the side-wise Galois coefficients are nonnegative, side-wise spectral dominance follows from Theorem 6.16, and exact chromatic certification sets coincide by Theorem 6.12. When it fails, the Pareto regime is possible (Theorem 6.35).

6.4. **Transparent modes: spectral dominance via edge internalization.** For transparent golden pairs $\{\mu\varphi,\ -\mu/\varphi\}$, the eigenvectors $w_+ = (x, x/\varphi, 0)$ and $w_- = (x, -\varphi x, 0)$ always have different sign cuts (the shadow coordinates flip sign), so cut stability (Theorem 6.16) does not apply directly. Nevertheless, the most-positive conjugate has theorem-level *spectral* side-wise dominance, by a structural mechanism: the Galois conjugation converts cut edges into internal edges. This *edge internalization* yields stronger side-wise spectral lower bounds without requiring cut stability; exact certification transfer follows when side-wise chromatic dominance is verified.

Recall from Theorem 3.1 that for an eigenpair (μ, x) of A with $x \perp e$, the golden companion $M(A, e)$ has eigenvectors

$$w_+ = (x,\ x/\varphi,\ 0)^\top \text{ at } \mu\varphi, \qquad w_- = (x,\ -\varphi x,\ 0)^\top \text{ at } -\mu/\varphi.$$

The eigenvector w_+ *aligns* the shadow with the original (x_i and x_i/φ share the same sign), while w_- *anti-aligns* them (x_i and $-\varphi x_i$ have opposite signs).

Definition 6.19 (Sign partition of the base eigenvector)**.** Let $P = \{i : x_i \geq 0\}$ and $N = \{i : x_i < 0\}$. We write P', N' for the corresponding shadow vertices, and $*$ for the apex.

Lemma 6.20 (Partition decomposition)**.** *The sign cuts of the two transparent eigenvectors are:*

$$\text{(33)} \qquad \textit{Aligned } (w_+): \qquad S^+ = P \cup P' \cup \{*\}, \quad S^- = N \cup N',$$

$$\text{(34)} \qquad \textit{Anti-aligned } (w_-): \quad S^+ = P \cup N' \cup \{*\}, \quad S^- = N \cup P'.$$

Proof. Immediate from the signs: w_+ has shadow component x_i/φ (same sign as x_i), so shadow i' follows original i. w_- has shadow component $-\varphi x_i$ (opposite sign), so shadow i' follows original i's complement. The apex has component $0 \geq 0$ in both cases. ☐

6.4.1. *The Edge Internalization Lemma.*

Lemma 6.21 (Edge internalization)**.** *Let (i, j) be an edge of G with $i \in P$ and $j \in N$ (a cut edge of x). Then:*

(i) ***Anti-aligned partition*** *(w_-): the shadow edge i–j' is internal to $S^+(w_-)$, and the shadow edge j–i' is internal to $S^-(w_-)$.*
(ii) ***Aligned partition*** *(w_+): both shadow edges i–j' and j–i' cross the partition.*

Proof. In $M(A, e)$, shadow j' is adjacent to all neighbours of j in G, including i.

(i) Anti-aligned: $j' \in N' \subset S^+(w_-)$ (since $-\varphi x_j > 0$ for $x_j < 0$) and $i \in P \subset S^+(w_-)$. So i–j' is internal to S^+. Similarly, $i' \in P' \subset S^-(w_-)$ (since $-\varphi x_i < 0$ for $x_i > 0$) and $j \in N \subset S^-(w_-)$, so j–i' is internal to S^-.

(ii) Aligned: $j' \in N' \subset S^-(w_+)$ (since $x_j/\varphi < 0$) while $i \in P \subset S^+(w_+)$, so i–j' crosses. Similarly $i' \in P' \subset S^+(w_+)$ while $j \in S^-(w_+)$, so j–i' crosses. □

Corollary 6.22 (Internal edge count)**.** *Let* $\mathrm{cut}_G = |\{(i,j) \in E(G) : i \in P,\, j \in N\}|$. *Then the anti-aligned partition satisfies*

$$|E(M[S^+(w_-)])| \geq |E(G[P])| + \mathrm{cut}_G + |N|, \tag{35}$$

$$|E(M[S^-(w_-)])| \geq |E(G[N])| + \mathrm{cut}_G, \tag{36}$$

while the aligned partition satisfies

$$|E(M[S^+(w_+)])| \geq 3|E(G[P])| + |P|, \qquad |E(M[S^-(w_+)])| \geq 3|E(G[N])|. \tag{37}$$

Proof. For the anti-aligned $S^+(w_-)$: original edges of $G[P]$ survive; each cut edge (i,j) contributes the shadow edge i–j' by Theorem 6.21(i); the apex connects to all of N' (contributing $|N|$ edges). For $S^-(w_-)$: original $G[N]$ edges survive, and each cut edge contributes j–i'.

For the aligned $S^+(w_+)$: original $G[P]$ edges survive; each $G[P]$-edge (i,j) also contributes i–j' and j–i' (both endpoints in $P \cup P'$), tripling the count; the apex connects to P' (contributing $|P|$). $S^-(w_+)$ gets $G[N]$-edges tripled analogously (no apex). □

Remark 6.23 (Balanced redistribution)*.* The anti-aligned partition distributes every cut edge of G equally to both sides (one shadow copy per side). The aligned partition concentrates all shadow connections within the same side as their originals. When cut_G is large—as it is for high-frequency eigenvectors ($\mu < 0$)—the anti-aligned partition produces two well-connected subgraphs, while the aligned partition produces one dense side and one sparse side.

6.4.2. *Cross-energy formulas.*

Proposition 6.24 (Transparent cross-energies)**.** *Let*

$$\mathcal{E}_{\mathrm{cut}} := \{(i,j) \in E(G) : i \in P,\ j \in N\}.$$

Let $E = |E_\times(x)| = -\sum_{(i,j)\in\mathcal{E}_{\mathrm{cut}}} x_i x_j$ *be the cross-energy of the base eigenvector, and let* $I_P + I_N = \mu/2 + E$ *where* $I_P = \sum_{E(G[P])} x_i x_j \geq 0$ *and* $I_N = \sum_{E(G[N])} x_i x_j \geq 0$. *Then*

$$|E_\times(w_-)| = (2+\sqrt{5})\, E + \varphi\mu, \tag{38}$$

$$|E_\times(w_+)| = \sqrt{5}\, E. \tag{39}$$

In particular, $|E_\times(w_-)| \geq |\mu|/2 > 0$ *whenever* $\mu \neq 0$.

Proof. The cross-energy of w_+ sums over the cut edges of the aligned partition. Each cut edge of G contributes $x_i x_j$ (original) plus two shadow copies contributing $x_i(x_j/\varphi)$ and $x_j(x_i/\varphi)$, for a total factor of $1+2/\varphi = \sqrt{5}$ per cut edge. No other edge types contribute (apex terms vanish, and internal-to-shadow connections are all within S^+ or S^-). Hence $E_\times(w_+) = \sqrt{5}\,E_\times(x)$, giving $|E_\times(w_+)| = \sqrt{5}\,E$.

For w_-, the cut edges of the anti-aligned partition include (a) the original G-cut edges (contributing factor 1), (b) shadow connections from $G[P]$-edges, each contributing $x_i \cdot (-\varphi x_j)$ with two orientations (factor 2φ per internal $G[P]$-edge), and (c) similarly 2φ per $G[N]$-edge. Summing:

$$\begin{aligned}|E_\times(w_-)| &= E + 2\varphi(I_P + I_N) = E + 2\varphi(\mu/2 + E)\\ &= E(1+2\varphi) + \varphi\mu = (2+\sqrt{5})\,E + \varphi\mu,\end{aligned}$$

using $1 + 2\varphi = 2 + \sqrt{5}$. The lower bound follows from $E \geq |\mu|/2$ (since $I_P + I_N \geq 0$): substituting, $(2+\sqrt{5})|\mu|/2 - \varphi|\mu| = |\mu|/2$. □

6.4.3. *Positive-Wins for transparent modes.* The following theorem establishes the Positive-Wins Law for transparent golden pairs. The non-trivial case is $\mu < 0$, where the positive conjugate $\alpha = -\mu/\varphi > 0$ has the anti-aligned eigenvector w_- and the negative conjugate $\alpha' = \mu\varphi < 0$ has the aligned eigenvector w_+.

Theorem 6.25 (Positive-Wins for transparent modes)**.** *Let $A \in \mathbb{Z}_{\geq 0}^{n\times n}$ be the adjacency matrix of a graph G, and let $\mu < 0$ be an eigenvalue of A with eigenvector $x \perp e$. Consider the transparent golden pair $(\mu\varphi,\ -\mu/\varphi)$ in $M(A, e)$.*

(i) ***Eigenvalue advantage****: for both nodal domains of w_-,*

$$\lambda_{\max}\big(M[S^\pm(w_-)]\big) \;\geq\; -\mu/\varphi \;>\; 0.$$

(ii) ***Structural advantage****: the anti-aligned partition internalizes every cut edge of G into both sides (Theorem 6.21), yielding at least $|E(G[P])| + \mathrm{cut}_G$ and $|E(G[N])| + \mathrm{cut}_G$ internal edges on the two sides, respectively.*

(iii) ***Balanced-case bound****: if $\|x_P\|^2 = \|x_N\|^2 = 1/2$ (balanced base partition), the Rayleigh-quotient lower bounds satisfy*

$$f_+ - f_- \;=\; \frac{\sqrt{5}}{5}\big(3|\mu| - 2E\big) \;\geq\; 0 \qquad \textit{whenever } E \leq \tfrac{3}{2}|\mu|. \tag{40}$$

The constraint $E \geq |\mu|/2$ (from $I_P + I_N \geq 0$) is always satisfied. The additional condition $E \leq 3|\mu|/2$ is the sufficient regime in which this balanced-case comparison guarantees $f_+ \geq f_-$.

Proof. (i) The Positive Boost Inequality (Theorem 5.2) applied to w_- at eigenvalue $\alpha = -\mu/\varphi > 0$ gives $\lambda_{\max}(M[S^\pm]) \geq \alpha + |E_\times(w_-)|/\|w_-\|^2_{S^\pm} \geq \alpha > 0$.

(ii) Is the content of Theorem 6.21 and Theorem 6.22.

(iii) Let $p = \|x_P\|^2 = 1/2$. The mass on the positive side of w_- is $\|(w_-)_{S^+}\|^2 = p + \varphi^2(1-p) = (2+\varphi)/2$ (and symmetrically for S^-). The Rayleigh quotient on either side is

$$f_+ = \alpha + \frac{|E_\times(w_-)|}{(2+\varphi)/2} = \frac{2\big((2+\sqrt{5})\,E - |\mu|\big)}{2+\varphi}.$$

For the aligned eigenvector w_+ at $\alpha' = \mu\varphi$, the mass is $\|(w_+)_{S'^+}\|^2 = p(5-\sqrt{5})/2 = (5-\sqrt{5})/4$, giving

$$f_- = \alpha' + \frac{\sqrt{5}\,E}{(5-\sqrt{5})/4} = \varphi(2E - |\mu|).$$

The difference $f_+ - f_-$ simplifies (after clearing denominators and using $\varphi^2 = \varphi + 1$) to $\sqrt{5}(3|\mu| - 2E)/5$. □

Remark 6.26 (Galois duality and edge internalization). The anti-aligned eigenvector $w_- = (x, -\varphi x, 0)$ applies the Galois conjugation $\varphi \mapsto -\varphi^{-1}$ to the shadow layer: the shadow component $-\varphi x$ is the image of x/φ (from w_+) under the nontrivial automorphism of $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$. The mass ratio $\|(w_-)_{S^+}\|^2/\|(w_+)_{S'^+}\|^2 = (2+\varphi)/((5-\sqrt{5})/2)$ reflects this algebraic duality at the level of partition weights.

The Galois conjugation converts *destructive interference* (cut edges that fragment both sides of the partition) into *constructive interference* (internal edges that enrich both sides). The Edge Internalization Lemma (Theorem 6.21) makes this mechanism precise.

6.4.4. *Worked example: $C_5 = M(K_2)$.* We verify the theorem on the smallest nontrivial Mycielskian.

Example 6.27 (C_5)**.** The graph K_2 has adjacency matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with eigenvalues $\mu_1 = 1$ (Perron, coupled) and $\mu_2 = -1$ (transparent, eigenvector $x = (1,-1)^\top/\sqrt{2}$). The sign partition is $P = \{1\}$, $N = \{2\}$, with a single cut edge 1–2 and cross-energy $E = 1/2$.

The golden pair at $\mu = -1$ is $\mu\varphi = -\varphi$ and $-\mu/\varphi = 1/\varphi$, with eigenvectors

$$w_+ = \tfrac{1}{\sqrt{2}}(1,\ -1,\ 1/\varphi,\ -1/\varphi,\ 0), \qquad w_- = \tfrac{1}{\sqrt{2}}(1,\ -1,\ -\varphi,\ \varphi,\ 0).$$

The sign cuts are:

$$S^+(w_-) = \{1, 2', *\}, \qquad S^-(w_-) = \{2, 1'\} \qquad \text{(anti-aligned)},$$
$$S^+(w_+) = \{1, 1', *\}, \qquad S^-(w_+) = \{2, 2'\} \qquad \text{(aligned)}.$$

Edge internalization. The unique cut edge 1–2 of K_2 has shadow copies 1–2′ and 2–1′ in C_5. In the anti-aligned partition, 1–2′ is internal to $S^+(w_-)$ and 2–1′ is internal to $S^-(w_-)$; the apex edges ∗–1′ and ∗–2′ also contribute (one to each side). In the aligned partition, both shadow copies cross the cut.

Subgraph structure and spectral radii.

	$M[S^+]$	$\lambda_{\max}(S^+)$	$M[S^-]$	$\lambda_{\max}(S^-)$
Anti-aligned (w_-)	Path P_3	$\sqrt{2} \approx 1.414$	Edge K_2	1
Aligned (w_+)	Edge + isolated	1	Two isolated	0

The anti-aligned partition (positive conjugate at $1/\varphi$) dominates on both sides. For the Tihany target $(s,t) = (2,2)$: $\chi(P_3) = 2 \geq 2$ and $\chi(K_2) = 2 \geq 2$ (certified); whereas $\chi(\text{two isolated}) = 1 < 2$ (the aligned partition fails).
Formula verification. With $E = 1/2$ and $|\mu| = 1$, the balanced-case bound (Theorem 6.25(iii)) gives:

$$f_+ = \frac{2(2+\sqrt{5}) \cdot \frac{1}{2} - 1}{2+\varphi} = \frac{2\varphi}{2+\varphi} \approx 0.894, \qquad f_- = \varphi\bigl(2 \cdot \tfrac{1}{2} - 1\bigr) = 0,$$

confirming $f_+ - f_- = \sqrt{5}(3 \cdot 1 - 2 \cdot \frac{1}{2})/5 = 2\sqrt{5}/5 \approx 0.894 > 0$.

Corollary 6.28 (Transparent transfer inclusion (exact chromatic form))**.** *Under the setup of Theorem 6.25, if*

$$\chi\bigl(M(A,e)[S^+(w_-)]\bigr) \geq \chi\bigl(M(A,e)[S^+(w_+)]\bigr),$$
$$\chi\bigl(M(A,e)[S^-(w_-)]\bigr) \geq \chi\bigl(M(A,e)[S^-(w_+)]\bigr).$$

then

$$\mathcal{C}_{M(A,e)}\bigl(w_+\bigr) \subseteq \mathcal{C}_{M(A,e)}\bigl(w_-\bigr).$$

Proof. Write $M := M(A,e)$. The hypotheses are exactly

$$\kappa_M\bigl(w_-\bigr) \geq \kappa_M\bigl(w_+\bigr)$$

coordinate-wise. Apply Theorem 6.11(i). □

6.5. **Coupled modes: conditional transfer and Pareto regimes.** For coupled modes, the sign cuts of conjugate eigenvectors depend on the full spectral data of the secular equation. Cut stability is not automatic: it must be verified from the coupled spectral coefficients. We develop a hierarchy of increasingly coarse sufficient conditions for cut stability, and exhibit a counterexample showing that the Pareto regime is unavoidable.

Theorem 6.29 (Spectral certificate for coupled cut stability)**.** *Let $A \in \mathbb{Z}_{\geq 0}^{n \times n}$ be symmetric, and let λ, λ' be a Galois-conjugate simple coupled pair of roots of the secular equation* (10) *for $M(A,e)$. With the notation of Theorem 4.1, define for $t \in \{\lambda, \lambda'\}$:*

$$x(t) = \sum_{i=1}^{n} \frac{p_i \mu_i}{D_i(t)}\, v_i, \qquad y(t) = \sum_{i=1}^{n} \frac{p_i (t - \mu_i)}{D_i(t)}\, v_i, \tag{41}$$

and $w(t) := (x(t), y(t), 1)$. Then the following are equivalent:

(i) $S^{\pm}(w(\lambda)) = S^{\pm}(w(\lambda'))$;

(ii) for every coordinate $r = 1, \ldots, n$,

$$x_r(\lambda) < 0 \iff x_r(\lambda') < 0 \quad \textit{and} \quad y_r(\lambda) < 0 \iff y_r(\lambda') < 0.$$

Under these equivalent conditions, the two coupled modes have identical certification sets (hence transfer in both directions).

Proof. The formulas for $x(t), y(t)$ are exactly the coupled-eigenvector coefficients from Theorem 4.1. Since the apex coordinate is fixed to 1 for both $w(\lambda), w(\lambda')$, sign-cut equality is equivalent to coordinate-wise agreement of negativity/nonnegativity on original and shadow layers, proving (i)⇔(ii).

If the cuts are equal, the induced principal submatrices on both sides are identical for the two modes; therefore the two chromatic pairs are identical, and by Theorem 6.6 their certification sets are equal. □

Theorem 6.30 (One-mode eigenprobe transfer class)**.** *Let $A \in \mathbb{Z}^{n\times n}$ be symmetric, and let $e \in \mathbb{R}^n \setminus \{0\}$ satisfy*

$$Ae = \kappa e$$

Define $D_\kappa(t) := t^2 - \kappa t - \kappa^2$. Let λ, λ' be a real Galois-conjugate coupled pair of eigenvalues of $M(A, e)$, and let $w(\lambda), w(\lambda')$ be the corresponding apex-normalized coupled eigenvectors. If

$$\lambda\lambda' > 0, \qquad D_\kappa(\lambda)\, D_\kappa(\lambda') > 0,$$

then

$$S^{\pm}\big(w(\lambda)\big) = S^{\pm}\big(w(\lambda')\big),$$

hence

$$\mathcal{C}_{M(A,e)}\big(w(\lambda)\big) = \mathcal{C}_{M(A,e)}\big(w(\lambda')\big).$$

This includes Perron-aligned probes and non-Perron mixed-sign eigenprobes.

Proof. Because $Ae = \kappa e$, only the κ-eigenspace is coupled (Theorem 4.4). Let $v_1 = e/\|e\|$; then $p_1 = \|e\|$ and $p_i = 0$ for $i \geq 2$. Using the coupled formulas in Theorem 4.1 at a coupled root t:

$$x(t) = \frac{\kappa}{D_\kappa(t)}\, e, \qquad y(t) = \frac{t}{\|e\|^2}\, e.$$

For each coordinate r, if $e_r = 0$ then $(x_r(t), y_r(t)) = (0, 0)$ for all t. If $e_r \neq 0$, then

$$\operatorname{sgn}(x_r(t)) = \operatorname{sgn}\!\Big(\frac{\kappa}{D_\kappa(t)}\Big) \operatorname{sgn}(e_r), \qquad \operatorname{sgn}(y_r(t)) = \operatorname{sgn}(t) \operatorname{sgn}(e_r).$$

Hence $\lambda\lambda' > 0$ and $D_\kappa(\lambda) D_\kappa(\lambda') > 0$ force coordinate-wise sign agreement between $w(\lambda)$ and $w(\lambda')$ on both layers. Apex signs are identical by normalization. Therefore the sign cuts coincide.

Certification-set equality follows from Theorem 6.12. □

Theorem 6.31 (Perron-dominance transfer class)**.** *Let $A \in \mathbb{Z}^{n\times n}_{\geq 0}$ be symmetric and irreducible, with orthonormal eigenbasis $\{v_i\}_{i=1}^n$ and eigenvalues $\mu_1 = \rho > \mu_2 \geq \cdots \geq \mu_n$. Let $v_1 > 0$ be the Perron vector, $\|v_1\| = 1$, and*

$$m := \min_{1 \leq r \leq n} (v_1)_r > 0.$$

Write the probe as

$$e = \alpha v_1 + u, \qquad u \perp v_1.$$

Let λ, λ' be a real Galois-conjugate coupled pair of eigenvalues of $M(A, e)$, with apex-normalized coupled eigenvectors $w(\lambda), w(\lambda')$. For $t \in \{\lambda, \lambda'\}$ define

$$f_t(\mu) := \frac{\mu}{t^2 - t\mu - \mu^2}, \qquad g_t(\mu) := \frac{t - \mu}{t^2 - t\mu - \mu^2},$$

and

$$M_f(t) := \max_{2 \le i \le n} |f_t(\mu_i)|, \qquad M_g(t) := \max_{2 \le i \le n} |g_t(\mu_i)|.$$

Assume:

(i) $f_\lambda(\rho)\, f_{\lambda'}(\rho) > 0$ *and* $g_\lambda(\rho)\, g_{\lambda'}(\rho) > 0$*;*

(ii)

$$|\alpha|\, m \min_{t \in \{\lambda, \lambda'\}} |f_t(\rho)| > \|u\| \max_{t \in \{\lambda, \lambda'\}} M_f(t);$$

(iii)

$$|\alpha|\, m \min_{t \in \{\lambda, \lambda'\}} |g_t(\rho)| > \|u\| \max_{t \in \{\lambda, \lambda'\}} M_g(t).$$

Then

$$S^{\pm}\big(w(\lambda)\big) = S^{\pm}\big(w(\lambda')\big),$$

hence

$$\mathcal{C}_{M(A,e)}\big(w(\lambda)\big) = \mathcal{C}_{M(A,e)}\big(w(\lambda')\big).$$

Proof. From Theorem 4.1, for any coupled root t,

$$x(t) = \sum_{i=1}^{n} p_i f_t(\mu_i)\, v_i, \qquad y(t) = \sum_{i=1}^{n} p_i g_t(\mu_i)\, v_i,$$

where $p_i = e^\top v_i$. Since $e = \alpha v_1 + u$ and $u \perp v_1$, write $u = \sum_{i=2}^{n} \beta_i v_i$, so $p_1 = \alpha$ and $p_i = \beta_i$ for $i \ge 2$. Therefore

$$x(t) = \alpha f_t(\rho)\, v_1 + r_x(t), \qquad r_x(t) := \sum_{i=2}^{n} \beta_i f_t(\mu_i)\, v_i,$$

$$y(t) = \alpha g_t(\rho)\, v_1 + r_y(t), \qquad r_y(t) := \sum_{i=2}^{n} \beta_i g_t(\mu_i)\, v_i.$$

Fix a coordinate r. By Cauchy–Schwarz,

$$\begin{aligned} |(r_x(t))_r| &\le M_f(t) \sum_{i=2}^{n} |\beta_i|\, |v_i(r)| \\ &\le M_f(t) \Big(\sum_{i=2}^{n} \beta_i^2\Big)^{1/2} \Big(\sum_{i=2}^{n} v_i(r)^2\Big)^{1/2} \le M_f(t)\|u\|, \end{aligned}$$

since $\sum_{i=2}^{n} v_i(r)^2 \le \sum_{i=1}^{n} v_i(r)^2 = 1$. Similarly, $|(r_y(t))_r| \le M_g(t)\|u\|$.

By (ii), $|\alpha f_t(\rho)|(v_1)_r \ge |\alpha f_t(\rho)| m > \|u\| M_f(t) \ge |(r_x(t))_r|$ for both $t \in \{\lambda, \lambda'\}$, so

$$\operatorname{sgn}(x_r(t)) = \operatorname{sgn}\big(\alpha f_t(\rho)\big) \quad \text{for all } r.$$

By (iii), analogously

$$\operatorname{sgn}(y_r(t)) = \operatorname{sgn}\big(\alpha g_t(\rho)\big) \quad \text{for all } r.$$

Now apply (i): the signs of $\alpha f_t(\rho)$ agree for $t = \lambda, \lambda'$, and the signs of $\alpha g_t(\rho)$ also agree. Hence the original-layer and shadow-layer sign patterns are identical between $w(\lambda)$ and $w(\lambda')$. Apex signs agree by normalization. Therefore $S^{\pm}(w(\lambda)) = S^{\pm}(w(\lambda'))$. Certification-set equality follows from Theorem 6.12. □

Corollary 6.32 (Spectral-window resolvent-margin criterion)**.** *Under the setup of Theorem 6.31, define*

$$\nu := \max_{2\le i\le n} |\mu_i|.$$

For $t \in \{\lambda, \lambda'\}$ set

$$\delta_+(t) := \operatorname{dist}\big(t/\varphi,\ [-\nu,\nu]\big), \quad \delta_-(t) := \operatorname{dist}\big(-\varphi t,\ [-\nu,\nu]\big), \quad \Gamma(t) := \delta_+(t)\delta_-(t).$$

Assume:

(i) $f_\lambda(\rho)\, f_{\lambda'}(\rho) > 0$ *and* $g_\lambda(\rho)\, g_{\lambda'}(\rho) > 0$*;*

(ii) $\Gamma(\lambda) > 0$ *and* $\Gamma(\lambda') > 0$*;*

(iii)
$$|\alpha|\, m \min_{t\in\{\lambda,\lambda'\}} |f_t(\rho)| > \|u\| \max_{t\in\{\lambda,\lambda'\}} \frac{\nu}{\Gamma(t)};$$

(iv)
$$|\alpha|\, m \min_{t\in\{\lambda,\lambda'\}} |g_t(\rho)| > \|u\| \max_{t\in\{\lambda,\lambda'\}} \frac{|t|+\nu}{\Gamma(t)}.$$

Then

$$S^{\pm}\big(w(\lambda)\big) = S^{\pm}\big(w(\lambda')\big),$$

hence

$$\mathcal{C}_{M(A,e)}\big(w(\lambda)\big) = \mathcal{C}_{M(A,e)}\big(w(\lambda')\big).$$

Proof. For fixed t, write

$$D_i(t) = t^2 - t\mu_i - \mu_i^2 = -(\mu_i - t/\varphi)(\mu_i + \varphi t).$$

Since $|\mu_i| \le \nu$ for $i \ge 2$ and $\Gamma(t) > 0$, we have

$$|\mu_i - t/\varphi| \ge \delta_+(t), \qquad |\mu_i + \varphi t| \ge \delta_-(t),$$

hence

$$|D_i(t)| \ge \Gamma(t) \qquad (i \ge 2).$$

Therefore, for $i \ge 2$,

$$|f_t(\mu_i)| = \frac{|\mu_i|}{|D_i(t)|} \le \frac{\nu}{\Gamma(t)}, \qquad |g_t(\mu_i)| = \frac{|t-\mu_i|}{|D_i(t)|} \le \frac{|t|+\nu}{\Gamma(t)}.$$

So

$$M_f(t) \le \frac{\nu}{\Gamma(t)}, \qquad M_g(t) \le \frac{|t|+\nu}{\Gamma(t)}.$$

Assumptions (iii)–(iv) imply assumptions (ii)–(iii) of Theorem 6.31; assumption (i) is the same. Applying Theorem 6.31 yields the claim. □

Corollary 6.33 (Perron-misalignment threshold form)**.** *Under the setup of Theorem 6.32, define*

$$\Theta_f := m\,\frac{\min_{t\in\{\lambda,\lambda'\}} |f_t(\rho)|}{\max_{t\in\{\lambda,\lambda'\}} \nu/\Gamma(t)}, \qquad \Theta_g := m\,\frac{\min_{t\in\{\lambda,\lambda'\}} |g_t(\rho)|}{\max_{t\in\{\lambda,\lambda'\}} (|t|+\nu)/\Gamma(t)}.$$

If

$$\frac{\|u\|}{|\alpha|} < \min\{\Theta_f, \Theta_g\},$$

then

$$S^{\pm}\big(w(\lambda)\big) = S^{\pm}\big(w(\lambda')\big), \qquad \mathcal{C}_{M(A,e)}\big(w(\lambda)\big) = \mathcal{C}_{M(A,e)}\big(w(\lambda')\big).$$

Proof. Assuming $|\alpha| > 0$, divide the inequalities in Theorem 6.32(iii)–(iv) by $|\alpha|$. The resulting pair of bounds is exactly $\|u\|/|\alpha| < \Theta_f$ and $\|u\|/|\alpha| < \Theta_g$. Apply Theorem 6.32. □

Remark 6.34 (Reduction of the coupled universal-winner question)*.* The coupled extension of the strong Positive-Wins claim asks whether every coupled conjugate pair admits a target-independent universal winner. Theorem 6.16 reduces this question to sign-stability of the eigenvector coordinates together with nonnegative side-wise Galois coefficients. Theorem 6.29 certifies this directly from $(\mu_i, p_i, \lambda, \lambda')$, without explicit coloring. Successively coarser sufficient conditions then follow: Theorem 6.30 (one-mode eigenprobes), Theorem 6.31 (Perron-dominant probes with resolvent margins), and Theorem 6.33 (single-ratio threshold form). Together with Theorem 6.11, this yields the coupled-mode classification theorem (Theorem 6.37).

Proposition 6.35 (Coupled counterexample to the unconditional form)**.** *Let $G_\star$ be the graph on vertices $\{1,\dots,7\}$ with*

$$E(G_\star) = \{(1,2),(1,6),(2,3),(2,4),(2,6),(2,7),(3,5),(3,6),(4,6),(4,7),(5,7),(6,7)\}.$$

Set $H = M(G_\star, \mathbf{1})$. Then:

- *(i) H has the irreducible quadratic orbit $\{\varphi, -\varphi^{-1}\}$;*
- *(ii) the corresponding eigenvectors are coupled (nonzero apex component);*
- *(iii) after normalising each by positive apex coordinate, the nodal-domain chromatic pairs are*

$$\big(\chi(H[S^+_\varphi]),\, \chi(H[S^-_\varphi])\big) = (3,3), \qquad \big(\chi(H[S^+_{-\varphi^{-1}}]),\, \chi(H[S^-_{-\varphi^{-1}}])\big) = (4,2).$$

Hence the target $(s,t) = (4,2)$ is certified by $-\varphi^{-1}$ but not by φ. Therefore the unconditional statement of Theorem 6.39 is false for coupled modes.

Proof. The reproducibility script `reproducibility/code/counterexample_coupled_positive_wins.py` constructs H exactly from the above edge list, confirms

$$\chi(H) = 5, \qquad \omega(H) = 4,$$

and identifies the factor $x^2 - x - 1 = (x-\varphi)(x+\varphi^{-1})$ in $\chi_H(x)$, yielding the orbit $\{\varphi, -\varphi^{-1}\}$. The same script verifies nonzero apex components for both eigenvectors and computes the induced chromatic numbers of their nodal sets by exact backtracking coloring, obtaining the two stated pairs $(3,3)$ and $(4,2)$. The certification claim for target $(4,2)$ follows immediately. □

Corollary 6.36 (Coupled Pareto behaviour)**.** *In the coupled pair of Theorem 6.35,*

$$\mathcal{C}_H(v_\varphi) \not\subseteq \mathcal{C}_H(v_{-\varphi^{-1}}) \quad \textit{and} \quad \mathcal{C}_H(v_{-\varphi^{-1}}) \not\subseteq \mathcal{C}_H(v_\varphi).$$

Hence the pair is in the Pareto regime of Theorem 6.8.

Proof. From Theorem 6.35, $v_{-\varphi^{-1}}$ certifies $(4,2)$ while v_φ does not, and v_φ certifies $(3,3)$ while $v_{-\varphi^{-1}}$ does not ($4 \geq 3$ but $2 \ngeq 3$). □

Theorem 6.37 (Coupled-mode classification and impossibility of universal winners)**.** *For integer symmetric nonnegative matrices, coupled conjugate pairs satisfy the following exact statement:*

(i) The unconditional Strong Positive-Wins form (Theorem 6.39) is false in coupled modes (Theorem 6.35).
(ii) For any conjugate pair $(v, \sigma(v))$, certification transfer and Pareto behavior are completely characterized by chromatic signatures:

$$\mathcal{C}_A(\sigma(v)) \subseteq \mathcal{C}_A(v) \iff \kappa_A(v) \geq \kappa_A(\sigma(v))$$

coordinate-wise, and Pareto iff the two signatures are incomparable (Theorem 6.11).
(iii) Consequently, universal scalar winner laws are impossible in general for coupled modes, and the strongest true universal law is transfer/Pareto classification together with structural transfer criteria (cut stability, side-wise homomorphic dominance, and coarse Perron-family certificates).

Proof. Part (i) is Theorem 6.35. Part (ii) is exactly Theorem 6.11. For (iii), the counterexample in (i) rules out a universal scalar winner law for coupled modes, while (ii) gives an exact classification for every pair; the listed structural criteria are provided by Theorems 6.14, 6.16 and 6.30 to 6.33. □

Corollary 6.38 (No target-independent coupled winner)**.** *There is no rule that assigns, to every coupled conjugate pair, a single universally winning member that transfers all certifiable targets of the other member for all integer symmetric nonnegative matrices.*

Proof. If such a rule existed, it would imply the unconditional transfer form of Theorem 6.39 on coupled pairs. This contradicts Theorems 6.35 and 6.36, where each member wins on a different target. □

6.6. **The three structural laws.** We formulate three conjectural laws governing which Galois orbit members produce effective eigenvector partitions. The first is proved in spectral form for transparent modes (Theorem 6.25), and can be promoted to certification transfer once side-wise chromatic dominance is verified (Theorem 6.28); it is shown to fail unconditionally for coupled modes (Theorem 6.35); the exact replacement is the Transfer/Pareto classification of Theorem 6.11.

Conjecture 6.39 (Strong Positive-Wins Law (unconditional form))**.** *Let $\mathcal{O} = \{\alpha_1 > \cdots > \alpha_d\}$ be a Galois orbit of degree $d \geq 2$ in* Spec(A) *for an integer symmetric nonnegative matrix A. If any α_j certifies a target partition, then α_1 certifies it with margin at least as large.*

Observation 6.40. The conjecture matches the original benchmark corpus (13/13 orbits with at least one successful member, across degrees 2, 3, and 5). Among 50 quadratic pairs in $\mathbb{Q}(\sqrt{D})$ for $D \in \{5, 17, 41\}$, the positive conjugate certifies in 32 cases and the negative conjugate alone in 0 cases. However, Theorem 6.35 shows that the unconditional form fails for coupled modes; the exact replacement is the Transfer/Pareto classification of Theorem 6.11.

Conjecture 6.41 (Monotone Filtration)**.** *Let $\mathcal{O} = \{\alpha_1 > \cdots > \alpha_d\}$ be a Galois orbit and fix a target (s, t). Partition success is monotone in orbit rank: there exists $k \in \{0, \ldots, d\}$ such that α_j certifies (s, t) if and only if $j \leq k$.*

Observation 6.42. Verified on 13/15 orbits (87%). The two exceptions involve margins ≈ 0 at the anomalous member.

Conjecture 6.43 (Redundancy–Degree Law)**.** *The fraction ρ of eigenvectors producing valid target partitions scales inversely with the maximal Galois orbit degree $d_{\max}$: $\rho \sim C/d_{\max}$.*

Observation 6.44.

$d_{\max}$	Average ρ
1 ($\mathbb{Q}$-spectrum)	81%
2 ($\mathbb{Q}(\sqrt{D})$)	67–83%
3 (cubic cyclotomic)	52%
5 (quintic cyclotomic)	44%

Remark 6.45 (Reproducibility of Theorems 6.40, 6.42 and 6.44). All observations were verified on adjacency matrices of Mycielski chains $M_k(K_2)$ for $2 \leq k \leq 8$, random d-regular graphs on $n \in \{20, 50, 100\}$ vertices (10 instances per (n, d) pair), and circulant graphs $C_n(S)$ for $n \leq 50$. Eigenvalues were computed in exact arithmetic using the characteristic polynomial's irreducible factorisation over $\mathbb{Q}$ (via `sympy.polys.polytools.factor`); partition certification used exact chromatic number computation for graphs on ≤ 30 vertices; the Positive Boost Inequality (Theorem 5.2) was used only as a spectral screening statistic. All source code, input graphs, and run instructions are available at `https://github.com/vfssantos/golden-resolvent-theory` (directory `reproducibility/`).

Definition 6.46 (Galois partition signature)**.** The *Galois partition signature* of a matrix A at a target partition specification is the function $\sigma\colon \{\text{Galois orbits}\} \to \{0, 1, \ldots, d\}$ mapping each orbit of degree d to the number k of its members that produce valid partitions. By Theorem 6.41, σ determines *which* members succeed (the top k by orbit rank).

7. The Chebyshev Ladder

The Galois Transfer Principle governs partition quality within a single golden companion. Replacing the golden ratio $\varphi = 2\cos(\pi/5)$ by the amplification constant of a deeper block structure leads to a natural generalisation of the golden companion operator to p intermediate layers.

Definition 7.1 (Generalised golden companion)**.** For $p \geq 1$, the *p-layer golden companion* of (A, e) has $(p+1)n+1$ rows and columns: layers $0, 1, \ldots, p$ each with n coordinates, plus an apex. Layer j is coupled to layer $j-1$ through A; the apex is coupled to layer p through e. Equivalently,

$$M_p(A,e) = \begin{pmatrix} A & A & 0 & \cdots & 0 & 0 \\ A & 0 & A & \ddots & \vdots & \vdots \\ 0 & A & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & A & \vdots \\ 0 & \cdots & 0 & A & 0 & e \\ 0^\top & \cdots & 0^\top & 0^\top & e^\top & 0 \end{pmatrix},$$

where the first $p+1$ block rows/columns are $n \times n$ layer blocks and the final row/column is the apex. Theorem 2.4 is the case $p = 1$.

Theorem 7.2 (Chebyshev Ladder)**.** *For the p-layer golden companion, every eigenvector x with $Ax = \mu x$ and $e^\top x = 0$ produces eigenvalues $\lambda = r\mu$ where r satisfies the ratio polynomial $R_p(r) = 0$ of degree $p + 1$. The ratio polynomials are:*

$$p = 1: \quad r^2 - r - 1 = 0, \tag{42}$$

$$p = 2: \quad r^3 - r^2 - 2r + 1 = 0, \tag{43}$$

$$p = 3: \quad r^4 - r^3 - 3r^2 + 2r + 1 = 0, \tag{44}$$

$$p = 4: \quad r^5 - r^4 - 4r^3 + 3r^2 + 3r - 1 = 0. \tag{45}$$

The dominant real root of R_p is $r_{\max} = 2\cos\bigl(\pi/(2p+3)\bigr)$.

Proof. For the p-layer companion with eigenvector x ($Ax = \mu x$, $e^\top x = 0$), seek eigenvectors $(x, \alpha_1 x, \ldots, \alpha_p x, 0)^\top$ at eigenvalue $\lambda = r\mu$. The block equations give the three-term recurrence

$$\alpha_{j+1} = r\alpha_j - \alpha_{j-1}, \quad \alpha_0 = 1, \quad \alpha_1 = r - 1, \tag{46}$$

with boundary condition $\alpha_{p-1} = r\alpha_p$.

The characteristic equation $t^2 - rt + 1 = 0$ has roots $t = e^{\pm i\theta}$ where $r = 2\cos\theta$. The general solution is

$$\alpha_j = \frac{\sin\big((j+1)\theta\big) - \sin(j\theta)}{\sin\theta} = \frac{\cos\big((j+\frac{1}{2})\theta\big)}{\cos(\theta/2)},$$

using the product-to-sum identity. The boundary condition $\alpha_{p-1} = r\alpha_p$ reduces to

$$\cos\Big((p+\tfrac{3}{2})\theta\Big) = 0 \implies \theta = \frac{(2k-1)\pi}{2p+3}, \quad k = 1, \dots, p+1.$$

This gives $p+1$ distinct values $r = 2\cos\theta \in (-2, 2)$. The dominant root is $r_{\max} = 2\cos\big(\pi/(2p+3)\big)$ (the case $k = 1$). □

Corollary 7.3 (Cyclotomic interpretation)**.** *The dominant amplification ratios form the sequence:*

p	$2p+3$	*Dominant root*	*Decimal*	*Splitting field*
1	*5*	$2\cos(\pi/5) = \varphi$	*1.6180*	$\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5)^+$
2	*7*	$2\cos(\pi/7)$	*1.8019*	$\mathbb{Q}(\zeta_7)^+$
3	*9*	$2\cos(\pi/9)$	*1.8794*	$\mathbb{Q}(\zeta_9)^+$
4	*11*	$2\cos(\pi/11)$	*1.9190*	$\mathbb{Q}(\zeta_{11})^+$

Each layer depth ascends one step on the Chebyshev ladder*: the p-layer operator introduces eigenvalues in the maximal real subfield* $\mathbb{Q}(\zeta_{2p+3})^+$ *of the* $(2p+3)$*-th cyclotomic field.*

In summary, the Chebyshev ladder extends the golden factorisation's core mechanism: each additional layer replaces φ by the next ratio-polynomial root, climbing from $\mathbb{Q}(\sqrt{5})$ through successively richer real cyclotomic fields toward the hyperbolic boundary $r = 2$.

Proposition 7.4 (Compositum principle)**.** *Let μ have minimal polynomial of degree d over $\mathbb{Q}$, and let the ratio polynomial R_p have irreducible degree δ. If $\mathbb{Q}(\mu) \cap \mathbb{Q}(r) = \mathbb{Q}$ (linear disjointness), then the amplified eigenvalue $\lambda = r\mu$ has $[\mathbb{Q}(\lambda) : \mathbb{Q}] = d\delta$.*

7.1. **The hyperbolic continuation and the spectral phase boundary.** The Chebyshev Ladder (Theorem 7.2) produces all its eigenvalue ratios in the interval $(-2, 2)$, because $|2\cos\theta| < 2$ for $\theta \in (0, \pi)$. The three-term recurrence (46) admits a natural analytic continuation beyond the upper spectral edge $r = 2$, obtained by the Wick rotation $\theta \mapsto i\kappa$ on the dominant positive branch. This continuation replaces the oscillatory (trigonometric) regime with an exponential (hyperbolic) regime, and the phase boundary $r = 2$ separates the two branches considered below.

Proposition 7.5 (Hyperbolic Chebyshev recurrence)**.** *The recurrence* (46) *with initial conditions* $\alpha_0 = 1$*,* $\alpha_1 = r - 1$ *has closed-form solution*

$$\alpha_j = \begin{cases} \dfrac{\cos\big((j+\frac{1}{2})\theta\big)}{\cos(\theta/2)}, & r = 2\cos\theta, \quad |r| < 2 \quad \text{(oscillatory)}, \\ 1, & r = 2 \quad \text{(constant)}, \\ \dfrac{\cosh\big((j+\frac{1}{2})\kappa\big)}{\cosh(\kappa/2)}, & r = 2\cosh\kappa, \quad r > 2 \quad \text{(exponential)}. \end{cases} \tag{47}$$

The three cases are related by the Wick rotation $\theta \mapsto i\kappa$*, under which* $\cos \to \cosh$*.*

Proof. The characteristic equation of the recurrence $\alpha_{j+1} = r\alpha_j - \alpha_{j-1}$ is $t^2 - rt + 1 = 0$, with discriminant $\Delta_t = r^2 - 4$.

Case $|r| < 2$**.** Set $r = 2\cos\theta$ with $\theta \in (0, \pi)$. The roots are $t = e^{\pm i\theta}$, and the general solution is $\alpha_j = A\cos(j\theta) + B\sin(j\theta)$. The initial conditions $\alpha_0 = 1$, $\alpha_1 = 2\cos\theta - 1$ give $A = 1$ and $B = (2\cos\theta - 1 - \cos\theta)/\sin\theta = (\cos\theta - 1)/\sin\theta = -\tan(\theta/2)$. Hence

$$\alpha_j = \cos(j\theta) - \tan(\theta/2)\sin(j\theta) = \frac{\cos(j\theta)\cos(\theta/2) - \sin(j\theta)\sin(\theta/2)}{\cos(\theta/2)} = \frac{\cos\big((j+\frac{1}{2})\theta\big)}{\cos(\theta/2)}.$$

Case $r = 2$**.** The characteristic equation has a double root $t = 1$, giving $\alpha_j = A + Bj$. From $\alpha_0 = 1$: $A = 1$. From $\alpha_1 = 1$: $B = 0$. Hence $\alpha_j = 1$ for all j.

Case $r > 2$**.** Set $r = 2\cosh\kappa$ with $\kappa > 0$. The roots are $t = e^{\pm\kappa}$, and $\alpha_j = Ae^{j\kappa} + Be^{-j\kappa}$. The initial conditions give $A + B = 1$ and $Ae^{\kappa} + Be^{-\kappa} = 2\cosh\kappa - 1$, so $A = (e^{\kappa} - 1)/(2\sinh\kappa)$ and $B = (1 - e^{-\kappa})/(2\sinh\kappa)$. Then

$$\begin{aligned} \alpha_j &= \frac{(e^{\kappa} - 1)e^{j\kappa} + (1 - e^{-\kappa})e^{-j\kappa}}{2\sinh\kappa} \\ &= \frac{\sinh\big((j{+}1)\kappa\big) - \sinh(j\kappa)}{\sinh\kappa} \\ &= \frac{2\cosh\big((j+\frac{1}{2})\kappa\big)\sinh(\kappa/2)}{2\sinh(\kappa/2)\cosh(\kappa/2)} = \frac{\cosh\big((j+\frac{1}{2})\kappa\big)}{\cosh(\kappa/2)}. \end{aligned}$$

The passage from the oscillatory to the exponential formula is the substitution $\theta = i\kappa$, under which $\cos(z) \to \cos(iz) = \cosh(z)$. □

Corollary 7.6 (Spectral phase boundary)**.** *The three regimes have the following qualitative behaviour:*

Regime	r	*Roots of* $t^2 - rt + 1$	$\lvert\alpha_j\rvert$	*Character*
Oscillatory	$\lvert r\rvert < 2$	$e^{\pm i\theta}$ *(unit circle)*	*bounded*	*algebraic*
Critical	$r = 2$	$1, 1$ *(double)*	*constant* $(= 1)$	*rational*
Exponential	$r > 2$	$e^{\pm\kappa}$ *(real,* > 0*)*	$\sim e^{j\kappa}$	*transcendental*

In the oscillatory regime, α_j changes sign; in the exponential regime, $\alpha_j > 0$ for all $j \geq 0$.

Proof. In the oscillatory case, $\cos\big((j+\frac{1}{2})\theta\big)$ has zeros at $(j+\frac{1}{2})\theta = \pi/2 + k\pi$, so α_j changes sign. In the exponential case, $\cosh(x) \geq 1 > 0$ for all real x, so $\alpha_j = \cosh\big((j+\frac{1}{2})\kappa\big)/\cosh(\kappa/2) \geq 1$ for all $j \geq 0$. □

Theorem 7.7 (Non-existence of apex resonance in the hyperbolic regime)**.** *The apex boundary condition of Theorem 7.2,*

$$\cos\Big((p+\tfrac{3}{2})\theta\Big) = 0, \tag{48}$$

which selects the Chebyshev Ladder roots $\theta_k = (2k-1)\pi/(2p+3)$ in the oscillatory regime, has no real solution *under the Wick rotation $\theta \mapsto i\kappa$: the equation $\cosh\big((p+\frac{3}{2})\kappa\big) = 0$ has no solution for $\kappa \in \mathbb{R}$.*

Consequently, the spectral selection mechanism of the Chebyshev Ladder—which produces eigenvalues at specific algebraic points in $\mathbb{Q}(\zeta_{2p+3})^+$—is a purely oscillatory phenomenon. It does not extend to the hyperbolic regime.

Proof. $\cosh(x) = (e^x + e^{-x})/2 \geq 1$ for all $x \in \mathbb{R}$. □

Remark 7.8 (The dual Chebyshev Ladder). The Wick rotation $\theta \mapsto i\kappa$ maps each rung of the Chebyshev Ladder to a *dual rung*:

$$r_p = 2\cos\Big(\frac{\pi}{2p+3}\Big) \quad \xrightarrow{\theta\mapsto i\kappa} \quad \tilde{r}_p = 2\cosh\Big(\frac{\pi}{2p+3}\Big).$$

The two families approach $r = 2$ from opposite sides:

p	r_p (oscillatory)	$\tilde{r}_p$ (exponential)	Growth $e^{(p+3/2)\pi/(2p+3)}$
1	$\varphi = 1.618$	2.408	$e^{5\pi/10} = e^{\pi/2} \approx 4.81$
2	1.802	2.205	$e^{7\pi/14} = e^{\pi/2} \approx 4.81$
3	1.879	2.123	≈ 4.81
4	1.919	2.082	≈ 4.81

For every p, the quantity $(p+\frac{3}{2}) \cdot \pi/(2p+3) = \pi/2$ is constant, so the total growth α_{p+1} evaluated at the dual rung $\tilde{r}_p$ satisfies

$$\alpha_{p+1}\big|_{r=\tilde{r}_p} = \frac{\cosh(\pi/2)}{\cosh\big(\pi/(2(2p+3))\big)} \xrightarrow{p\to\infty} \cosh(\pi/2) \approx 2.509. \tag{49}$$

This constant asymptotic amplification contrasts with the oscillatory ladder, where $\alpha_{p+1} = 0$ at the corresponding rung r_p (the apex boundary condition).

Remark 7.9 (Additive versus multiplicative self-similarity). The spectral phase boundary $r = 2$ separates two types of algebraic self-similarity in the Chebyshev recurrence:

- **Oscillatory regime** ($r < 2$): governed by $\varphi = 2\cos(\pi/5)$ at the first rung. The defining relation $\varphi^2 = \varphi + 1$ is *additive self-similarity*: the square decomposes as a sum. Eigenvalues lie in cyclotomic fields $\mathbb{Q}(\zeta_{2p+3})^+$ and the sequences $\{\alpha_j\}$ are bounded.

- **Exponential regime** ($r > 2$): governed by e^{κ} where $\kappa = \pi/(2p+3)$ at the dual rung. The defining relation $\frac{d}{dx}e^x = e^x$ is *multiplicative self-similarity*: the derivative equals the function. The sequences $\{\alpha_j\}$ grow as $e^{j\kappa}$, and the amplification ratios are transcendental.

The Chebyshev Ladder's spectral selection mechanism (Theorem 7.2) operates exclusively in the oscillatory regime, where the apex boundary condition $\cos\big((p+\frac{3}{2})\theta\big) = 0$ has solutions. The exponential regime has no apex resonances (Theorem 7.7), but its recurrence coefficients are universally sign-preserving (Theorem 7.6).

8. Stability Analysis and Iterated Operators

8.1. **A secular-weight barrier lemma (finite depth).** We isolate a clean finite-depth stability statement that serves as a mathematical template for time-varying operator families. The key inputs are: (i) a uniform lower barrier on the secular weight (simplicity margin), and (ii) a uniform oscillatory margin for the Chebyshev recurrence coefficient (distance from the hyperbolic phase boundary).

Step 1: explicit secular function. Fix a real symmetric $A \in \mathbb{R}^{n\times n}$ with eigenpairs (μ_i, v_i) and a probe vector $e \in \mathbb{R}^n$. Write $p_i := e^\top v_i$ and

$$D_i(\lambda) := \lambda^2 - \lambda\mu_i - \mu_i^2 = (\lambda - \mu_i\varphi)(\lambda + \mu_i/\varphi). \tag{50}$$

Define the *secular function*

$$s(\lambda; \mathbf{p}^2) := \sum_{i=1}^{n} \frac{p_i^2(\lambda - \mu_i)}{D_i(\lambda)} - \lambda, \tag{51}$$

so that coupled eigenvalues are exactly the roots $s(\lambda; \mathbf{p}^2) = 0$ (cf. Theorem 4.1 and (19)).

Following Theorem 4.8, define the *secular weight*

$$W(\lambda) := 1 + \sum_{i=1}^{n} p_i^2 \, \frac{(\lambda - \mu_i)^2 + \mu_i^2}{D_i(\lambda)^2}. \tag{52}$$

At a simple coupled root λ_* we have $W(\lambda_*) \neq 0$ and the participation sensitivities satisfy

$$\frac{\partial \lambda_*}{\partial (p_k^2)} = \frac{N_k(\lambda_*)}{W(\lambda_*)}, \qquad N_k(\lambda) := \frac{\lambda - \mu_k}{D_k(\lambda)}. \tag{53}$$

Step 2: explicit Chebyshev recurrence and coefficient r. We work with the three-term recurrence from the Chebyshev Ladder (Theorem 7.2):

$$x_{j+1} = r\, x_j - x_{j-1}, \qquad x_0 = 1, \quad x_1 = r - 1, \tag{54}$$

where the recurrence coefficient is the (time-dependent) ratio parameter r (in the p-layer companion, $r = \lambda/\mu$ for the corresponding transparent branch eigenvalue $\lambda = r\mu$). For the purposes of a time-varying reduced model, we allow $r = r(t)$ and write $x_j(t)$ for the resulting finite-depth ladder vector.

Step 3: leader diagnostic. Fix a depth $m \in \mathbb{N}$ and a nonempty set of *active channels* $K \subseteq \{1, \dots, n\}$. Given a (time-varying) coupled root $\lambda_*(t)$ and probe participations $p_i(t)$, define the depth-m leader diagnostic

$$\mathcal{L}_m(t) := \Big(x_0(t), \dots, x_m(t)\,;\, \Big\{\frac{\partial \lambda_*(t)}{\partial (p_k(t)^2)}\Big\}_{k \in K}\Big), \tag{55}$$

with max norm

$$\|\mathcal{L}_m(t)\|_\infty := \max\Big\{\max_{0 \le j \le m} |x_j(t)|,\ \max_{k \in K} \Big|\frac{\partial \lambda_*(t)}{\partial (p_k(t)^2)}\Big|\Big\}. \tag{56}$$

We say there is *finite-time leader blow-up (at depth m)* at time T if $\limsup_{t \uparrow T} \|\mathcal{L}_m(t)\|_\infty = +\infty$.

Lemma 8.1 (Oscillatory ladder bound with tracked constant)**.** *Fix $\varepsilon \in (0, 2)$ and $m \in \mathbb{N}$. Assume $|r| \le 2 - \varepsilon$ in (54). Then for all $0 \le j \le m$,*

$$|x_j| \le C_{\mathrm{osc}}(\varepsilon) \quad \textit{where} \quad C_{\mathrm{osc}}(\varepsilon) := \csc\Big(\tfrac{1}{2}\arccos(1 - \varepsilon/2)\Big) = \frac{2}{\sqrt{\varepsilon}}. \tag{57}$$

In particular, $\max_{0 \le j \le m} |x_j| \le C_{\mathrm{osc}}(\varepsilon)$ *uniformly in m (for fixed ε).*

Proof. Since $|r| \le 2 - \varepsilon < 2$, write $r = 2\cos\theta$ with $\theta \in [\theta_0, \pi - \theta_0]$ where $\theta_0 := \arccos(1 - \varepsilon/2) \in (0, \pi)$. By Theorem 7.5 (oscillatory case (47)), the solution of (54) is

$$x_j = \frac{\cos((j + \frac{1}{2})\theta)}{\cos(\theta/2)}.$$

Hence $|x_j| \le 1/|\cos(\theta/2)|$. Over $\theta \in [\theta_0, \pi - \theta_0]$ we have $\cos(\theta/2) \ge \cos((\pi - \theta_0)/2) = \sin(\theta_0/2)$, so $|x_j| \le 1/\sin(\theta_0/2) = C_{\mathrm{osc}}(\varepsilon)$. The identity $C_{\mathrm{osc}}(\varepsilon) = 2/\sqrt{\varepsilon}$ follows from $\sin^2(\theta_0/2) = (1 - \cos\theta_0)/2 = \varepsilon/4$. □

Lemma 8.2 (Sensitivity bound under secular-weight and pole-separation barriers)**.** *Fix $w_0 > 0$ and $\delta > 0$, and let λ_* be a simple coupled root of $s(\lambda; \mathbf{p}^2) = 0$. Assume the* secular-weight barrier

$$|W(\lambda_*)| \ge w_0, \tag{58}$$

and the pole-separation barrier *on the active channels K:*

$$\min_{k \in K} |D_k(\lambda_*)| \ge \delta. \tag{59}$$

Then for every $k \in K$,

$$\Big|\frac{\partial \lambda_*}{\partial (p_k^2)}\Big| \le \frac{|\lambda_*| + |\mu_k|}{\delta\, w_0}. \tag{60}$$

Consequently, if additionally $|\lambda_*| \le \Lambda$ and $\max_{k \in K} |\mu_k| \le M$, then

$$\max_{k \in K} \Big|\frac{\partial \lambda_*}{\partial (p_k^2)}\Big| \le \frac{\Lambda + M}{\delta\, w_0}. \tag{61}$$

Proof. From Theorem 4.8(i) we have $\partial\lambda_*/\partial(p_k^2) = N_k(\lambda_*)/W(\lambda_*)$ with $N_k(\lambda) = (\lambda - \mu_k)/D_k(\lambda)$. Thus, using (58) and (59),

$$\left|\frac{\partial\lambda_*}{\partial(p_k^2)}\right| \le \frac{|N_k(\lambda_*)|}{|W(\lambda_*)|} \le \frac{|\lambda_* - \mu_k|}{\delta\, w_0} \le \frac{|\lambda_*| + |\mu_k|}{\delta\, w_0}.$$

The compact bound (61) follows immediately. □

Concretely, leader blow-up means that eigenvalue sensitivities diverge: an infinitesimal change in the coupling vector produces an unbounded shift in the coupled spectrum. The next proposition identifies three explicit barriers that together prevent this pathology.

Proposition 8.3 (Secular-weight barrier + oscillatory margin ⇒ no leader blow-up (finite depth))**.** *Fix $\varepsilon \in (0, 2)$, $m \in \mathbb{N}$, and a nonempty active channel set K. Let $t \mapsto (A(t), e(t))$ be a time-varying family with a chosen simple coupled root $\lambda_*(t)$ of $s(\lambda; \mathbf{p}(t)^2) = 0$ on $t \in [0, T)$. Assume the three barriers hold on $[0, T)$:*

- *(i)* ***Secular-weight barrier:*** $|W(\lambda_*(t))| \ge w_0 > 0$.
- *(ii)* ***Oscillatory margin:*** $|r(t)| \le 2 - \varepsilon$, *where $r(t)$ is the recurrence coefficient in* (54).
- *(iii)* ***Pole separation on active channels:*** $\min_{k\in K} |D_k(\lambda_*(t))| \ge \delta > 0$.

If furthermore $|\lambda_*(t)| \le \Lambda$ and $\max_{k\in K} |\mu_k(t)| \le M$ for all $t \in [0, T)$, then the depth-m leader diagnostic $\mathcal{L}_m(t)$ from (55) *is uniformly bounded:*

$$\sup_{0\le t<T} \|\mathcal{L}_m(t)\|_\infty \;\le\; \max\left\{C_{\text{osc}}(\varepsilon),\; \frac{\Lambda + M}{\delta\, w_0}\right\}. \tag{62}$$

In particular, there is no finite-time leader blow-up (at depth m) on $[0, T)$.

Proof. By Theorem 8.1 and the oscillatory margin (ii), for all $t \in [0, T)$ we have $\max_{0\le j\le m} |x_j(t)| \le C_{\text{osc}}(\varepsilon)$. By Theorem 8.2 and assumptions (i)+(iii) plus the uniform bounds on λ_* and μ_k, we have for all $t \in [0, T)$,

$$\max_{k\in K} \left|\frac{\partial\lambda_*(t)}{\partial(p_k(t)^2)}\right| \le \frac{\Lambda + M}{\delta\, w_0}.$$

Taking the maximum of the two components yields (62). □

Remark 8.4 (Three-barrier interpretation). Theorem 8.3 has a direct intrinsic Golden Resolvent Theory reading: (i) $W(\lambda_*)$ is the simplicity/sensitivity denominator governing the magnitude of participation derivatives in (53); (ii) $r = 2$ is the intrinsic phase boundary separating oscillatory and hyperbolic Chebyshev continuation; and (iii) $|D_k(\lambda_*)|$ is a pole-distance regularity quantity on channel k, preventing near-pole amplification of $N_k(\lambda_*)$.

8.2. **Dynamic Pareto switching.** In this subsection, $w_+(t)$ and $w_-(t)$ denote the top conjugate candidates. Here $B_L^\pm(t)$ and $B_H^\pm(t)$ denote the low-band and high-band scores assigned to candidate $w_\pm(t)$, respectively. This subsection models a time-varying tradeoff between two certification surrogate bands; the weight $\theta(t)$ below encodes the current relative emphasis between the low-band and high-band scores. We write

$$\Delta_L(t) := B_L^+(t) - B_L^-(t), \qquad \Delta_H(t) := B_H^+(t) - B_H^-(t),$$

for the low-band and high-band score gaps, and use *persistent Pareto conflict* to mean $\Delta_L(t) > 0 > \Delta_H(t)$.

Definition 8.5 (Dynamic candidate-dominance gap)**.** Let $\theta : [0, T) \to [0, 1]$ be an effective weight and define

$$D(t) := \theta(t)\,\Delta_L(t) + (1 - \theta(t))\,\Delta_H(t). \tag{63}$$

When $\Delta_L(t) > 0 > \Delta_H(t)$, define the critical threshold

$$\theta_c(t) := \frac{-\Delta_H(t)}{\Delta_L(t) - \Delta_H(t)} \in (0, 1). \tag{64}$$

Theorem 8.6 (Threshold equivalence in persistent Pareto conflict (candidate form))**.** *Assume $\Delta_L(t) > 0 > \Delta_H(t)$ for all $t \in [0, T)$. Then*

$$D(t) = \big(\Delta_L(t) - \Delta_H(t)\big)\,\big(\theta(t) - \theta_c(t)\big), \tag{65}$$

and therefore

$$\operatorname{sign} D(t) = \operatorname{sign}\big(\theta(t) - \theta_c(t)\big). \tag{66}$$

In particular, candidate-dominance switching is exactly threshold crossing of θ through θ_c.

Proof. By definition of θ_c,

$$\begin{aligned} D(t) &= \theta\Delta_L + (1-\theta)\Delta_H = \theta(\Delta_L - \Delta_H) + \Delta_H \\ &= (\Delta_L - \Delta_H)\left(\theta - \frac{-\Delta_H}{\Delta_L - \Delta_H}\right) = (\Delta_L - \Delta_H)(\theta - \theta_c). \end{aligned}$$

Since $\Delta_L - \Delta_H > 0$, signs agree. □

Remark 8.7 (Transfer window versus conflict-driven switching). Within the dynamic gap model, fixed sign of D on an interval is a *transfer-type dominance window* for one top conjugate candidate. Sign changes of D are exactly conflict-driven switches through the Pareto threshold θ_c, by Theorem 8.6.

Proposition 8.8 (Candidate-switching stability under transversality)**.** *Let $\psi := \theta - \theta_c \in C^1([0, T])$ and set $D := (\Delta_L - \Delta_H)\psi$, where $\Delta_L - \Delta_H \geq \gamma > 0$ on $[0, T]$. Assume the zero set $Z := \{t \in (0, T) : \psi(t) = 0\} = \{t_1, \ldots, t_q\}$ is finite and transversal:*

$$|\dot{\psi}(t_i)| \geq m_0 > 0, \qquad i = 1, \ldots, q. \tag{67}$$

Then there exists $\varepsilon_0 > 0$ *such that for any perturbation* $\widetilde{\psi} \in C^1([0,T])$ *and* $\widetilde{\Delta} \in C^0([0,T])$ *with*

$$\|\widetilde{\psi} - \psi\|_{C^1} + \|\widetilde{\Delta} - (\Delta_L - \Delta_H)\|_{C^0} < \varepsilon_0, \qquad \widetilde{\Delta} \geq \gamma/2, \tag{68}$$

the perturbed gap $\widetilde{D} := \widetilde{\Delta}\,\widetilde{\psi}$ *has exactly* q *transversal zeros in* $(0,T)$*, with one zero near each* t_i*, and the same sign pattern on the complementary intervals. Consequently, the number of candidate-dominance switches is locally stable under small* C^1 *perturbations.*

Proof. By transversality and compactness, each t_i has a neighborhood where ψ is strictly monotone and $|\dot{\psi}| \geq m_0/2$. For small enough $\|\widetilde{\psi} - \psi\|_{C^1}$, the implicit-function theorem (or one-dimensional monotonicity argument) gives a unique zero $\widetilde{t}_i$ near each t_i, still transversal. Outside the union of these neighborhoods, $|\psi|$ has a positive minimum, so for small perturbation $\widetilde{\psi}$ keeps the same sign and no extra zeros appear. Since $\widetilde{\Delta} \geq \gamma/2 > 0$, $\operatorname{sign}(\widetilde{D})$ equals $\operatorname{sign}(\widetilde{\psi})$, giving the same sign pattern and the same switch count. □

Corollary 8.9 (Recurrence implies positive candidate-switching rate)**.** *Assume the hypotheses of Theorem 8.6 and let* $\psi := \theta - \theta_c$ *be continuous on* $[0,\infty)$*. Suppose there exist* $\delta, \tau > 0$ *such that every interval* $[k\tau, (k+1)\tau]$*,* $k \in \mathbb{N}$*, contains times* s_k^-, s_k^+ *with*

$$\psi(s_k^-) \leq -\delta, \qquad \psi(s_k^+) \geq \delta. \tag{69}$$

If every zero of D *is transversal, then the switching count*

$$N_{\mathrm{sw}}(T) := \#\{t \in (0,T) : D(t) = 0,\ \dot{D}(t) \neq 0\} \tag{70}$$

satisfies

$$N_{\mathrm{sw}}(T) \geq \lfloor T/\tau \rfloor, \qquad \liminf_{T\to\infty} \frac{N_{\mathrm{sw}}(T)}{T} \geq \frac{1}{\tau} > 0. \tag{71}$$

Proof. Fix k with $(k+1)\tau < T$. By hypothesis, there are points of opposite sign for ψ inside $[k\tau, (k+1)\tau]$; continuity gives at least one zero of ψ, hence of D, in that interval. Transversality ensures each such zero is a genuine switch. Summing over disjoint intervals yields $N_{\mathrm{sw}}(T) \geq \lfloor T/\tau \rfloor$, and dividing by T gives the lower-rate bound. □

Conjecture 8.10 (Endogenous recurrence mechanism (candidate-switching form))**.** *For autonomous deterministic candidate dynamics with top conjugate candidates in persistent Pareto conflict, there exist constants* $\delta, \tau > 0$ *(independent of the observation horizon) such that the recurrence hypothesis of Theorem 8.9 holds. Consequently,* $\liminf_{T\to\infty} N_{\mathrm{sw}}(T)/T > 0$.

8.3. **Iterated operators and degree growth.** Define an iterated chain by $A_1 = A_0$ and $A_{k+1} = M(A_k, e_k)$ for a sequence of coupling vectors.

Definition 8.11 (Apex degree)**.** The *apex degree* d_k is the degree of the highest-degree irreducible factor of $\det(\lambda I - A_k)$ over $\mathbb{Q}$.

Conjecture 8.12 (Degree growth). *For the* standard chain

$$A_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \qquad e_k = \mathbf{1} \text{ for all } k,$$

the apex degree satisfies $d_{k+1} = 2d_k + 1$ *for* $k \geq 3$.

Remark 8.13 (Evidence for Theorem 8.12). Verified computationally for $k \leq 8$ using exact factorisation of the characteristic polynomial over $\mathbb{Q}$. The recurrence $d_{k+1} = 2d_k + 1$ holds exactly for $k = 3, 4, 5, 6, 7, 8$, producing apex degrees $3, 7, 15, 31, 63, 127$.

Remark 8.14. The orbit signature reveals a universal pattern: one high-degree apex orbit, a sea of degree-2 golden orbits, and (at odd chain steps) rational eigenvalues. The golden orbits proliferate at every step and never disappear—they are the "fossil record" of every previous application of the operator.

9. The Sweep Framework

A generic ordering-and-threshold framework is introduced to analyse how certification targets behave along eigenvector-induced vertex sweeps. This provides a bridge from orbit-level spectral statements to side-wise partition diagnostics.

Definition 9.1 (Eigenvector sweep). Let $A \in \mathbb{R}^{n\times n}$ be real symmetric with eigenvector v. Order the indices $u_1, \dots, u_n$ so that $v_{u_1} \geq \cdots \geq v_{u_n}$. For $0 \leq k \leq n$, define

$$S_k = \{u_1, \dots, u_k\}, \qquad T_k = \{u_{k+1}, \dots, u_n\}.$$

Definition 9.2 (Monotone functions on sweeps). Let $\chi\colon 2^{\{1,\dots,n\}} \to \mathbb{Z}_{\geq 0}$ be a set function satisfying:

- (M1) $\chi(\varnothing) = 0$,
- (M2) $\chi(S) \leq \chi(S \cup \{v\})$ for all S, v (monotonicity under vertex addition),
- (M3) $\chi(S \cup T) \leq \chi(S) + \chi(T)$ for disjoint S, T (subadditivity).

The chromatic number of a graph satisfies (M1)–(M3).

Lemma 9.3 (Inertia bound). *For any set function* χ *satisfying (M1)–(M3), any sweep ordering, and any partition* $\{1, \dots, n\} = S_k \cup T_k$*:*

$$h(k) := \chi(S_k) + \chi(T_k) \geq \chi(\{1, \dots, n\}) \quad \text{for all } 0 \leq k \leq n.$$

Proof. By subadditivity, $\chi(\{1, \dots, n\}) = \chi(S_k \cup T_k) \leq \chi(S_k) + \chi(T_k) = h(k)$. □

Definition 9.4 (One-step gap). Fix targets $s, t \geq 1$ with $s+t = \chi(\{1, \dots, n\})+1$. Let k_0 be the first index with $\chi(S_{k_0}) \geq s$. The *one-step gap* occurs at k_0 if $\chi(T_{k_0}) = t - 1$: the function χ on the complement drops below t at the exact step where the growing side first reaches s.

Lemma 9.5 (Unit-deficit bound). *With notation as above:*

(i) *For all* $k < k_0$*:* $\chi(S_k) \leq s - 1$ *and* $\chi(T_k) \geq t$.

(ii) At $k = k_0$: $\chi(S_{k_0}) \geq s$ and $\chi(T_{k_0}) \geq t - 1$.
The targets (s, t) are certified at k_0 unless the one-step gap occurs.

Proof. Part (i): by definition $\chi(S_k) \leq s - 1$ for $k < k_0$. By Theorem 9.3, $\chi(T_k) \geq \chi(V) - \chi(S_k) \geq (s + t - 1) - (s - 1) = t$. Part (ii): $\chi(S_{k_0}) \geq s$ by definition. By inertia, $\chi(T_{k_0}) \geq (s + t - 1) - \chi(S_{k_0})$; the tightest case $\chi(S_{k_0}) = s$ gives $\chi(T_{k_0}) \geq t - 1$. □

Remark 9.6 (Stiffness and the one-step gap). Eigenvectors at large positive eigenvalues are "stiff": their components vary slowly across entries coupled by large matrix elements. Moving one index from T to S cannot fragment a stiff partition, because the transferred index has nearly the same eigenvector value as its neighbours. For transparent modes of golden companion operators, this mechanism is made precise by the Edge Internalization Lemma (Theorem 6.21) and proved in Theorem 6.25. The strong unconditional Positive-Wins form (Theorem 6.39) fails in general for coupled modes (Theorem 6.35), but a conditional form is still expected: under transfer hypotheses such as cut stability, the stiffest orbit member should be the most effective partition source because it best resists the one-step gap.

10. Arithmetic Invariants

We return to the general setting of a real symmetric matrix A with golden companion $M(A, e)$. The invariants below depend only on the golden ratio algebra $\mathbb{Z}[\varphi]$ and hold for arbitrary (A, e).

10.1. **The Galois centroid and shell depth.** The golden resolvent factorisation produces eigenvalue pairs $\{\mu\varphi,\ -\mu/\varphi\}$ at each transparent mode. Their absolute amplification factors φ and φ^{-1} determine a natural arithmetic invariant.

Definition 10.1 (Galois centroid)**.** The *Galois centroid* of the golden pair is

$$\gamma \;=\; \frac{\varphi + \varphi^{-1}}{2} \;=\; \frac{\sqrt{5}}{2}. \tag{72}$$

Definition 10.2 (Golden partial sums)**.** For $m \geq 1$, the *golden partial sum* is

$$S_m \;=\; \sum_{j=1}^{m} \varphi^{-j} \;=\; \varphi(1 - \varphi^{-m}). \tag{73}$$

Proposition 10.3 (Shell depth bound)**.** *The Galois centroid equals the $3 \to 4$ transition ratio of the golden partial sums:*

$$\gamma \;=\; \frac{S_4}{S_3}. \tag{74}$$

Moreover, $n = 3$ is the unique positive integer such that

$$\gamma^n \;<\; \sqrt{2} \;<\; \gamma^{n+1}. \tag{75}$$

Proof. Using $\varphi^{-1} = (\sqrt{5}-1)/2$ and the identity $\varphi - 1 = \varphi^{-1}$:

$$S_3 = \varphi^{-1} + \varphi^{-2} + \varphi^{-3} = \sqrt{5} - 1,$$
$$S_4 = S_3 + \varphi^{-4} = \tfrac{1}{2}(5 - \sqrt{5}).$$

Then $S_4/S_3 = (5-\sqrt{5})/\big(2(\sqrt{5}-1)\big)$. Rationalising by $(\sqrt{5}+1)$:

$$\frac{(5-\sqrt{5})(\sqrt{5}+1)}{2(5-1)} = \frac{4\sqrt{5}}{8} = \frac{\sqrt{5}}{2} = \gamma.$$

For the inequality, $\gamma^3 = 5\sqrt{5}/8$ and $\gamma^4 = 25/16$. Squaring both sides of $\gamma^3 < \sqrt{2}$:

$$\frac{125}{64} < 2 \iff 125 < 128.$$

Squaring $\gamma^4 > \sqrt{2}$:

$$\frac{625}{256} > 2 \iff 625 > 512.$$

Since $\gamma > 1$ and $\gamma^3 < \sqrt{2} < \gamma^4$, the integer $n = 3$ is unique. □

Remark 10.4 (The threshold $\sqrt{2}$ from the Positive Boost). The threshold $\sqrt{2}$ arises intrinsically from the golden companion. In the base case $C_5 = M(K_2)$ (Theorem 6.27), the anti-aligned partition at the positive conjugate $-\mu/\varphi$ induces the path P_3 as the positive nodal domain subgraph, with spectral radius $\lambda_{\max}(P_3) = 2\cos(\pi/4) = \sqrt{2}$. The Positive Boost Inequality (Theorem 5.2) certifies this as the minimal spectral density achievable by the golden companion at the first nontrivial step.

Remark 10.5 (Position on the Chebyshev Ladder). The Chebyshev Ladder (Theorem 7.2) places the golden amplification $\varphi = 2\cos(\pi/5)$ at the first rung ($p = 1$). The threshold $\sqrt{2} = 2\cos(\pi/4)$ sits strictly between the zeroth rung and the first:

$$1 \;=\; 2\cos\frac{\pi}{3} \;<\; \sqrt{2} \;=\; 2\cos\frac{\pi}{4} \;<\; \varphi \;=\; 2\cos\frac{\pi}{5}.$$

The shell depth $n = 3$ is determined by the diophantine conditions $5^3 < 2^7$ and $5^4 > 2^9$, encoding the position of $\log_2 5 \approx 2.322$ between $9/4 = 2.25$ and $7/3 \approx 2.333$.

10.2. **Spectral spread and the different ideal.** The spectral spread of a golden pair connects the spectral theory of the golden companion to the arithmetic of $\mathbb{Z}[\varphi]$.

Definition 10.6 (Spectral spread)**.** For a transparent golden pair $\{\mu\varphi,\ -\mu/\varphi\}$ in $\mathrm{Spec}\big(M(A,e)\big)$, the *spectral spread* is

$$\Delta_\lambda(\mu) \;=\; \mu\varphi - \big(-\mu/\varphi\big) \;=\; \mu\big(\varphi + \varphi^{-1}\big) \;=\; \mu\sqrt{5} \;=\; \mu\sqrt{\Delta}, \tag{76}$$

where $\Delta = 5$ is the field discriminant of $\mathbb{Q}(\sqrt{5})$. At unit base eigenvalue, $\Delta_\lambda(1) = \sqrt{\Delta} = 2\gamma$, where γ is the Galois centroid (Theorem 10.1).

Proposition 10.7 (Spectral-arithmetic duality)**.** *The spectral spread of the golden pair at unit base eigenvalue equals the generator of the different ideal of $\mathbb{Z}[\varphi]$ over $\mathbb{Z}$:*

$$\Delta_\lambda(1) \;=\; \varphi + \varphi^{-1} \;=\; \sqrt{5} \;=\; \operatorname{gen}\big(\mathfrak{D}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}\big). \tag{77}$$

Proof. The different ideal of a quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ (so $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{d})/2]$ and $\Delta_K = d$) is $\mathfrak{D}_{K/\mathbb{Q}} = (\sqrt{d})$ [7, Ch. 2]. For $d = 5$: $\mathfrak{D} = (\sqrt{5})$. The spectral identity $\varphi + \varphi^{-1} = \sqrt{5}$ follows from $\varphi = (1+\sqrt{5})/2$ and $\varphi^{-1} = (\sqrt{5}-1)/2$. □

Remark 10.8 (Two faces of $\sqrt{5}$). The identity $\Delta_\lambda(1) = \operatorname{gen}(\mathfrak{D})$ expresses a duality between spectral theory and algebraic number theory:

- **Spectrally**: $\sqrt{5}$ is the gap between the two eigenvalues produced by the golden resolvent factorisation at a single base eigenvalue—the distance between the "physical" mode $\mu\varphi$ and the "internal" mode $-\mu/\varphi$.
- **Arithmetically**: $\sqrt{5}$ generates the different ideal $\mathfrak{D}$, which measures the deviation of $\mathbb{Z}[\varphi]$ from self-duality under the trace form $\operatorname{Tr}_{K/\mathbb{Q}}(xy)$.

Both arise from the same algebraic identity $\varphi-\sigma(\varphi) = \varphi+\varphi^{-1} = \sqrt{5}$, where $\sigma\colon \varphi \mapsto -\varphi^{-1}$ is the Galois conjugation (Theorem 2.3). The codifferent $\mathfrak{D}^{-1} = (1/\sqrt{5})$ appears naturally as a discriminant correction in spectral applications of $\mathbb{Z}[\varphi]$.

11. Discussion

11.1. **Graph chromatic partitions.** When A is the adjacency matrix of a graph G and $e = \mathbf{1}$, the golden companion $M(A, \mathbf{1})$ is the Mycielskian $M(G)$. The theory developed here applies directly:

- Eigenvector threshold cuts of A (or pullback sweeps from $M(A, \mathbf{1})$) produce vertex partitions whose chromatic properties are controlled by Galois orbits.
- The Positive Boost explains why nodal domains of positive eigenvalues inherit chromatic density.
- The secular equation determines the complete spectrum of $M(G)$, and the complementarity principle identifies which eigenvector types handle which graph families.

11.2. **Open problems.** The coupled universal-winner question is settled by Theorems 6.37 and 6.38: the unconditional coupled extension is false, and transfer/Pareto classification provides the exact strongest replacement. The remaining open problems concern the broader structural program:

(1) Prove the Monotone Filtration (Theorem 6.41) for circulant matrices.

(2) Determine whether the degree-growth recurrence $d_{k+1} = 2d_k + 1$ holds exactly for all k or only asymptotically.
(3) Characterise which initial matrices cause the generalised ratio polynomial R_p to factor over $\mathbb{Q}$.
(4) Extend the Chebyshev ladder to other block structures (e.g. cones, suspensions, lexicographic products) and determine their spectral amplification constants.
(5) Determine the Galois groups of the apex polynomials for general parameters.
(6) Prove that for a golden companion with golden-geometric base spectrum ($\mu_j \sim \mu_0\varphi^{-j}$), the cumulative secular residue over m shells grows as γ^m (the Galois centroid rate) rather than as the linear partial sum S_m, due to resolvent feedback in the secular equation (10). This would establish that the shell depth bound (Theorem 10.3) governs the spectral energy, not merely the geometric participation.
(7) Establish the discriminant correction for the full Chebyshev Ladder: at layer depth p, the splitting field is $K_p = \mathbb{Q}(\zeta_{2p+3})^+$ with discriminant Δ_p, and the lattice-dual correction should be $\delta_p = \Delta_p^{-1/2}$. Verify for $p = 2$: $K_2 = \mathbb{Q}(\zeta_7)^+$, $\Delta_2 = 49$, $\delta_2 = 1/7$. (At $p = 1$, the correction is $1/\sqrt{5}$ from the codifferent of $\mathbb{Z}[\varphi]$, cf. Theorem 10.7.)
(8) Apply the golden companion to crystallographic root systems, Dynkin-diagram adjacency matrices, and projection operators.

11.3. **Concluding remarks.** The golden resolvent factorisation provides a single algebraic identity from which the complete spectral theory of block-structured operators with golden companion structure can be derived. The six theorem clusters developed here—from the elementary matrix identity to the Chebyshev Ladder—form a self-contained framework connecting operator spectra to the arithmetic of $\mathbb{Q}(\sqrt{5})$ and its cyclotomic generalisations. The Transfer/Pareto classification settles the coupled universal-winner question definitively, while the three conjectural structural laws point toward a deeper orbit-level theory governing partition effectiveness across all Galois degrees. The arithmetic invariants—the Galois centroid, spectral spread, and connection to the different ideal—suggest that the interplay between spectral factorisation and algebraic number theory extends well beyond the golden ratio to the full Chebyshev hierarchy.

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